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Derivation of Equations for Flexible Multibody Systems in Terms of Quasi-Coordinates from the Extended Hamilton's Principle

Early derivations of the equations of motion for single rigid bodies, single flexible bodies, and flexible multibody systems in terms of quasi-coordinates have been carried out in two stages. The first consists of the use of the extended Hamilton's principle to derive standard Lagrange's equations in terms of generalized coordinates and the second represents a transformation of the Lagrange's equations to equations in terms of quasi-coordinates. In this article, hybrid (ordinary and partial) differential equations for flexible multibody systems are derived in terms of quasi-coordinates directly from the extended Hamilton's principle. The approach has beneficial implications in an eventual spatial discretization of the problem.

INTRODUCTION

Multibody systems are ubiquitous in engineering. Indeed, automobiles, construction equipment, agricultural equipment, aircraft, rotorcraft, robots, space vehicles, etc., can be modeled as multibodies. Of particular interest here are articulated multibodies, characterized by the fact that the bodies are hinged to permit rotational motions of one body relative to another. The literature on multibody systems is concerned to a large extent with land vehicles and robots and to a lesser extent with rotorcraft and space vehicles. In studies by Wittenburg (1977), Huston, Passerello, and Harlow (1978), Hollerbach (1980), Haug (1984, 1989), Roberson and Schwertassek (1988), Huston (1990), and Lin (1991) the bodies are assumed to be rigid. In studies by Ho (1977), Huston (1980), Book (1984), Yoo and Haug (1986a,b), Meirovitch and Quinn (1987), Changizi and Shabana (1988), Meirovitch, Stemple, and Kwak (1990), and Meirovitch and Lim (1992) flexibility is included. The interest lies here in flexible multibody systems.

There are two dominant issues in the treatment of articulated flexible multibody systems, namely modeling of the flexibility and the choice of coordinates for rotational motions. For the most part, flexible bodies are characterized by distributed parameters, such as the mass and stiffness. The motion of distributed-parameter systems is described by partial differential equations. On the other hand, translational and rotational rigid-body motions are described by

Shock and Vibration, Vol. 1, No. 2, pp. 107–119 (1993)

CCC 1070-9622/93/020107-13

ordinary differential equations. Systems of simultaneous ordinary and partial differential equations are referred to as *hybrid*. Difficulties in producing solutions for hybrid differential equations have prompted many investigators to approximate distributed systems through spatial discretization. To this end, the use of the classical Rayleigh–Ritz method or the finite element method is quite common, as discussed by Meirovitch (1980). Both methods are based on the same theory and assume approximate solutions in the form of finite series of admissible functions. One of the main differences between the two is that in the classical Raleigh–Ritz method the admissible functions are global functions, extending over the entire length of the body, and in the finite element method they are local functions, extending over small subdomains of the body. As far as the rotational motions are concerned, it is common practice to describe them by means of Euler-type angles, which permits the use of standard Lagrange's equations. This works well for planar motions. For more complex three-dimensional motions, the use of Euler-type angles complicates the equations of motion unduly. This can be traced to the fact that the kinetic energy takes a relatively complicated form when expressed in terms of such angles. Indeed, the kinetic energy takes the simplest form when expressed in terms of quasi-coordinates. In this regard, it is recalled from Meirovitch (1970) that angular velocities about body axes represent time derivatives of quasi-coordinates. Meirovitch and Nelson (1966), Kwak and Meirovitch (1992), Quinn and Chang (1989), and Quinn (1990) demonstrate that for more complex problems an approach using quasi-coordinates offers many advantages.

Discretized models of distributed-parameter systems are only approximations, as they replace systems with an infinite number of degrees of freedom by systems with a finite number of degrees of freedom, which amounts to truncation. The accuracy of the approximation depends on two related factors, the nature of the admissible functions and the degree of truncation. Quite often, an accurate discrete model requires a very large number of degrees of freedom. In extending the classical Rayleigh–Ritz method to flexible multibody systems, there is also the question of choosing suitable admissible functions, as discussed by Meirovitch and Kwak (1991). The choice of admissible functions can be put on a more rational basis if the boundary-value problem, that is, the differential equations and the boundary conditions, were available. These and perhaps other reasons have prompted a renewed interest in distributed models. Consistent with this, and using the approach of Meirovitch (1980), hybrid differential equations of motion have been derived in Low (1987) and Lee and Junkins (1992).

Hybrid differential equations in terms of quasi-coordinates have been derived for the first time in Meirovitch and Nelson (1966) for a spinning body consisting of a rigid hub with flexible appendages and generalized in Meirovitch (1991) to a wholly flexible body undergoing rigid-body translations, rigid-body rotations, and elastic deformations. The general theory of Meirovitch (1991) has been extended in Meirovitch and Stemple (1993) to articulated flexible multibody systems.

The derivation of equations of motion in terms of quasi-coordinates was carried out in Meirovitch (1970), Meirovitch (1991), and Meirovitch and Stemple (1993) in two stages. The first consists of the use of the extended Hamilton's principle to derive standard Lagrange's equations in terms of generalized coordinates. Then, a transformation from standard Lagrange's equations in terms of generalized coordinates to equations in terms of quasi-coordinates is carried out. We note that Meirovitch (1970) is concerned with a single rigid body, Meirovitch (1991) with a single flexible body, and Meirovitch and Stemple (1993) with flexible multibody systems.

This paper addresses the same problem as that of Meirovitch and Stemple (1993), namely, the derivation of the hybrid equations of motion for flexible multibody systems in terms of quasi-coordinates. However, for the first time the hybrid equations in terms of quasi-coordinates are derived directly from the extended Hamilton's principle, that is, without the intermediate step of deriving standard Lagrange's equations in terms of generalized coordinates. The new formulation is more efficient than that in Meirovitch and Stemple (1993), but it should be noted that it benefited from some relations developed there. Another advantage is that, in an eventual spatial discretization of the problem, certain benefits accrue if the discretization can be carried out by working directly with the extended Hamilton's principle.

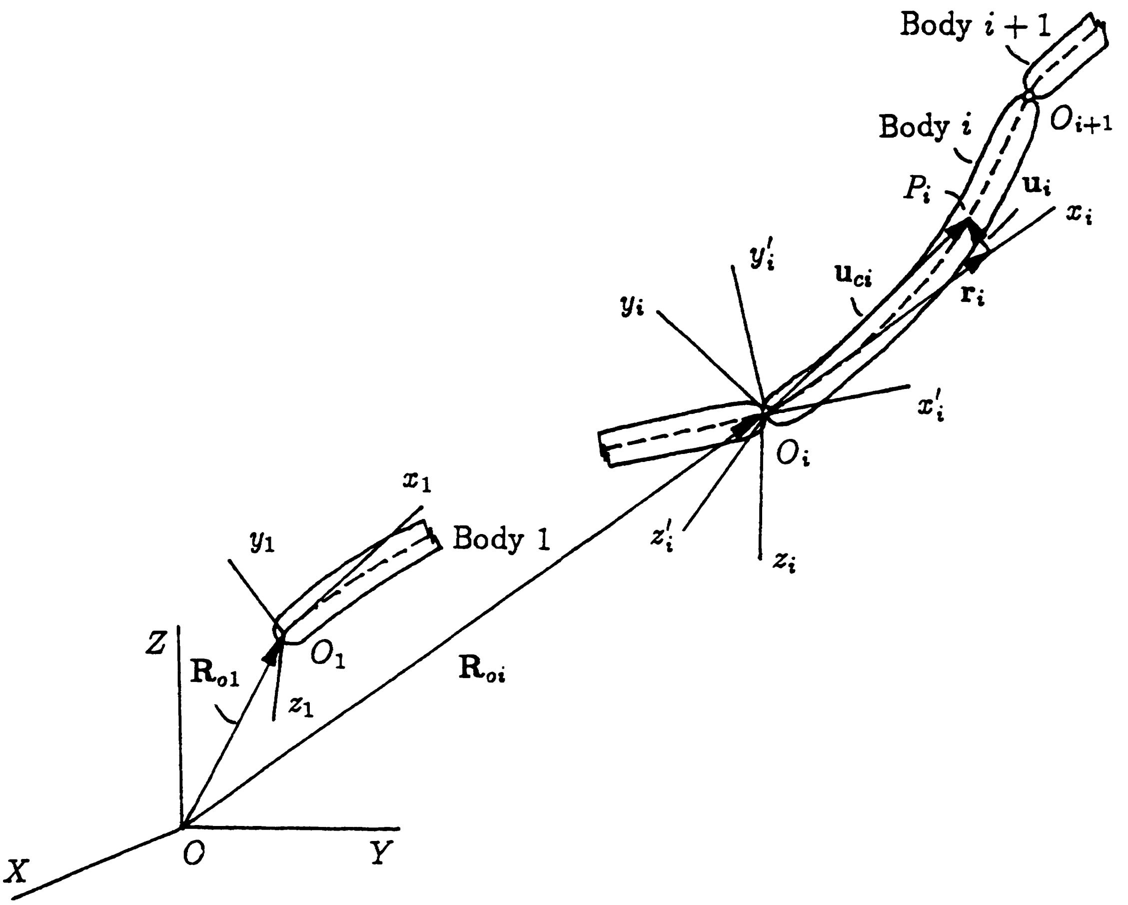

FIGURE 1 Flexible multibody system.

KINEMATICS

We are concerned with a multibody system consisting of a chain of articulated, slender, flexible links i ($i = 1, 2, \ldots, N$), where links $i-1$ and i are hinged at O_i, as shown in Fig. 1. To describe the motion, we conceive of sets of body axes $x_iy_iz_i$ with the origin at O_i and with axis x_i coinciding with the centroidal axis of body i in undeformed state. Essentially, assuming that the links are straight when undeformed, axis x_i passes through O_i and O_{i+1}. As the body deforms, axis x_i remains tangent to the centroidal axis at O_i. At the same time, we conceive of sets of body axes $x_i'y_i'z_i'$ with the origin at O_i but with x_i' embedded in body $i-1$ and tangent to the centroidal axis of body $i-1$ at O_i. We refer to $x_i'y_i'z_i'$ as intermediate axes. Although we will work mostly with body axes, we will find it convenient at times to work with the inertial reference frame XYZ with the origin at O.

We consider a typical point P_i on the deformed centroidal axis of body i and write the position vector from O to P_i in the form

$$\mathbf{R}_i = C_i^* \mathbf{R}_{Oi} + \mathbf{r}_i + \mathbf{u}_i, \quad i = 1, 2, \ldots, n \quad (1)$$

where C_i^* is the matrix of direction cosines of axes $x_iy_iz_i$ relative to axes $x_{i-1}y_{i-1}z_{i-1}$, $\mathbf{R}_{Oi} = [X_{Oi} \quad Y_{Oi} \quad Z_{Oi}]^T$ is the radius vector from O to O_i in terms of components along axes $x_{i-1}y_{i-1}z_{i-1}$, $\mathbf{r}_i$ is the position vector from O_i to P_i when body i is undeformed and $\mathbf{u}_i$ is the elastic displacement of P_i relative to the body axes $x_iy_iz_i$, and we note that $\mathbf{R}_i$, $\mathbf{r}_i$, and $\mathbf{u}_i$ are all in terms of components along $x_iy_iz_i$.

The assumption that the links are slender and originally straight implies that the nominal position of P_i when body i is undeformed is defined by the spatial variable x_i alone. We assume that the links are inextensible and that the only translational elastic displacements are due to bending in the y_i and z_i directions. Hence, the vectors $\mathbf{r}_i$ and $\mathbf{u}_i$ can be expressed in the explicit form

$$\mathbf{r}_i = [x_i \quad 0 \quad 0]^T, \quad \mathbf{u}_i = \mathbf{u}_i(x_i, t) = [0 \quad u_{yi}(x_i, t) \quad u_{zi}(x_i, t)]^T, \quad i = 1, 2, \ldots, N. \quad (2a,b)$$

We will find it convenient to introduce the notation

$$\mathbf{u}_{ci} = \mathbf{r}_i + \mathbf{u}_i = [x_i \quad u_{yi} \quad u_{zi}]^T, \quad i = 1, 2, \ldots, N \quad (3)$$

so that Eq. (1) can be rewritten as

$$\mathbf{R}_i = C_i^* \mathbf{R}_{Oi} + \mathbf{u}_{ci}, \quad i = 1, 2, \ldots, N. \quad (4)$$

The radius vector $\mathbf{R}_{Oi}$ can be expressed in terms of the position of the preceding $i - 1$ bodies in the chain as follows:

$$\mathbf{R}_{Oi} = C_{i-1}^* R_{O,i-1} + \mathbf{u}_{c,i-1}(l_{i-1}, t), \quad i = 2, 3, \ldots, N \quad (5)$$

where l_{i-1} is the length of body $i - 1$. Equation (5) represents recursive relations that can be used to eliminate redundant coordinates. Indeed, recursive use of Eq. (5) permits us to conclude that only one of the vectors $\mathbf{R}_{Oi}$ is independent. We choose $\mathbf{R}_{O1}$ as the independent position vector, where $\mathbf{R}_{O1} = \mathbf{R}_{O1}(t)$ is simply the radius vector from O to the origin O_1 of body axes $x_1y_1z_1$; it is measured relative to the inertial axes, because axes $x_0y_0z_0$ coincide with axes XYZ.

At this point, we turn our attention to the rotational motion. In the first place, we introduce a set of body axes $\xi_i\eta_i\zeta_i$ (Fig. 2) attached to a typical beam cross-section originally with the centroid on x_i, lying in the y_iz_i-plane and moving with the beam cross-section as body i deforms. In this regard, we note that $\xi_{i-1}(l_{i-1}, t)\eta_{i-1}(l_{i-1}, t)\zeta_{i-1}(l_{i-1}, t)$ coincide with $x_i'y_i'z_i'$. We assume that the beam undergoes torsion about x_i, in addition to bending about y_i and z_i. Denoting the angle of twist by ψ_{xi} and the angles of rotation due to bending by ψ_{yi} and ψ_{zi}, respectively, and assuming that the three angles are small, we conclude that the rotation of angles $\xi_i\eta_i\zeta_i$ relative to axes $x_iy_iz_i$ can be expressed by the vector

$$\boldsymbol{\psi}_i = [\psi_{xi} \quad \psi_{yi} \quad \psi_{zi}]^T, \quad i = 1, 2, \ldots, N. \quad (6)$$

In describing rotational motions and vector products, it is often convenient to use skew symmetric matrices derived from vectors. As an example, if a typical vector $\mathbf{r}$ has the components x, y, and z, the associated skew symmetric matrix has the form

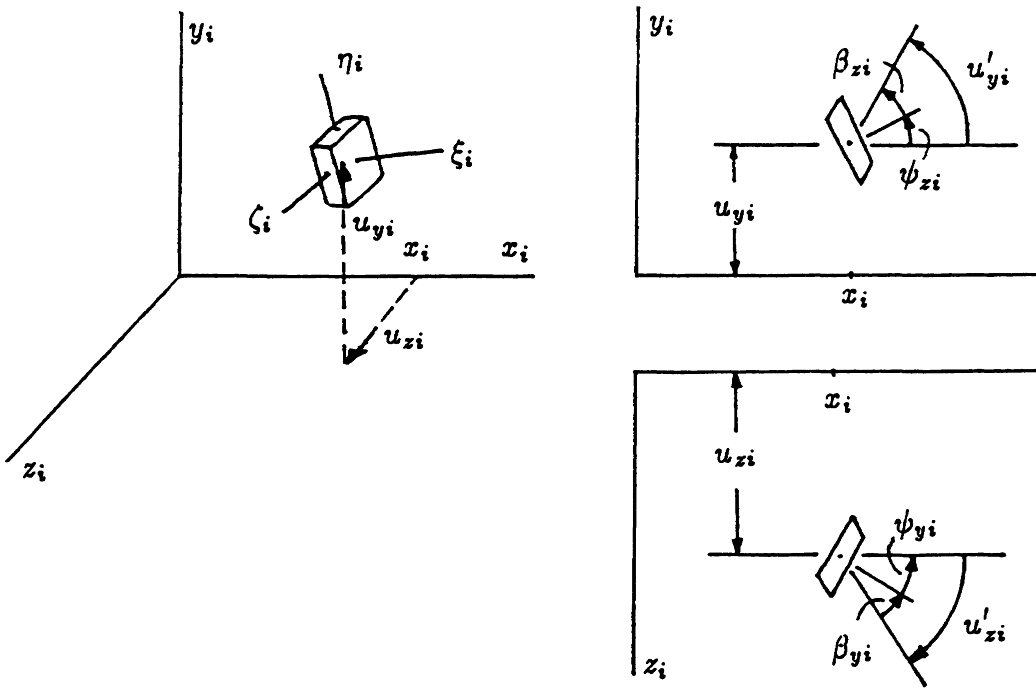

FIGURE 2 Bending displacements.

$$\tilde{r} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}. \quad (7)$$

In view of this definition, the matrix of direction cosines of axes $\xi_i\eta_i\zeta_i$ relative to axes $x_iy_iz_i$ for small angular motions reduces to

$$E_i(x_i, t) = I - \tilde{\psi}_i(x_i, t), \quad i = 1, 2, \ldots, N. \quad (8)$$

Next, we assume that axes $x_iy_iz_i$ are obtained from axes $x_i'y_i'z_i'$ through the rotations θ_{ij}, and denote by $C_i(\boldsymbol{\theta}_i)$ the matrix of direction cosines of $x_iy_iz_i$ relative to $x_i'y_i'z_i'$, where $\boldsymbol{\theta}_i = [\theta_{i1} \ \theta_{i2} \ \theta_{i3}]^T$. Then, the matrix of direction cosines of axes $x_iy_iz_i$ relative to axes $x_{i-1}y_{i-1}z_{i-1}$ is simply

$$C_i^* = C_iE_{i-1}(l_{i-1}, t). \quad (9)$$

From kinematics, the velocity vector of the typical point P_i in terms of the rotating axes $x_iy_iz_i$ has the form

$$\mathbf{V}_i = \mathbf{V}_{Oi} + \tilde{\boldsymbol{\Omega}}_{ri}\mathbf{u}_{ci} + \mathbf{v}_i = \mathbf{V}_{Oi} + \tilde{u}_{ci}^T\boldsymbol{\Omega}_{ri} + \mathbf{v}_i, \quad i = 1, 2, \ldots, N \quad (10)$$

where $\mathbf{V}_{Oi}$ is the velocity vector of the origin O_i, $\boldsymbol{\Omega}_{ri}$ is the angular velocity vector of axes $x_iy_iz_i$ relative to axes XYZ and $\mathbf{v}_i = \dot{\mathbf{u}}_{ci} = \dot{\mathbf{u}}_i$ is the elastic velocity vector relative to $x_iy_iz_i$, all vectors in terms of components along $x_iy_iz_i$. Moreover, the angular velocity vector of the cross-sectional axes $\xi_i\eta_i\zeta_i$ relative to the inertial space is simply

$$\boldsymbol{\Omega}_i = \boldsymbol{\Omega}_{ri} + \boldsymbol{\Omega}_{ei}(x_i, t), \quad i = 1, 2, \ldots, N \quad (11)$$

where

$$\boldsymbol{\Omega}_{ei}(x_i, t) = \dot{\boldsymbol{\psi}}_i(x_i, t), \quad i, 2, \ldots, N \quad (12)$$

is identified as the elastic angular velocity vector of the cross-sectional axes.

At this point, we wish to eliminate redundant velocities. To this end, we note that the velocity vector of point O_i can be written in the recursive form

$$\begin{aligned}\mathbf{V}_{Oi} &= C_i^*\mathbf{V}_{i-1}(l_{i-1}, t) \\ &= C_i^*[\mathbf{V}_{O,i-1} + \tilde{u}_{c,i-1}^T(l_{i-1}, t)\boldsymbol{\Omega}_{r,i-1} \\ &\quad + \mathbf{v}_{i-1}(l_{i-1}, t)] \\ &= C_{i1}^*\boldsymbol{\Omega}_{O1} + \sum_{j=1}^{N} \gamma_{i-j-1}C_{ij}^*[\tilde{u}_{cj}^T(l_j, t)\boldsymbol{\Omega}_{rj} \\ &\quad + \mathbf{v}_j(l_j, t)], \quad i = 2, 3, \ldots N \end{aligned} \quad (13)$$

where C_{ij}^* can be identified as the matrix of direction cosines of axes $x_iy_iz_i$ relative to axes $x_jy_jz_j$ and it satisfies the relations

$$C_{ij}^* = \prod_{k=j+1}^{i} C_k^*, \quad 1 \le j < i \le N \quad (14a)$$

$$C_{i,i-1}^* = C_i^*, \quad C_{ii}^* = I, \quad 1 \le i \le N \quad (14b)$$

$$C_{ij}^{*T} = C_{ji}^*, \quad C_{ik}^*C_{kj}^* = C_{ij}^*, \quad 1 \le i, j, k \le N \quad (14c,d)$$

where I is the identity matrix, and

$$\gamma_k = \begin{cases} 0 & \text{for} \quad k < 0 \\ 1 & \text{for} \quad k = 0, 1, 2, \ldots \end{cases}. \quad (15)$$

Similarly, letting $\boldsymbol{\omega}_i$ be the angular velocity vector of axes $x_iy_iz_i$ relative to axes $x_i'y_i'z_i'$ in terms of $x_iy_iz_i$ components and using Eqs. (11), the angular velocity vector of axes $x_iy_iz_i$ relative to the inertial space can be expressed by means of the recursive formulae

$$\begin{aligned}\boldsymbol{\Omega}_{ri} &= C_i^*\boldsymbol{\Omega}_{i-1}(l_{i-1}, t) + \boldsymbol{\omega}_i \\ &= C_i^*[\boldsymbol{\Omega}_{r,i-1} + \boldsymbol{\Omega}_{e,i-1}(l_{i-1}, t)] + \boldsymbol{\omega}_i \\ &= \sum_{j=1}^{N} C_{ij}^*[\gamma_{i-j}\boldsymbol{\omega}_j + \gamma_{i-j-1}\boldsymbol{\Omega}_{ej}(l_j, t)], \\ i &= 1, 2, \ldots, N \end{aligned} \quad (16)$$

where we recall that axes $x_0y_0z_0$ coincide with the inertial axes XYZ.

DERIVATION OF HYBRID EQUATIONS

The motion of a flexible multibody system can be described in terms of rigid-body displacements of given sets of body axes and elastic deformations relative to the body axes, where the rigid-body displacements depend on time alone and the elastic deformations depend on spatial coordinates and time. As a result, the equations of motion represent a hybrid set consisting of ordinary differential equations for the rigid-body motions and partial differential equations for the elastic motions, where the latter are complemented by boundary conditions. In the preceding section, the rigid-body velocity vectors were expressed in terms of components along body axes, so that they represent nonintegrable combinations of time derivatives of inertial displacements. These body axes velocity vectors are generally referred to in Meirovitch (1970) as time derivatives of quasi-coordinates. In earlier works by Meirovitch (1970, 1991), hybrid equations for a single flexible body were derived from the extended Hamilton's principle by first deriving standard hybrid Lagrange's equations and then transforming the standard equations to equations in terms of quasi-coordinates. The same approach as in Meirovitch (1970, 1991) was used in Meirovitch and Stemple (1993) to derive hybrid equations in terms of quasi-coordinates for flexible multibody systems. In this section, we propose to derive hybrid equations for flexible multibody systems directly from the extended Hamilton's principle. This approach has the advantage that it permits a more natural spatial discretization of the hybrid equations of motion.

The extended Hamilton's principle can be stated in the form (Meirovitch, 1970)

$$\int_{t_1}^{t_2} (\delta L + \overline{\delta W})dt = 0, \quad \delta \mathbf{R}_{O1} = \mathbf{0},$$
$$\delta \boldsymbol{\theta}_i = \delta \mathbf{u}_i = \delta \boldsymbol{\psi}_i = \mathbf{0}, \quad i = 1, 2, \ldots, N \quad \text{at} \quad t = t_1, t_2, \tag{17}$$

where $L = T - V$ is the Lagrangian, in which T is the kinetic energy and V is the potential energy, and $\overline{\delta W}$ is the virtual work due to nonconservative forces. Hence, before we can use Eq. (17) to derive the equations of motion, it is necessary to produce expressions for T, V, and $\overline{\delta W}$.

We express the kinetic energy in the general form

$$T = \sum_{i=1}^{N} \int_0^{l_i} \hat{T}_i dx_i \tag{18}$$

where $\hat{T}_i$ is the kinetic energy density of body i. We assume that the kinetic energy density consists of two parts, one due to the translation of an element of mass associated with the cross-sectional area in the nominal position x_i and one due to the rotation of the same element of mass. Hence, the kinetic energy density of body i has the expression

$$\hat{T}_i = \frac{1}{2}(\rho_i \mathbf{V}_i^T \mathbf{V}_i + \boldsymbol{\Omega}_i^T \hat{J}_{ci} \boldsymbol{\Omega}_i), \quad i = 1, 2, \ldots, N \tag{19}$$

where ρ_i is the mass density and

$$\hat{J}_{ci} = \text{diag}[\hat{J}_{xixi} \quad \hat{J}_{yiyi} \quad \hat{J}_{zizi}], \quad i = 1, 2, \ldots, N \tag{20}$$

is the moment of inertia density matrix, in which $\hat{J}_{xixi}$, $\hat{J}_{yiyi}$, and $\hat{J}_{zizi}$ are cross-sectional mass moment of inertia densities. We note that because the elastic deformations are small, the latter are equal to $\hat{J}_{\xi i \xi i}$, $\hat{J}_{\eta i \eta i}$, and $\hat{J}_{\zeta i \zeta i}$, respectively. Inserting Eqs. (10) and (11) into Eqs. (19), we can write the kinetic energy density in the more explicit form

$$\begin{aligned}\hat{T}_i = \frac{1}{2}[&\rho_i \mathbf{V}_{Oi}^T \mathbf{V}_{Oi} + \boldsymbol{\Omega}_{ri}^T \hat{J}_{ti} \boldsymbol{\Omega}_{ri} + \rho_i \mathbf{v}_i^T \mathbf{v}_i \\ &+ \dot{\boldsymbol{\psi}}_i^T \hat{J}_{ci} \dot{\boldsymbol{\psi}}_i + 2\mathbf{V}_{Oi}^T \tilde{\hat{S}}_i^T \boldsymbol{\Omega}_{ri} \\ &+ 2\rho_i \mathbf{V}_{Oi}^T \mathbf{v}_i + 2\boldsymbol{\Omega}_{ri}^T (\tilde{\hat{S}}_i \mathbf{v}_i + \hat{J}_{ci} \dot{\boldsymbol{\psi}}_i)], \\ i = 1, 2, \ldots, N &\end{aligned} \tag{21}$$

in which $\mathbf{V}_{Oi}$ is given by Eq. (13) and $\boldsymbol{\Omega}_{ri}$ by Eq. (16). Moreover,

$$\hat{J}_{ti} = \hat{J}_{ci} + \hat{J}_i \tag{22}$$

is the total moment of inertia density matrix, where, recalling Eq. (3),

$$\hat{J}_i = \rho_i \tilde{u}_{ci} \tilde{u}_{ci}^T = \rho_i \begin{bmatrix} u_{yi}^2 + u_{zi}^2 & -x_i u_{yi} & -x_i u_{zi} \\ -x_i u_{yi} & x_i^2 + u_{zi}^2 & -u_{yi} u_{zi} \\ -x_i u_{zi} & -u_{yi} u_{zi} & x_i^2 + u_{yi}^2 \end{bmatrix}. \tag{23}$$

In addition, $\tilde{\hat{S}}_i$ can be obtained from

$$\hat{\mathbf{S}}_i = \rho_i \mathbf{u}_{ci} = \rho_i [x_i \quad u_{yi} \quad u_{zi}]^T \tag{24}$$

which is recognized as the vector of first moment of inertia densities.

The potential energy arises commonly from two sources, gravity and flexibility. The interest lies here in a multibody system in space, in which case the gravitational effects are due to differential gravity, as shown in Meirovitch (1970). Because these effects are very small, we assume that the potential energy is equal to the strain energy. As indicated in the preceding section, the bodies undergo torsion about x_i and bending about y_i and z_i, where the torsional angle is denoted by ψ_{xi}, the bending translational displacements are denoted by u_{yi} and u_{zi}, and the bending rotations by ψ_{yi} and ψ_{zi}. In addition, we assume that the bodies undergo shearing distortions, where the shearing distortion angles are denoted by β_{yi} and β_{zi}. From Fig. 2, we conclude that the relations between the bending transitional and rotational displacements and the shearing distortion angles are

$$u'_{yi} = \psi_{zi} + \beta_{zi}, \quad u'_{zi} = -\psi_{yi} - \beta_{yi} \quad (25\text{a,b})$$

where primes denote partial derivatives with respect to x_i. From mechanics of materials, the relation between the twisting moment M_{xi} and the torsional angle ψ_{xi} is given by

$$M_{xi} = k_{xi} G_i I_{xi} \psi'_{xi} \quad (26)$$

where k_{xi} is a factor depending on the shape of the cross-section and $G_i I_{xi}$ is the torsional rigidity, in which G_i is the shear modulus and I_{xi} is the area polar moment of inertia about x_i. Note that for a circular cross-section the shape factor is equal to 1. Moreover, the bending moments are related to the bending rotational displacements by

$$M_{yi} = E_i I_{yi} \psi'_{yi}, \quad M_{zi} = E_i I_{zi} \psi'_{zi} \quad (27\text{a,b})$$

in which E_i is the Young's modulus and I_{yi} and I_{zi} are area moments of inertia about centroidal axes parallel to y_i and z_i, respectively, and the shearing forces are related to the shearing distortion angles according to

$$Q_{yi} = k_{yi} G_i A_i \beta_{zi}, \quad Q_{zi} = -k_{zi} G_i A_i \beta_{yi} \quad (28\text{a,b})$$

where k_{yi} and k_{zi} are factors depending on the shape of the cross-sectional area A_i.

The strain energy can be expressed as

$$V = \sum_{i=1}^{N} \int_0^{l_i} \hat{V}_i dx_i \quad (29)$$

where

$$\hat{V}_i = \frac{1}{2}(M_{xi}\psi'_{xi} + M_{yi}\psi'_{yi} + M_{zi}\psi'_{zi} + Q_{yi}\beta_{zi} - Q_{zi}\beta_{yi}), \quad i = 1, 2, \ldots, N \quad (30)$$

is the strain energy density for body i. Inserting Eqs. (26)–(28) into Eqs. (30), the strain energy density takes the form

$$\hat{V}_i = \frac{1}{2}[k_{xi}G_iI_{xi}(\psi'_{xi})^2 + E_iI_{yi}(\psi'_{yi})^2 + E_iI_{zi}(\psi'_{zi})^2 + k_{yi}G_iA_i(u'_{yi} - \psi_{zi})^2 + k_{zi}G_iA_i(u'_{zi} + \psi_{yi})^2], \quad i = 1, 2, \ldots, N. \quad (31)$$

The nonconservative virtual work is due to actuator forces and torques and can be written in the form

$$\overline{\delta W} = \sum_{i=1}^{N} [\mathbf{M}_{Oi}^{*T}\delta\theta_i^* + \int_0^{l_i} (\mathbf{f}_i^T\delta\mathbf{R}_i^* + \mathbf{m}_i^T\delta\mathbf{\Theta}_i^*)dx_i] \quad (32)$$

where $\mathbf{M}_{Oi}^*$ are torques acting at the joints O_i, $\mathbf{f}_i$ are distributed forces, and $\mathbf{m}_i$ are distributed torques, both acting on body i, and note that all three are in terms of $x_iy_iz_i$ components; $\delta\boldsymbol{\theta}_i^*$, $\delta\mathbf{R}_i^*$, and $\delta\mathbf{\Theta}_i^*$ are associated virtual displacement vectors. By analogy with Eqs. (10) and (11), the latter two can be written as

$$\delta\mathbf{R}_i^* = \delta\mathbf{R}_{Oi}^* + \tilde{r}_i^T\delta\mathbf{\Theta}_{ri}^* + \delta\mathbf{u}_i,$$
$$\delta\mathbf{\Theta}_i^* = \delta\mathbf{\Theta}_{ri}^* + \delta\boldsymbol{\psi}_i, \quad i = 1, 2, \ldots, N. \quad (33\text{a,b})$$

Introducing Eqs. (33) into Eq. (32), the virtual work becomes

$$\overline{\delta W} = \sum_{i=1}^{N} [\mathbf{M}_{Oi}^{*T}\delta\boldsymbol{\theta}_i^* + \mathbf{F}_{ri}^{*T}\delta\mathbf{R}_{Oi}^* + \mathbf{M}_{ri}^{*T}\delta\boldsymbol{\Theta}_{ri}^* + \int_0^{l_i} (\mathbf{f}_i^T\delta\mathbf{u}_i + \mathbf{m}_i^T\delta\psi_i)dx_i] \quad (34)$$

where

$$\mathbf{F}_{ri}^* = \int_0^{l_i} \mathbf{f}_i dx_i, \quad \mathbf{M}_{ri}^* \int_0^{l_i} (\tilde{r}_i\mathbf{f}_i + \mathbf{m}_i)dx_i \quad (35\text{a,b})$$

are resultant forces and torques acting on member i, respectively.

At this point, we wish to use the extended Hamilton's principle, Eq. (17), to derive generic equations of motion. To this end, we recall that all position and velocity vectors of the joints O_i ($i = 2, 3, \ldots, N$) depend on $\mathbf{R}_{O1}$ and $\mathbf{V}_{O1}$. Hence, using Eqs. (21) and (31), we can write the Lagrangian in the general functional form

$$L = L[\mathbf{R}_{O1}, \mathbf{V}_{O1}, \boldsymbol{\theta}_i, \boldsymbol{\omega}_i, \mathbf{u}_i, \mathbf{u}_i', \dot{\mathbf{u}}_i, \mathbf{u}_i(l_i, t), \dot{\mathbf{u}}_i(l_i, t), \boldsymbol{\psi}_i, \boldsymbol{\psi}_i', \dot{\boldsymbol{\psi}}_i, \boldsymbol{\psi}_i(l_i, t), \dot{\boldsymbol{\psi}}_i(l_i, t)] \quad (36)$$

so that the variation in the Lagrangian can be written as

$$\begin{aligned}\delta L = {} & \delta \mathbf{R}_{O1}^T \frac{\partial L}{\partial \mathbf{R}_{O1}} + \delta \mathbf{V}_{O1}^T \frac{\partial L}{\partial \mathbf{V}_{O1}} + \sum_{i=1}^{N} \Bigg[\delta \boldsymbol{\theta}_i^T \frac{\partial L}{\partial \boldsymbol{\theta}_i} \\ & + \delta \boldsymbol{\omega}_i^T \frac{\partial L}{\partial \boldsymbol{\omega}_i} + \int_0^{l_i} \left(\delta \mathbf{u}_i^T \frac{\partial \hat{L}_i}{\partial \mathbf{u}_i} + \delta \mathbf{u}_i'^T \frac{\partial \hat{L}_i}{\partial \mathbf{u}_i'} \right. \\ & \left. + \delta \dot{\mathbf{u}}_i^T \frac{\partial \hat{L}_i}{\partial \dot{\mathbf{u}}_i} \right) dx_i + \delta \mathbf{u}_i^T(l_i, t) \frac{\partial L}{\partial \mathbf{u}_i(l_i, t)} \\ & + \delta \dot{\mathbf{u}}_i^T(l_i, t) \frac{\partial L}{\partial \dot{\mathbf{u}}_i(l_i, t)} + \int_0^{l_i} \left(\delta \boldsymbol{\psi}_i^T \frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}_i} \right. \\ & \left. + \delta \boldsymbol{\psi}_i'^T \frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}_i'} + \delta \dot{\boldsymbol{\psi}}_i^T \frac{\partial \hat{L}_i}{\partial \dot{\boldsymbol{\psi}}_i} \right) dx_i + \delta \boldsymbol{\psi}_i^T(l_i, t) \\ & \times \frac{\partial L}{\partial \boldsymbol{\psi}_i(l_i, t)} + \delta \dot{\boldsymbol{\psi}}_i^T(l_i, t) \frac{\partial L}{\partial \dot{\boldsymbol{\psi}}_i(l_i, t)} \Bigg] \end{aligned} \quad (37)$$

where $\hat{L}_i$ is the Lagrangian density for body i. Before proceeding any farther, we note that $\mathbf{R}_{O1}$ is in terms of inertial components and $\boldsymbol{\theta}_i$ are Eulerian-type angles. On the other hand, $\mathbf{V}_{O1}$ and $\boldsymbol{\omega}_i$ are quasi-velocity vectors, so that they are in terms of body-axes components. As shown in Meirovitch and Stemple (1993), the relation between the two types of quantities has the form

$$\mathbf{V}_{O1} = C_1 \dot{\mathbf{R}}_{O1}, \quad \boldsymbol{\omega}_i = D_i \dot{\boldsymbol{\theta}}_i, \quad i = 1, 2, \ldots, N \quad (38a,b)$$

where $C_1 = C_1(\boldsymbol{\theta}_1)$ is the matrix of direction cosines of axes $x_1 y_1 z_1$ relative to axes XYZ and $D_i = D_i(\boldsymbol{\theta}_i)$ are transformation matrices.

It may appear paradoxical, but to derive equations in terms of quasi-coordinates, it is necessary to express the variations in the quasi-velocities in terms of variations in the actual coordinates and velocities. To this end, we consider Eq. (38a) and write

$$\delta \mathbf{V}_{O1} = \delta C_1 \dot{\mathbf{R}}_{O1} + C_1 \delta \dot{\mathbf{R}}_{O1}. \quad (39)$$

But, from Meirovitch and Stemple (1993), we have

$$\dot{C}_1 = \tilde{\omega}_1^T C_1 \quad (40)$$

where $\tilde{\omega}_1$ is a skew symmetric matrix associated with $\boldsymbol{\omega}_1$ [see (Eq. (7)]. By analogy with Eq. (40), we can write

$$\delta C_1 = \delta \tilde{\theta}_1^{*T} C_1 \quad (41)$$

where $\delta \tilde{\theta}_1^*$ is a skew symmetric matrix of quasi-virtual displacements. The concept becomes obvious if we consider Eq. (38b) and write

$$\delta \boldsymbol{\theta}_i^* = D_i \delta \boldsymbol{\theta}_i, \quad i = 1, 2, \ldots, N. \quad (42)$$

Clearly, $\delta \tilde{\theta}_1^*$ corresponds to the vector of quasi-virtual displacements $\delta \boldsymbol{\theta}_1^*$. Hence, inserting Eq. (41) into Eq. (39) and considering Eqs. (38), as well as Eq. (42) with $i = 1$, we obtain

$$\begin{aligned}\delta \mathbf{V}_{O1} &= \delta \tilde{\theta}_1^{*T} C_1 \dot{\mathbf{R}}_{O1} + C_1 \delta \dot{\mathbf{R}}_{O1} \\ &= \delta \tilde{\theta}_1^{*T} \mathbf{V}_{O1} + C_1 \delta \dot{\mathbf{R}}_{O1} \\ &= \tilde{V}_{O1} \delta \boldsymbol{\theta}_1^* + C_1 \delta \dot{\mathbf{R}}_{O1} \\ &= \tilde{V}_{O1} D_1 \delta \boldsymbol{\theta}_1 + C_1 \delta \dot{\mathbf{R}}_{O1}. \end{aligned} \quad (43)$$

Similarly, we consider Eq. (38b) and write

$$\delta \boldsymbol{\omega}_i = \delta D_i \dot{\boldsymbol{\theta}}_i + D_i \delta \dot{\boldsymbol{\theta}}_i. \quad (44)$$

But, from Meirovitch and Stemple (1993), we obtain

$$\dot{D}_i = \frac{\partial (D_i \dot{\boldsymbol{\theta}}_i)}{\partial \boldsymbol{\theta}_i^T} + \tilde{\omega}_i^T D_i \quad (45)$$

so that by analogy we can write

$$\delta D_i = \frac{\partial (D_i \delta \boldsymbol{\theta}_i)}{\partial \boldsymbol{\theta}_i^T} + \delta \tilde{\theta}_i^{*T} D_i. \quad (46)$$

Inserting Eq. (46) into Eq. (44) and using Eq. (42), it can be shown after some operations that

$$\begin{aligned}\delta \boldsymbol{\omega}_i &= \left[\frac{\partial (D_i \delta \boldsymbol{\theta}_i)}{\partial \boldsymbol{\theta}_i^T} + \delta \tilde{\theta}_i^{*T} D_i \right] \dot{\boldsymbol{\theta}}_i + D_i \delta \dot{\boldsymbol{\theta}}_i \\ &= \dot{D}_i \delta \boldsymbol{\theta}_i + \delta \tilde{\theta}_i^{*T} \boldsymbol{\omega}_i + D_i \delta \dot{\boldsymbol{\theta}}_i \\ &= \dot{D}_i \delta \boldsymbol{\theta}_i + \tilde{\omega}_i \delta \boldsymbol{\theta}_i^* + D_i \delta \dot{\boldsymbol{\theta}}_i \\ &= (\dot{D}_i + \tilde{\omega}_i D_i) \delta \boldsymbol{\theta}_i + D_i \delta \dot{\boldsymbol{\theta}}_i. \end{aligned} \quad (47)$$

At this point, we are ready to proceed with the extended Hamilton's principle. To this end, we use Eqs. (43) and (47) and carry out the operations term-by-term as follows:

$$\int_{t_1}^{t_2} \delta \mathbf{V}_{O1}^T \frac{\partial L}{\partial \mathbf{V}_{O1}} dt$$

$$= \int_{t_1}^{t_2} (-\delta \boldsymbol{\theta}_1^T D_1 \tilde{V}_{O1} + \delta \dot{\mathbf{R}}_{O1}^T C_1^T) \frac{\partial L}{\partial \mathbf{V}_{O1}} dt$$

$$= -\int_{t_1}^{t_2} \delta \boldsymbol{\theta}_1^T D_1 \tilde{V}_{O1} \frac{\partial L}{\partial \mathbf{V}_{O1}} dt + \delta \mathbf{R}_{O1}^T C_1 \frac{\partial L}{\partial \mathbf{V}_{O1}}\bigg|_{t_1}^{t_2}$$

$$-\int_{t_1}^{t_2} \delta \mathbf{R}_{O1}^T \frac{d}{dt}\left(C_1^T \frac{\partial L}{\partial \mathbf{V}_{O1}}\right) dt$$

$$= -\int_{t_1}^{t_2} \delta \boldsymbol{\theta}_1^T D_1^T \tilde{V}_{O1} \frac{\partial L}{\partial \mathbf{V}_{O1}} dt - \int_{t_1}^{t_2} \delta \mathbf{R}_{O1}^T \left[\dot{C}_1^T \frac{\partial L}{\partial \mathbf{V}_{O1}} + C_1^T \frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{V}_{O1}}\right)\right] dt$$

$$= -\int_{t_1}^{t_2} \delta \boldsymbol{\theta}_1^T D_1^T \tilde{V}_{O1} \frac{\partial L}{\partial \mathbf{V}_{O1}} dt - \int_{t_1}^{t_2} \delta \mathbf{R}_{O1}^T C_1^T \left[\tilde{\omega}_1 \frac{\partial L}{\partial \mathbf{V}_{O1}} + \frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{V}_{O1}}\right)\right] dt \quad (48)$$

$$\int_{t_1}^{t_2} \delta \boldsymbol{\omega}_i^T \frac{\partial L}{\partial \boldsymbol{\omega}_i} dt$$

$$= \int_{t_1}^{t_2} [\delta \boldsymbol{\theta}_i^T (\dot{D}_i^T - D_i^T \tilde{\omega}_i) + \delta \dot{\boldsymbol{\theta}}_i^T D_i^T] \frac{\partial L}{\partial \boldsymbol{\omega}_i} dt$$

$$= \int_{t_1}^{t_2} \delta \boldsymbol{\theta}_i^T (\dot{D}_i^T - D_i^T \tilde{\omega}_i) \frac{\partial L}{\partial \boldsymbol{\omega}_i} dt + \delta \boldsymbol{\theta}_i^T D_i^T \frac{\partial L}{\partial \boldsymbol{\omega}_i}\bigg|_{t_1}^{t_2}$$

$$- \int_{t_1}^{t_2} \delta \boldsymbol{\theta}_i^T \frac{d}{dt}\left(D_i^T \frac{\partial L}{\partial \boldsymbol{\omega}_i}\right) dt$$

$$= -\int_{t_1}^{t_2} \delta \boldsymbol{\theta}_i^T D_i^T \left[\tilde{\omega}_i \frac{\partial L}{\partial \boldsymbol{\omega}_i} + \frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{\omega}_i}\right)\right] dt \quad (49)$$

$$\int_0^{l_i} \delta \mathbf{u}_i'^T \frac{\partial \hat{L}_i}{\partial \mathbf{u}_i'} dx_i = \delta \mathbf{u}_i^T \frac{\partial \hat{L}_i}{\partial \mathbf{u}_i'}\bigg|_0^{l_i} - \int_0^{l_i} \delta \mathbf{u}_i^T \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial \mathbf{u}_i'}\right) dx_i \quad (50)$$

$$\int_{t_1}^{t_2} \delta \dot{\mathbf{u}}_i^T \frac{\partial \hat{L}_i}{\partial \dot{\mathbf{u}}_i} dt = \delta \mathbf{u}_i^T \frac{\partial \hat{L}_i}{\partial \dot{\mathbf{u}}_i}\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta \mathbf{u}_i^T \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_i}{\partial \dot{\mathbf{u}}_i}\right) dt$$

$$= -\int_{t_1}^{t_2} \delta \mathbf{u}_i^T \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_i}{\partial \dot{\mathbf{u}}_i}\right) dt \quad (51)$$

$$\int_{t_1}^{t_2} \delta \dot{\mathbf{u}}_i^T(l_i, t) \frac{\partial L}{\partial \dot{\mathbf{u}}(l_i, t)} dt$$

$$= \delta \mathbf{u}_i^T(l_i, t) \frac{\partial L}{\partial \dot{\mathbf{u}}(l_i, t)}\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta \mathbf{u}_i^T(l_i, t) \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\mathbf{u}}(l_i, t)}\right] dt$$

$$= -\int_{t_1}^{t_2} \delta \mathbf{u}_i^T(l_i, t) \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\mathbf{u}}(l_i, t)}\right] dt. \quad (52)$$

Similar equations to Eqs. (50)–(52) can be obtained for $\boldsymbol{\psi}_i$ by simply replacing $\mathbf{u}_i$ by $\boldsymbol{\psi}_i$ in Eqs. (50)–(52).

Consistent with the above derivations, we must express the virtual work, Eq. (34), in terms of the same variations in the actual coordinates. This requires some extensive operations, as $\delta \mathbf{R}_{O1}^*$ and $\delta \boldsymbol{\Theta}_{ri}^*$ involve coordinates associated with bodies preceding in the chain. To this end, we use the analogy with Eq. (13) and write

$$\delta \mathbf{R}_{Oi}^* = C_{i1}^* \delta \mathbf{R}_{O1}^* + \sum_{j=1}^{N} [\Gamma_{ij} \delta \boldsymbol{\theta}_j^* + \Gamma_{i,j+1} \delta \boldsymbol{\psi}_j(l_j, t) + \gamma_{i-j-1} C_{ij}^* \delta \mathbf{u}_j(l_j, t)],$$

$$j = 1, 2, \ldots, N \quad (53)$$

where

$$\Gamma_{ij} = \sum_{k=j}^{i-1} C_{ik}^* \tilde{u}_{ck}^T(l_k, t) C_{kj}^* \quad (54)$$

and we note that second-order terms in the virtual displacements have been ignored in Eqs. (53). Moreover, by analogy with Eqs. (16), we can write

$$\delta \boldsymbol{\Theta}_{ri}^* = \sum_{j=1}^{N} C_{ij}^* [\gamma_{i-j} \delta \boldsymbol{\theta}_j^* + \gamma_{i-j-1} \delta \boldsymbol{\psi}_j(l_j, t)],$$

$$i = 1, 2, \ldots, N. \quad (55)$$

Inserting Eqs. (53) and (55) into Eq. (34), we can rewrite the virtual work in the form

$$\overline{\delta W}$$

$$= \sum_{i=1}^{N} \left[\mathbf{M}_{Oi}^{*T} \delta \boldsymbol{\theta}_i^* + \mathbf{F}_{ri}^{*T} \delta \mathbf{R}_{Oi}^* + \mathbf{M}_{ri}^{*T} \delta \boldsymbol{\Theta}_{ri}^* + \int_0^{l_i} (\mathbf{f}_i^T \delta \mathbf{u}_i + \mathbf{m}_i^T \delta \boldsymbol{\psi}_i) dx_i\right]$$

$$= \sum_{i=1}^{N} \left\langle \mathbf{M}_{Oi}^{*T} \delta \boldsymbol{\theta}_i^* + \mathbf{F}_{ri}^{*T} \left\{C_{i1}^* \delta \mathbf{R}_{O1}^* + \sum_{j=1}^{N} [\Gamma_{ij} \delta \boldsymbol{\theta}_j^* + \Gamma_{i,j+1} \delta \boldsymbol{\psi}_j(l_j, t) + \gamma_{i-j-1} C_{ij}^* \delta \mathbf{u}_j(l_j, t)]\right\}\right.$$

$$+ \mathbf{M}_{ri}^{*T} \left\{ \sum_{j=1}^{N} C_{ij}^{*}[\gamma_{i-j}\delta\boldsymbol{\theta}_j^{*} + \gamma_{i-j-1}\delta\boldsymbol{\psi}_j(l_j, t)] \right\}$$

$$+ \int_0^{l_i} (\mathbf{f}_i^T\delta\mathbf{u}_i + \mathbf{m}_i^T\delta\boldsymbol{\psi}_i)dx_i \Big\rangle$$

$$= \mathbf{F}_1^{*T}\delta\mathbf{R}_{O1}^{*} + \sum_{i=1}^{N} \left[\mathbf{M}_i^{*T}\delta\boldsymbol{\theta}_i^{*} + \int_0^{l_i} (\mathbf{f}_i^T\delta\mathbf{u}_i + \mathbf{m}_i^T\delta\boldsymbol{\psi}_i)dx_i + \mathbf{U}_i^T\delta\mathbf{u}_i(l_i, t) + \boldsymbol{\Psi}_i^T\delta\boldsymbol{\psi}_i(l_i, t) \right] \tag{56}$$

in which

$$\mathbf{F}_1^{*} = \sum_{i=1}^{N} C_{1i}^{*}\mathbf{F}_{ri}^{*} \tag{57a}$$

$$\mathbf{M}_i^{*} = \mathbf{M}_{Oi}^{*} + \sum_{j=1}^{N} (\Gamma_{ji}^T\mathbf{F}_{rj}^{*} + \gamma_{j-1}C_{ij}^{*}\mathbf{M}_{rj}^{*}), \quad i = 1, 2, \ldots, N \tag{57b}$$

$$\mathbf{U}_i = \sum_{j=1}^{N} \gamma_{j-i-1}C_{ij}^{*}\mathbf{F}_{rj}^{*}, \quad i = 1, 2, \ldots, N-1 \tag{57c}$$

$$\boldsymbol{\Psi}_i = \sum_{j=1}^{N} (\Gamma_{j,i+1}^T\mathbf{F}_{rj}^{*} + \gamma_{j-i-1}C_{ij}^{*}\mathbf{M}_{rj}^{*}), \quad i = 1, 2, \ldots, N-1 \tag{57d}$$

are forces and torques associated with the quasi-virtual displacements.

At this point, we have all the ingredients required for the extended Hamilton's principle. Indeed, inserting Eqs. (37) and (56) into Eq. (17), considering Eqs. (42) and (48)–(52), following the usual steps, and performing simple matrix multiplications when appropriate, we obtain the desired hybrid equations in terms of quasi-coordinates

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{V}_{O1}}\right) + \tilde{\omega}_1 \frac{\partial L}{\partial \mathbf{V}_{O1}} - C_1 \frac{\partial L}{\partial \mathbf{R}_{O1}} = \mathbf{F}_1^{*} \tag{58a}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{\omega}_1}\right) + \tilde{V}_{O1}\frac{\partial L}{\partial \mathbf{V}_{O1}} + \tilde{\omega}_1\frac{\partial L}{\partial \boldsymbol{\omega}_1} - D_1^{-T}\frac{\partial L}{\partial \boldsymbol{\theta}_1} = \mathbf{M}_1^{*} \tag{58b}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{\omega}_i}\right) + \tilde{\omega}_i \frac{\partial L}{\partial \boldsymbol{\omega}_i} - D_i^{-T}\frac{\partial L}{\partial \boldsymbol{\theta}_i} = \mathbf{M}_i^{*}, \quad i = 2, 3, \ldots, N \tag{58c}$$

$$-\frac{\partial \hat{L}_i}{\partial \mathbf{u}_i} + \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial \mathbf{u}_i'}\right) + \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_i}{\partial \dot{\mathbf{u}}_i}\right) = \mathbf{f}_i, \quad i = 1, 2, \ldots, N \tag{58d}$$

$$-\frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}_i} + \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}_i'}\right) + \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_i}{\partial \dot{\boldsymbol{\psi}}_i}\right) = \mathbf{m}_i, \quad i = 1, 2, \ldots, N \tag{58e}$$

where Eqs. (58d) and (58e) are subject to the boundary conditions

$$\mathbf{u}_i(0, t) = \mathbf{0}, \quad \boldsymbol{\psi}_i(0, t) = \mathbf{0}, \quad i = 1, 2, \ldots, N \tag{59a,b}$$

$$-\left.\frac{\partial \hat{L}_i}{\partial \mathbf{u}_i'}\right|_{x_i=l_i} + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\mathbf{u}}_i(l_i, t)}\right] - \frac{\partial L}{\partial \mathbf{u}_i(l_i, t)} = \mathbf{U}_i(l_i, t), \quad i = 1, 2, \ldots, N-1 \tag{59c}$$

$$-\left.\frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}_i'}\right|_{x_i=l_i} + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\boldsymbol{\psi}}_i(l_i, t)}\right] - \frac{\partial L}{\partial \boldsymbol{\psi}_i(l_i, t)} = \boldsymbol{\Psi}_i(l_i, t), \quad i = 1, 2, \ldots, N-1 \tag{59d}$$

$$\frac{\partial \hat{L}_N}{\partial \mathbf{u}_N'} = \mathbf{0}, \quad \frac{\partial \hat{L}_N}{\partial \boldsymbol{\psi}_N'} = \mathbf{0}, \quad x_N = l_N \tag{59e,f}$$

HYBRID STATE EQUATIONS

Equations (58) represent second-order (in time) configuration equations. In control design, it is common practice to work with first-order state equations. First-order equations can be derived conveniently by introducing the momenta

$$\mathbf{p}_{VO1} = \frac{\partial L}{\partial \mathbf{V}_{O1}} = \sum_{i=1}^{N} \frac{\partial \mathbf{V}_{Oi}^T}{\partial \mathbf{V}_{O1}}\frac{\partial L_i}{\partial \mathbf{V}_{Oi}} = \sum_{i=1}^{N} C_{1i}^{*}\frac{\partial L_i}{\partial \mathbf{V}_{Oi}} = \sum_{i=1}^{N} C_{1i}^{*}\left(m_i\mathbf{V}_{Oi} + \tilde{S}_i^T\boldsymbol{\Omega}_{ri} + \int_0^{l_i} \rho_i\mathbf{v}_i dx_i\right) \tag{60a}$$

$$\mathbf{p}_{\omega j} = \frac{\partial L}{\partial \boldsymbol{\omega}_j} = \sum_{i=1}^{N}\left(\frac{\partial \mathbf{V}_{Oi}^T}{\partial \boldsymbol{\omega}_j}\frac{\partial L_i}{\partial \mathbf{V}_{Oi}} + \frac{\partial \boldsymbol{\Omega}_{ri}^T}{\partial \boldsymbol{\omega}_j}\frac{\partial L_i}{\partial \boldsymbol{\Omega}_{ri}}\right) = \sum_{i=1}^{N}\left(\Gamma_{ij}^T\frac{\partial L_i}{\partial \mathbf{V}_{Oi}} + \gamma_{i-j}C_{ij}^{*}\frac{\partial L_i}{\partial \boldsymbol{\Omega}_{ri}}\right)$$

$$= \sum_{i=1}^{N}\left\{\Gamma_{ij}^T\left(m_i\mathbf{V}_{Oi} + \tilde{S}_i^T\boldsymbol{\Omega}_{ri} + \int_0^{l_i} \rho_i\mathbf{v}_i dx_i\right)\right.$$

$$+ \gamma_{i-j} C_{ij}^* \left[J_{ti} \boldsymbol{\Omega}_{ri} + \tilde{S}_i \mathbf{V}_{Oi} + \int_0^{l_i} (\tilde{\hat{S}}_i \mathbf{v}_i + \hat{J}_{ci} \boldsymbol{\Omega}_{ei}) dx_i \right] \Bigg\}, \quad j = 1, 2, \ldots, N \tag{60b}$$

where m_i is the total mass of body i and $\mathbf{V}_{Oi}$ and $\boldsymbol{\Omega}_{ri}$ are given by Eqs. (13) and (16), respectively, as well as the momentum densities

$$\hat{\mathbf{p}}_{vi} = \frac{\partial \hat{L}_i}{\partial \mathbf{v}_i} = \frac{\partial \hat{L}_i}{\partial \dot{\mathbf{u}}_i} = \rho_i \mathbf{V}_{Oi} + \tilde{\hat{S}}_i^T \boldsymbol{\Omega}_{ri} + \rho_i \mathbf{v}_i, \quad i = 1, 2, \ldots, N \tag{60c}$$

$$\hat{\mathbf{p}}_{\Omega ei} = \frac{\partial \hat{L}_i}{\partial \boldsymbol{\Omega}_{ei}} = \frac{\partial \hat{L}_i}{\partial \dot{\boldsymbol{\psi}}_i} = \hat{J}_{ci}(\boldsymbol{\Omega}_{ri} + \boldsymbol{\Omega}_{ei}), \quad i = 1, 2, \ldots, N. \tag{60d}$$

Then, using Eqs. (38) and inserting Eqs. (60) into Eqs. (58), the hybrid state equations in terms of quasi-coordinates can be written as

$$\dot{\mathbf{R}}_{O1} = C_1^T \mathbf{V}_{O1}, \quad \dot{\boldsymbol{\theta}}_i = D_i^{-1} \boldsymbol{\omega}_i \tag{61a,b}$$

$$\dot{\mathbf{u}}_i = \mathbf{v}_i, \quad \dot{\boldsymbol{\psi}}_i = \boldsymbol{\Omega}_{ei}, \quad i = 1, 2, \ldots, N \tag{61c,d}$$

$$\dot{\mathbf{p}}_{VO1} = -\tilde{\omega}_1 \mathbf{p}_{VO1} + \mathbf{F}_1^* \tag{61e}$$

$$\dot{\mathbf{p}}_{\omega 1} = \tilde{V}_{O1} \mathbf{p}_{VO1} - \tilde{\omega}_1 \mathbf{p}_{\omega 1} + \mathbf{M}_1^* \tag{61f}$$

$$\dot{\mathbf{p}}_{\omega i} = \tilde{\omega}_i \mathbf{p}_{\omega i} + \mathbf{M}_i^*, \quad i = 2, 3, \ldots, N \tag{61g}$$

$$\dot{\hat{\mathbf{p}}}_{vi} = -\frac{\partial \hat{L}_i}{\partial \mathbf{u}_i} + \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial \mathbf{u}_i'}\right) + \mathbf{f}_i, \quad i = 1, 2, \ldots, N \tag{61h}$$

$$\dot{\hat{\mathbf{p}}}_{\Omega ei} = -\frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}_i} + \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}_i'}\right) + \mathbf{m}_i, \quad i = 1, 2, \ldots, N \tag{61i}$$

where Eqs. (61h) and (61i) are subject to the boundary conditions (59).

DISCRETIZATION CONSIDERATIONS

Equations (61) represent a set of simultaneous hybrid (ordinary and partial) differential equations describing the motion of our articulated flexible multibody system. Moreover, the elastic displacements are subject to boundary conditions (59). The equations may have a relatively simple appearance, but this appearance is highly misleading. Indeed, the equations are nonlinear and hybrid. Expressing them as first-order equations in terms of quasi-momenta gives them a neat compact form, but the quasi-momenta themselves have relatively complicated expressions. Although the object of this paper, namely, the derivation of hybrid equations of motion for articulated flexible multibody systems, has been accomplished, some thoughts on possible solutions are in order. Clearly, the state of the art does not permit a closed-form solution of Eqs. (61), so that an approximate solution is a virtual necessity.

The presence of partial differential equations, Eqs. (61h) and (61i), implies that the dimension of the problem is infinite. Hence, the first task is to reduce the system to a finite-dimensional one. As indicated in Meirovitch (1980), this task can be carried out by means of the classical Rayleigh–Ritz method or the finite element method. In both cases, we assume a solution for the elastic displacement and velocity vectors in the form

$$\mathbf{u}_i(x_i, t) = \Phi_{ui}(x_i)\boldsymbol{\xi}_{ui}(t), \quad \dot{\mathbf{u}}_i(x_i, t) = \mathbf{v}_i(x_i, t) = \Phi_{ui}(x_i)\boldsymbol{\eta}_{ui}(t), \quad i = 1, 2, \ldots, N \tag{62a,b}$$

$$\boldsymbol{\psi}_i(x_i, t) = \Phi_{\psi i}(x_i)\boldsymbol{\xi}_{\psi i}(t), \quad \dot{\boldsymbol{\psi}}_i(x_i, t) = \boldsymbol{\Omega}_{ei}(x_i, t) = \Phi_{\psi i}(x_i)\boldsymbol{\eta}_{\psi i}(t), \quad i = 1, 2, \ldots, N \tag{62c,d}$$

where

$$\dot{\boldsymbol{\xi}}_{ui}(t) = \boldsymbol{\eta}_{ui}(t), \quad \dot{\boldsymbol{\xi}}_{\psi i}(t) = \boldsymbol{\eta}_{\psi i}(t), \quad i = 1, 2, \ldots, n \tag{63a,b}$$

in which $\Phi_{ui}(x_i)$ and $\Phi_{\psi i}(x_i)$ are matrices of admissible functions (Meirovitch, 1980) and $\boldsymbol{\xi}_{ui}(t)$ and $\boldsymbol{\xi}_{\psi i}(t)$ are vectors of generalized coordinates. In the finite element method, the admissible functions have the form of low-degree polynomials defined over small subdomains of the elastic members. One of the drawbacks of the finite element method is that, to achieve reasonable accuracy, the number of degrees of freedom of the approximations tends to be very large, as discussed in Meirovitch (1980). This being a particularly critical issue here, it appears that the classical Rayleigh–Ritz method is more suitable for the task at hand. In the classical Rayleigh–Ritz method, the admissible functions are global functions, defined over the entire length of the elastic members. It is shown in Meirovitch and Kwak (1991) that the convergence characteristics can be improved dramatically by taking the admissible functions in the form of quasi-comparison functions, defined as linear combinations of admissible functions capable of satisfying all the boundary conditions of the problem. In choosing quasi-comparison functions, guidance should be sought from the boundary conditions, Eqs. (59).

The discretization process amounts to introducing Eqs. (62) into Eqs. (61h) and (61i), multiplying by Φ_{ui}^T and $\Phi_{\psi i}^T$, respectively, and integrating over the length of the elastic members. The same process must be carried out in conjunction with Eqs. (60c) and (60d), resulting in the generalized momenta $\mathbf{p}_{\eta ui}$ and $\mathbf{p}_{\eta\psi i}(t)$, respectively. The above discretization process can be carried out, perhaps more efficiently, by introducing Eqs. (62) directly into the extended Hamilton's principle, Eq. (17).

Finally, the question can be asked as to need for deriving partial differential equations, particularly if the second discretization process is used. In the first place, whereas the current state of the art does not permit closed-form solutions, one must not preclude such possibilities in the future. Moreover, the complete formulation of the hybrid equations of motion is likely to provide valuable insights into the problem, thus helping produce better approximate solutions.

SUMMARY AND CONCLUSIONS

In deriving equations of motion for articulated flexible multibody systems, it is convenient to introduce reference frames embedded in the individual bodies and known as body axes. Then the motion can be regarded as consisting of the rigid-body motions of the individual reference frames and the elastic motions measured relative to these frames. The rigid-body rotations can be described by Euler-type angles and the corresponding equations of motion have the form of standard Lagrange's equations of motion, which can be obtained by means of the extended Hamilton's principle. Of course, the same approach involves the elastic displacements as well, so that the result is a set of hybrid (ordinary and partial) differential equations together with appropriate boundary conditions. In many cases, there are many advantages to describing the rigid-body motions (translations and rotations) in terms of quasi-coordinates. Equations of motion in terms of quasi-coordinates have been obtained from standard Lagrange's equations through a (not so trivial) use of a transformation of coordinates.

In this article, hybrid differential equations of motion for flexible multibody systems in terms of quasi-coordinates are derived directly from the extended Hamilton's principle, thus obviating the task of deriving standard Lagrange's equations. The approach has beneficial implications in an eventual spatial discretization of the partial differential equations.

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