INTRODUCTION

A major avenue for sound generation is through the action of vibrating structural surfaces. One example is the low-frequency interior noise (called boom) that occurs in automobile passenger compartments. This is a particularly complicated problem because the sound comes from the road, engine, and power train excitations transmitted through the structure and into the cabin. The sound is transmitted due to the structural vibration of the surrounding surfaces of the enclosure, and is especially strong when the structural vibration modes and acoustic modes of the enclosure are coupled. This phenomenon occurs between 20 and 60 Hz in most passenger cars. Boom noise has become a critical issue as the automobile body has become lighter in order to save weight, thus leading to increased noise levels.

The coupling effect is a critical problem in automobile design because it so strongly affects booming noise, due to the structural–acoustic coupling of an automotive cabin. There currently exist two methods by which one can control acoustics in an active manner. The first is to inject “antisound” by means of speakers, that is, sound that destructively interferes with the offending noise. The second approach is to use a distributed sensor/actuator and implement a control strategy to control the structural vibrations that affect acoustic modes. If the structural vibration modes can be controlled, the acoustic response will be suppressed.

The control of flexible structure has been studied for several years now. Piezoelectric material was first used to measure the vibration of a mechanical system by Lee and Moon (1990), who along with other researchers (Chiang, and O’Sullivan, 1991) tailored a piezoelectric sensor to measure a single vibration mode. The piezoelectric sensor was designed to be very sensitive to a desired mode while remaining insensitive to the remaining modes. In this way, the sensor filtered out the contribution of the undesired modes. Because each sensor is tailored to detect a single mode, if multiple modes are to be measured, a single sensor is needed for each mode. To mea-
sure a single mode while rejecting the remaining modes, the width of the sensor must be proportional to the second derivative of the desired mode shape. Because it is very difficult to shape the piezoelectric sensor in accordance with this second derivative, an alternative approach is needed. To date, little work has been done to develop a systematic approach for finding the optimal location of sensor/actuators, but much effort has been put into creating a definition of controllability and observability (Arbel, 1981; Hamden and Nayfeh, 1989). Lim (1992) proposed a method for finding optimal sensor and actuator locations. This approach is based on projecting eigenvectors into the intersection subspace of the controllability and observability subspaces of each collocated sensor–actuator pair. Maghami and Joshi (1993) suggested the optimization of a function of the singular values of the Hankel matrix as a criterion for sensor/actuator placement. Each of these methods applies to the case of a discrete actuator and sensor pair.

In addition to the problem of optimal sensor location, it is also necessary to find the optimal location of a distributed actuator. The distributed actuator is well-suited for the control of vibration modes because it is space efficient, light-weight, and requires no external supports. Bailey and Hubbard (1987) first applied a distributed actuator to control the first vibration mode of a clamped-free beam. They demonstrated a drastic change in the decay rate of the impulse response when a distributed actuator is used as an active damper. Crawley and de Luis (1986) experimentally evaluated the effectiveness of using a piezoelectric actuator as a control source.

In this article we consider the design problem of locating a distributed sensor/actuator to maximize performance indices with regard to measurement and control of structural vibrations in a low frequency range. One objective of this work is to find the optimal location of a uniform piezoelectric sensor to sense effectively the specific vibration modes of a flexible structure. The method used is based on the properties of the observability grammian matrix because this matrix represents the degree of observability of a system’s modes. A second objective of this work is to find the optimal location of a piezoceramic actuator to control effectively the specific vibration modes of a flexible structure. The method used is based on the properties of the controllability grammian matrix, which represents the degree of controllability of the modes. In this work, the model is assumed to be a linear, second-order dynamic system. Another assumption is that we shall predetermine which are the important vibration modes that must be controlled via optimal feedback control.

**CONSTITUTIVE EQUATIONS OF PIEZOELECTRIC MATERIAL**

In this section, the governing equations for a piezoelectric material are derived. The constitutive equations for piezoelectric materials can be expressed in terms of the piezoelectric constants that couple the mechanical strain, stress, electric field, and electric displacement based on the IEEE standard on piezoelectricity (1987):

\[
\sigma_{ij} = C_{ijkl} e_{kl} - E_{ik} E_k \quad (1)
\]

\[
D_i = \epsilon_{ik} e_{kl} + E_{ik} E_k \quad (2)
\]

where \(\sigma_{ij}\) is the stress tensor, \(e_{kl}\) is the strain tensor, \(E_k\) is the electric field, \(D_i\) is the electric displacement (or electric flux density, the charge distributed uniformly over a surface), and \(C_{ijkl}\), \(\epsilon_{ik}\), and \(E_{ik}\) indicate the stiffness matrix, the piezoelectric stress/charge constant matrix, and the piezoelectric permittivity constant of a piezoelectric material. Because the stress and strain matrices \(\sigma_{ij}\) and \(e_{kl}\) are symmetric, the matrix notation can be replaced by vector notation \(\sigma_p\) and \(e_p\), such that 11, 22, 33, 12, 13, and 23 of the subscripts \(ij\) and \(kl\) matches to 1, 2, 3, 4, 5, and 6 of the subscripts \(p\) and \(q\). Similarly, the stiffness matrix \(C_{ijkl}\) and the piezoelectric stress/charge constant matrix \(e_{kl}\) can be expressed in the reduced-order matrix notation \(C_{pq}\) and \(e_p\):

\[
\sigma_p = C_{pq} e_q - E_{kp} E_k \quad (3)
\]

\[
D_i = \epsilon_{pq} e_q + E_{ik} E_k \quad (4)
\]

The subscript in \(\sigma_p\) and \(e_p\) denotes the normal stress and strain for \(p = 1, 2, 3\) and the shear stress and strain for \(p = 4, 5, 6\). The piezoelectric stress/charge matrix of piezoelectric material \(e_pq\) can be expressed as \(C_{pq} = C_{pq} e_p d_p\), where \(d_p\) represents the piezoelectric strain/charge matrix of the piezoelectric material and is defined as the ratio of the induced strain to the applied field in the stress-free state. As an alternative form, it can also be defined as the ratio of charge per unit area to the applied stress (with no external elec-
The stiffness matrix $c_{pq}^E$ is given by

$$
[c_{pq}^E] = \begin{bmatrix}
\lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu
\end{bmatrix}
$$

(5)

where

$$
\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1 + \nu)}
$$

are two Lamé's constants. The dielectric permittivity matrix $\epsilon_{\delta\delta}$ can be expressed as

$$
[\epsilon_{\delta\delta}] = \begin{bmatrix}
\epsilon_{11} & 0 & 0 \\
0 & \epsilon_{22} & 0 \\
0 & 0 & \epsilon_{33}
\end{bmatrix}
$$

(6)

For a thin film of a polarized material (taking the z-direction as the poling direction), the piezoelectric strain constant $d_{ps}$ is given by Sessler's work (1981) as

$$
d_{ps} = \begin{bmatrix}
0 & 0 & 0 & d_{15} & 0 \\
0 & 0 & 0 & d_{15} & 0 \\
d_{31} & d_{32} & d_{33} & 0 & 0
\end{bmatrix}
$$

(7)

Thus, for linear isotropic materials under plane stress, the governing equations become

$$
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{bmatrix} = [C]
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3
\end{bmatrix} - [C]
\begin{bmatrix}
d_{31}E_1 \\
d_{32}E_2 \\
d_{33}E_3
\end{bmatrix}
$$

(8)

where

$$
D_1 = \epsilon_5E_3 + d_{31}\sigma_1 + d_{32}\sigma_2
$$

(9)

and where $E$ is Young's modulus and $\nu$ is the Poisson's ratio of the piezoelectric material.

The strains are related to the displacements $u$, $v$, and $w$ in $x$, $y$, and $z$ directions. The displacement in the $x$-direction from the undeformed to the deformed neutral axis is $u_0$. It is assumed that the point $C$ on the neutral axis undergoes a linear displacement $u_0$ in the $x$-direction, the line perpendicular to the neutral axis, $BD$, remains straight and perpendicular to the neutral axis, and the displacement of any point on $BD$ in the $x$-direction is given by the linear relationship,

$$
u = u_0 - z\alpha \beta = u_0 - z\alpha \frac{\partial w_0}{\partial x}
$$

(11)

as shown in Fig. 1.

Generally, the displacement $u$ in the $x$-direction for any point at a distance $z$ from the neutral axis is given by

$$
u = u_0 - z\frac{\partial w_0}{\partial x}
$$

(12)
The subscript 0 indicates that the deformation is expressed with respect to the undeformed coordinate system. Similarly, \( v \), the displacement in the y-direction for any point at a distance \( z \) from the neutral axis, is given by

\[
v = v_0 - z \frac{\partial w_0}{\partial y}.
\]  

(13)

From linear elasticity, the strains are as follows:

\[
e_1 = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w_0}{\partial x^2}
\]

(14)

\[
e_2 = \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} - z \frac{\partial^2 w_0}{\partial y^2}
\]

(15)

and

\[
e_3 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - z \frac{\partial^2 w_0}{\partial x \partial y}
\]

(16)

Thus,

\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix} = \begin{pmatrix}
\frac{\partial u_0}{\partial x} \\
\frac{\partial v_0}{\partial y} \\
\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}
\end{pmatrix} - z \begin{pmatrix}
\frac{\partial^2 w_0}{\partial x^2} \\
\frac{\partial^2 w_0}{\partial y^2} \\
\frac{\partial^2 w_0}{\partial x \partial y}
\end{pmatrix}.
\]

(17)

Consider the induced charge due to a mechanical deformation. From Gauss' law, it is known that the charge enclosed by a surface \( S \) is

\[
q = \int_S \vec{D} \cdot d\vec{\sigma}
\]

(18)

where \( \vec{D} \) is the electric displacement vector and \( d\vec{\sigma} \) is the differential area vector normal to \( S \). Because charge is built up on the surface of a piezoelectric lamina when it experiences mechanical strain, we can use an equivalent circuit model for a piezoelectric sensor, as shown in Fig. 2, to relate the closed-circuit charge signal measured from the surface electrode to the strain in the beam.

By substituting Eq. (8) and Eq. (17) into Eq. (9), \( D_3 \), the electrical displacement in the z-direction can be derived:

\[
D_3 = \left[ \epsilon_{33} - \frac{E}{1 - \nu^2} (d_{31}^2 + d_{32}^2 + 2\nu d_{31} d_{32}) \right] E_3
\]

\[+ \frac{E}{1 - \nu^2} (d_{31} + \nu d_{32}) e_1 \]

\[+ \frac{E}{1 - \nu^2} (d_{32} + \nu d_{31}) e_2 \]

\[= \epsilon_{33} E_3 + e_{31} e_1 + e_{32} e_2 \]

(19)

where

\[
\epsilon_{33}^M = \epsilon_{33} - \frac{E}{1 - \nu^2} (d_{31}^2 + d_{32}^2 + 2\nu d_{31} d_{32}) \]


\[
\begin{pmatrix}
e_{31} \\
e_{32}
\end{pmatrix} = \frac{E}{1 - \nu^2} \begin{pmatrix} 1 & \nu \\ \nu & 1 \end{pmatrix} \begin{pmatrix} d_{31} \\
d_{32}\end{pmatrix}.
\]

The total induced charge \( q \) on the piezoelectric surfaces can be obtained by integrating the electric displacement \( D_3 \) over the domain covered by two sides of the piezoelectric sensor

\[
q = \int \int_{S_0} D_3 \, dx \, dy
\]

\[= \epsilon_{33} E_3 S_0 + e_{31} \int \int_{S_0} \left( \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w_0}{\partial x^2} \right) \, dx \, dy
\]

\[+ e_{32} \int \int_{S_0} \left( \frac{\partial v_0}{\partial y} - z \frac{\partial^2 w_0}{\partial y^2} \right) \, dx \, dy.
\]

(20)

Because we are interested in the charge signal in a dynamic range, we can set \( E_3 = 0 \). Then the charge signal in a closed circuit induced by the piezoelectric material can be expressed by
Method for Optimal Location of Distributed Sensor/Actuators

\[ q = e_{31} \int_{S_0} \left( \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w_0}{\partial x^2} \right) dx \, dy \]
+ \[ e_{32} \int_{S_0} \left( \frac{\partial u_0}{\partial y} - z \frac{\partial^2 w_0}{\partial y^2} \right) dx \, dy \]
\[ = e_{31} \int_{S_0} \frac{\partial u_0}{\partial x} dx \, dy + e_{32} \int_{S_0} \frac{\partial^2 u_0}{\partial y^2} dx \, dy \]
- \[ \bar{z} \left( e_{31} \int_{S_0} \frac{\partial^2 w_0}{\partial x^2} dx \, dy \right. \]
+ \[ e_{32} \int_{S_0} \frac{\partial^2 w_0}{\partial y^2} dx \, dy \) \] (21)

where \( \bar{z} \) is the distance between the center of the piezoelectric sensor and the neutral axis of the beam-sensor system (Fig. 3).

From the definition of the neutral axis (the axis with invariant length under bending), it is seen that \( u_0 = v_0 = 0 \). The induced charge signal is given by

\[ q = -\bar{z} \left( e_{31} \int_{S_0} \frac{\partial^2 w_0}{\partial x^2} dx \, dy \right. \]
+ \[ e_{32} \int_{S_0} \frac{\partial^2 w_0}{\partial y^2} dx \, dy \) \] (22)

For a beam,

\[ q = -\bar{z} e_{31} \int_{S_0} \frac{\partial^2 w_0}{\partial x^2} dx \, dy \]
\[ = -\bar{z} e_{31} b_{pe} \int_{S_0} \frac{\partial^2 w_0}{\partial x^2} dx \] (23)

where \( b_{pe} \) is the width of the piezoelectric sensor.

FIGURE 3 Coordinates in beam-distributed sensor system.

GOVERNING EQUATIONS FOR A DISTRIBUTED ACTUATOR

A piezoceramic actuator elongates or contracts as the polarity changes when it is exposed to an alternating voltage. A voltage applied across the piezoceramic actuator induces an interface stress and strain between the piezoelectric actuator and the underlying structure. Assuming a stress-free state initially, the induced strain of piezoceramic actuator under an applied voltage \( V \) will be defined by

\[ \varepsilon_r = \frac{d_{31}}{t_a} V \] (24)

where \( d_{31} \) is the ratio of the induced strain to the applied field in the stress-free state, and \( t_a \) is the thickness of the piezoceramic actuator. The stress distribution over cross section is discontinuous at the interface between the piezoceramic and the beam. Because the piezoceramic actuator is attached on one side of the beam, the stress distribution across the beam is not symmetric about the centerline of the beam. Therefore the stress distribution across the beam can be expressed as a combination of pure bending stress and pure extensional stress.

The strain distribution across the beam, as shown in Fig. 4, is

\[ \varepsilon = \kappa z + \varepsilon_0 \] (25)

where \( \varepsilon_0 \) is the extensional strain and \( \kappa \) is the curvature of the beam-distributed actuator system.

FIGURE 4 Strain distribution in beam-distributed actuator system.
The stress distribution across the beam, as shown in Fig. 5, can be expressed by

$$\sigma_b = E_b(\kappa z + \epsilon_0)$$  \hspace{1cm} (26)

where $E_b$ is the Young’s modulus of the beam. Similarly, the stress distribution across the distributed actuator is a combination of pure bending stress, pure extensional stress, and the induced stress by the piezoceramic actuator under an applied voltage $V$.

$$\sigma_p = E_p(\kappa z + \epsilon_0 - \epsilon_p)$$  \hspace{1cm} (27)

where $E_p$ is the Young’s modulus of the distributed actuator. The frequency range under consideration in the vibration control area is much lower in comparison to the thickness and length resonance frequencies of the distributed actuator, which are typically over 20 kHz. Therefore, its effect can also be neglected in the dynamic behavior of composite structure. Requiring the extensional force and bending moment across the combined system to be in equilibrium, gives:

$$\int_{-h/2}^{h/2} \sigma_b z \, dz + \int_{h/2}^{h/2+\epsilon_0} \sigma_p \, dz = 0$$  \hspace{1cm} (28)

and

$$\int_{-h/2}^{h/2} \sigma_b z \, dz + \int_{h/2}^{h/2+\epsilon_0} \sigma_p z \, dz = 0.$$  \hspace{1cm} (29)

From Eq. (28)

$$E_b \epsilon_0 h + E_p(\epsilon_0 - \epsilon_p) t + \frac{1}{2} \kappa E_p t(h + t) = 0.$$  \hspace{1cm} (30)

and from Eq. (29)

$$\kappa \left[ \frac{1}{12} E_b h^3 + \frac{1}{3} E_p \left( t^3 + \frac{1}{2} h t^2 + \frac{3}{4} h^2 t \right) \right] + \frac{1}{2} E_p \epsilon_0 t(h + t) = \frac{1}{2} E_p \epsilon_p t(h + t).$$  \hspace{1cm} (31)

Introducing two nondimensional numbers, $\beta = E_p/E_b$ and $k = t/h$, Eq. (30) and Eq. (31) can be expressed as

$$(1 + \beta k) \epsilon_0 + \frac{1}{2} \kappa h \beta k (1 + k) = \beta \epsilon_p$$  \hspace{1cm} (32)

FIGURE 5 Stress distribution in beam-distributed actuator system.

$$\kappa h \left[ \frac{1}{12} \left( k^3 + \frac{3}{2} k^2 + \frac{3}{4} k \right) \right] + \frac{1}{2} \beta k (1 + k) \epsilon_0 = \frac{1}{2} \beta k (1 + k) \epsilon_p.$$  \hspace{1cm} (33)

From Eq. (32) and Eq. (33) $\kappa h$ and $\epsilon_0$ can be obtained as follows:

$$\kappa h = K_1 \epsilon_p$$  \hspace{1cm} (34)

$$\epsilon_0 = K_2 \epsilon_p$$  \hspace{1cm} (35)

where the coefficients $K_1$ and $K_2$ are

$$K_1 = \frac{6(\beta k + \beta k^2)}{1 + 4\beta k + 6\beta k^2 + 4\beta k^3 + \beta k^4}$$  \hspace{1cm} (36)

$$K_2 = \frac{\beta k + \beta k^4}{1 + 4\beta k + 6\beta k^2 + 4\beta k^3 + \beta k^4}.$$  \hspace{1cm} (37)

The induced bending moment can be obtained by integrating $\sigma_b$ over the cross section of the beam. The extensional strain component $\epsilon_0$ does not contribute to the bending moment. The line moment acting on an edge of the actuator is

$$m_s = \int_{-h/2}^{-h/2} \sigma_b z \, dz = C_0 \epsilon_{pe} \frac{b_{ss} b_{ea}}{W_b} = \bar{C} V \frac{b_{ss} b_{ea}}{W_b}.$$  \hspace{1cm} (38)

where $C_0$ and $\bar{C}$ are constants, which depend on the properties of the piezoceramic actuator and beam, $\sigma_b$ is the bending stress in the beam, $b_{ss}$ is the width of the piezoceramic actuator, and $W_b$ is the width of the beam. By oscillating the voltage applied to the actuator, the induced line moment $m_s$ will oscillate at the same frequency. The
Mathematical Modeling of Beam with a Distributed Sensor/Actuator

The mathematical modeling of clamped-free beams, simply supported beams, and clamped beams are considered. A piezoelectric film is attached to one side of the beam as a sensor (Fig. 7) and a piezoceramic plate is bonded to the other side of the beam as an actuator (Fig. 8). The equation of motion for a beam can be represented as follows:

\[ EI \frac{\partial^4 w}{\partial x^4} + m' \ddot{w} = \frac{\partial^2 m_c}{\partial x^2} \]  

(41)

where \( m_c \) is a line moment due to a control input and \( m' \) is the mass per unit length of the beam. Assuming that the width of the beam and that of the distributed actuator are the same, the moment \( m_c \) due to a distributed actuator can be expressed as

\[ m_c = \hat{C} V (x - x_{1a})^0 - (x - x_{2a})^0 \]  

(42)

where \( x_{1a} \) and \( x_{2a} \) are two end points of the distributed actuator. The function \((x - x_0)^0\) denotes the unit step function at \( x = x_0 \) and the subscript \( c \) indicates the control input. Using the modal representation

\[ w(x, t) = \sum_{n=1}^{N} \eta_n(t) \phi_n(x) \]

the partial differential equation can be transformed to
The right-hand side can be represented as follows:

\[
\frac{\partial^2 m_c}{\partial x^2} = \tilde{C} V (x - x'_{la})^{-2} - (x - x'_{lb})^{-2} = \tilde{C} V [\delta(x - x'_{la}) - \delta'(x - x'_{lb})].
\]

(44)

Multiplying by \( \phi_m(x) \), integrating over the entire length of the beam, and using orthogonality leads to

\[
m' \ddot{\eta}_m + EI \left( \frac{\lambda_m}{l} \right)^4 \eta_m = b_m u(t)
\]

where \( u(t) \) is applied control input \( V \). The representation of \( b_m u(t) \) is defined by

\[
b_m u(t) = \int_0^l \phi_m \frac{\partial^2 m_c}{\partial x^2} = \tilde{C} V \left( \left. \frac{d\phi_m}{dx} \right|_{x=x'_{la}} - \left. \frac{d\phi_m}{dx} \right|_{x=x'_{lb}} \right).
\]

(46)

Next, the system equation of the beam can be expressed in a state-space form as

\[
\dot{x} = Ax + Bu
\]

where

\[
x(t) = \begin{bmatrix}
\eta_1 \\
\eta_2 \\
\vdots \\
\eta_N \\
\dot{\eta}_1 \\
\dot{\eta}_2 \\
\vdots \\
\dot{\eta}_N
\end{bmatrix},
\]

The \( A \) and \( B \) matrices are defined as follows:

\[
A = \begin{bmatrix}
0 & I \\
-\Omega^2 & -2\zeta \Omega
\end{bmatrix}_{2N \times 2N}
\]

\[
B = \begin{bmatrix}
0 \\
\mathbf{b}
\end{bmatrix}_{2N \times 1}
\]

(48)

(49)

where \( \zeta \) and \( \Omega \) are diagonal matrices of modal damping factors and the corresponding natural frequencies.

Because a piezoceramic plate induces a mechanical strain when it is charged, it can be used as a vibration actuator. Assuming that a piezoceramic actuator runs from \( x = x'_{la} \) to \( x = x'_{lb} \) when it is attached to the beam, the effective line moment \( b_m \) to the \( m \)th eigenmode of the beam due to the unit input voltage can be expressed as follows:

\[
b_m = \frac{T_m}{m' A_m^2}
\]

(50)

where \( T_m \) is given by

\[
T_m = \tilde{C} \left( \left. \frac{d\phi_m}{dx} \right|_{x=x'_{la}} - \left. \frac{d\phi_m}{dx} \right|_{x=x'_{lb}} \right).
\]

(51)

Note that the matrix \( B \) is a function of the position at two ends of the piezoceramic array.

The piezoelectric film generates an electric charge when it is deformed; therefore the piezoelectric film attached to the vibrating structure can be used as a vibration sensor. Assuming that a piezoelectric sensor runs from \( x = x_{ls} \) to \( x = x_{ls} \), the induced charge signal \( y(t) \) due to the modal deflections \( \phi_m \), \( 1 \leq m \leq N \), can be expressed as follows:

\[
y(t) = -z_0 e_3 b_p \sum_{m=1}^N \eta_m(t) \int_{x_{ls}}^{x_{ls}} d^2 \phi_m \frac{dx}{dx} - \left. \frac{d\phi_m}{dx} \right|_{x=x_{ls}} \eta_m(t)
\]

(52)

\[
y(t) = -z_0 e_3 b_p \sum_{m=1}^N \left( \left. \frac{d\phi_m}{dx} \right|_{x=x_{ls}} - \left. \frac{d\phi_m}{dx} \right|_{x=x_{ls}} \right) \eta_m(t)
\]

(53)

\[
y(t) = \sum_{m=1}^N K_{qm} \eta_m(t)
\]

(54)

where \( z_0 \) is the distance from the neutral axis of the beam to the centerline of the piezoelectric film, coefficient \( K_{qm} \) is defined as

\[
K_{qm} = -z_0 e_3 b_p \left( \left. \frac{d\phi_m}{dx} \right|_{x=x_{ls}} - \left. \frac{d\phi_m}{dx} \right|_{x=x_{ls}} \right)
\]
and $\varepsilon_{ij}^0$ is a piezoelectric stress/charge constant. The matrix $C$ can then be constructed as the row vector:

$$
C(x_{1a}, x_{2a}) = [K_{q1} K_{q2} \cdots K_{qN} 0 0 \cdots 0]_{1\times(2N)}.
$$

(55)

Note that matrix $C$ is a function of the position of the two ends of the piezoelectric film.

**OBSERVABILITY AND CONTROLLABILITY GRAMMIAN MATRICES**

**Observability**

The state-space representation of a constant-coefficient, linear system is given as follows:

$$
\dot{x} = Ax + Bu \\
y = Cx + Du.
$$

(56)

The state $x(t)$ can be obtained from the given initial state $x(t_0)$ through the state transition matrix:

$$
x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)Bu(\tau) \, d\tau
$$

(57)

where the transition matrix $\Phi(t, t_0) = e^{At-t_0}$. The sensing output from a uniform piezoelectric sensor $y(t)$ is given by

$$
y(t) = C\Phi(t, t_0)x(t_0) + \int_{t_0}^{t} C\Phi(t, \tau)Bu(\tau) \, d\tau.
$$

(58)

Because the input $u(t)$ is assumed to be known, we can assume $u(t) = 0$ without loss of generality, yielding

$$
y(t) = C\Phi(t, t_0)x(t_0).
$$

(59)

If the columns of $C\Phi(t, t_0)$ are linearly independent over $[t_0, t_f]$, then

$$
[C\Phi(t, t_0)x(t_0)]^T y(t) = [C\Phi(t, t_0)x(t_0)]^T C\Phi(t, t_0)x(t_0)
$$

$$
= \int_{t_0}^{t_f} \Phi(t, t_0)^T C^TY(t) \, dt
$$

(60)

where $\int_{t_0}^{t_f} \Phi(t, t_0)^T C^TY(t) \, dt$ is the observability grammian matrix $W^o(t_f, t_0)$.

From Eq. (61), the initial state $x(t_0)$ can be expressed in terms of the observability grammian matrix $W^o(t_f, t_0)$ and the measured signal $y(t)$.

$$
x(t_0) = W^o(t_f, t_0)^{-1} \int_{t_0}^{t_f} \Phi(t, t_0)^T C^TY(t) \, dt.
$$

(62)

After obtaining $x(t_0)$ and evaluating Eq. (57), the state trajectory $x(t)$ over $[t_0, t_f]$ can be observed. It is seen that the state $x(t)$, $t \in [t_0, t_f]$, explicitly depends on both $W^o(t_f, t_0)$ and the measured signal $y(t)$ based on Eqs. (62) and (57). Because the model is asymptotically stable for all linear damped flexible structures, the equilibrium solution $W^s$ can be obtained from the following equation.
If a knowledge of some of the modes of the structure are not required because they do not strongly affect the system’s structural vibration or are not strongly coupled to the acoustic field, we need not consider those modes. If their associated frequencies are between the natural frequencies of modes that do strongly couple, it would be desirable for the sensor to ignore them and only respond to those modes that couple with the acoustics. Let us separate the problem into two distinct subsystems. Subsystem 1 is composed of the important modes whose grammian matrix eigenvalues must be increased (thus magnifying their effect on the sensor) and subsystem 2 consists of the unimportant modes whose grammian matrix eigenvalues must be decreased (thus minimizing their contribution).

Subsystem 1:

\[
\begin{align*}
\dot{x}_1 &= A_1 x_1 \\
y_1 &= C_1 x_1 .
\end{align*}
\]

Subsystem 2:

\[
\begin{align*}
\dot{x}_2 &= A_2 x_2 \\
y_2 &= C_2 x_2 .
\end{align*}
\]

The observability grammian matrices \( W_{1e}^o \) and \( W_{2e}^o \) of these systems satisfy the following Lyapunov equations

\[
\begin{align*}
W_{1e}^o A_1 + A_1^T W_{1e}^o &= -C_1^T C_1 \\
W_{2e}^o A_2 + A_2^T W_{2e}^o &= -C_2^T C_2
\end{align*}
\]

Because the observability grammian matrix indicates the degree of the observability of the modes, it is proposed that the minimum eigenvalue of \( W_{1e}^o \) be maximized to increase the degree of observability of important modes. In a similar manner, we require that the maximum eigenvalue of \( W_{2e}^o \) be minimized in order to decrease the degree of observability of the unimportant modes. Mathematically, these conditions can be expressed as follows:

\[
\begin{align*}
\text{Subsystem 1:} \\
&\max_{0 \leq x_{1L} < x_{1U}, 0 \leq x_{2L} < x_{2U}} \min_i \lambda_i(W_{1e}^o) \\
&= \min_{0 \leq x_{1L} < x_{1U}, 0 \leq x_{2L} < x_{2U}} \left[ -\min_j \lambda_j(W_{2e}^o) \right].
\end{align*}
\]

\[
\begin{align*}
\text{Subsystem 2:} \\
&\min_{0 \leq x_{1L} < x_{1U}, 0 \leq x_{2L} < x_{2U}} \max_j \lambda_j(W_{2e}^o).
\end{align*}
\]

The final optimization problem is to determine \( x_{1s} \) and \( x_{2s} \) in order to minimize the global performance index \( J^o \). The performance index \( J^o \) is the sum of two performance indices, \( J_1^o \) and \( J_2^o \), defined by

\[
\begin{align*}
J_1^o &= -\min_i \lambda_i(W_{1e}^o) \\
J_2^o &= -\min_j \lambda_j(W_{2e}^o).
\end{align*}
\]

Thus

\[
J^o = J_1^o + J_2^o = (-\min_i \lambda_i(W_{1e}^o)) + \max_j \lambda_j(W_{2e}^o).
\]

### Controllability

The control input will be chosen so that after some finite time the state must be zero. Moreover, the control energy applied to the control actuator must be minimized. The state \( x(t) \) can be obtained from the given initial state \( x(t_0) \),

\[
x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)Bu(\tau) \, d\tau.
\]

The controllability grammian matrix \( W^c(t_f, t_0) \) is defined as

\[
W^c(t_f, t_0) = \int_{t_0}^{t_f} \Phi(t, t_0)^TB^T\Phi(t, t_0) \, dt.
\]

The measure of controllability to be used for finding the optimal placement of the distributed actuator can be defined by

\[
J = \max_{x} \min_{i} \lambda_i(W^c(t_f, t_0)).
\]
Because the model is asymptotically stable for all linear damping flexible structures, the equilibrium solution $W_e$ can be obtained from the Lyapunov equation

$$W_e = \lim_{t \to \infty} W^c(t_f, t_0) = \int_{t_0}^{t_f} \Phi(t, t_0)^T BB^T \Phi(t, t_0) \, dt. \quad (79)$$

$W_e$ satisfies the following Lyapunov equation

$$AW_e + W_e A^T = -BB^T. \quad (80)$$

The remaining derivation exactly parallels that of the previous section so we shall simply present the results. The final optimization problem is to design $X_{1a}$ and $X_{2a}$ to minimize the global performance index $J$. The performance index $J$ is a combination of two performance indices $J_1$ and $J_2$ defined by

$$J_1 = - \min_i \lambda_i(W_{1w}) \quad (81)$$

$$J_2 = \max_j \lambda_j(W_{2w}). \quad (82)$$

The global performance index $J$ that must be minimized over the domain composed of the design variables $x_{1a}$ and $x_{2a}$ is defined by

$$J = J_1 + J_2 = [- \min_i \lambda_i(W_{1w})] + \max_j \lambda_j(W_{2w}). \quad (83)$$

**OPTIMIZATION FOR OBSERVABILITY AND CONTROLLABILITY GRAMMINT MATRICES**

The observability gramian matrix $W_{1w}$ represents the degree of observability of the modes that must be controlled and the controllability gramian matrix $W_{2w}$ indicates the degree of controllability of the modes. The minimal eigenvalue of the observability gramian matrix $W_{1w}$ must be made as large as possible to observe the important modes well and the maximum eigenvalue of a different observability gramian matrix $W_{2w}$ must be made as small as possible to not detect the unimportant modes. In the case of the controllability gramian matrix the minimal eigenvalue of $W_{1w}$ should be large to control the important mode well and the maximum eigenvalue of $W_{2w}$ should be small, so as to not excite the unimportant modes. Thus the final optimization problem for the observability gramian matrix can be expressed as follows:

$$\min_{0 \leq x_{1a}, t_{1a}, t_{2a}, t_{3a} \leq L_b} J^o. \quad (84)$$

The important modes must be chosen before proceeding with the optimization algorithm to determine the optimal location of the distributed sensor.

The numerical procedure for obtaining the optimal locations of the distributed piezoelectric sensor is as follows

Step 1: Guess $x_{1s}$ and $x_{2s}$, the locations of the two end points of the piezoelectric sensor.

Step 2: Compute the matrices $C_1(x_{1s}, x_{2s})$ and $C_2(x_{1s}, x_{2s})$ functions of $x_{1s}$ and $x_{2s}$, in $y = Cx$.

Step 3: Solve the Lyapunov equations corresponding to the unimportant modes and the important modes,

$$W_{1w}A_1 + A_1^T W_{1w} = -C_1^T C_1$$

$$W_{2w}A_2 + A_2^T W_{2w} = -C_2^T C_2$$

to obtain $W_{1w}$ and $W_{2w}$.

Step 4: Check the minimum eigenvalue of $W_{1w}$

Step 5: Check the maximum eigenvalue of $W_{2w}$

Step 6: If the optimum condition is satisfied, stop. If not, change the positions and go to the 2nd step.

The flow chart for the optimization corresponding to observability gramian matrix is given in Fig. 9.

The numerical procedure for obtaining the optimal locations for the distributed actuator is very similar to that of finding the optimal location for the distributed sensor. The flow chart for the optimization corresponding to the controllability gramian matrix is given in Fig. 10.

**SIMULATION: CLAMPED-FREE, SIMPLY SUPPORTED, AND CLAMPED-CLAMPED BEAM**

The beam used in this example has the following properties: length: $L_b = 0.381$ m; width: $b = 0.0381$ m; thickness: $t_b = 3.175 \times 10^{-3}$ m; density: $\rho_b = 2,700$ kg/m$^3$, Young’s modulus: $E_b = 60$ Gpa. The properties of the piezoelectric sensor are as follows: width: $b_p = 0.01$ m; thickness: $t_p = 110e-6$ m; piezoelectric strain/charge constant: $d_{31} = 23e-12$ m/V; Young’s modulus: $E_p = 2$ Gpa;
Poisson’s ratio: $\nu_p = 0.3$; piezoelectric stress/charge constants: $e_{11}^0$ and $e_{32}^0$.

The eigenfunctions of a simply supported beam are given as follows:

$$
\phi_m(x) = \sqrt{2} \sin \left( \frac{m\pi x}{L_b} \right). \tag{85}
$$

The eigenfunctions of a clamped-free beam are

$$
\phi_m(x) = \cosh(\beta_m x) - \cos(\beta_m x) - \alpha_m [\sinh(\beta_m x) - \sin(\beta_m x)] \tag{86}
$$

$$
\alpha_m = \frac{\cosh(\beta_m L_b) + \cos(\beta_m L_b)}{\sinh(\beta_m L_b) + \sin(\beta_m x)} \tag{87}
$$

$$
\cos(\beta_m L_b) \cosh(\beta_m L_b) = -1. \tag{88}
$$

The eigenfunctions of a clamped-clamped beam are

$$
\phi_m(x) = \cosh(\beta_m x) - \cos(\beta_m x) \tag{89}
$$

$$
- \alpha_m [\sinh(\beta_m x) - \sin(\beta_m x)]
$$

$$
\alpha_m = \frac{\cosh(\beta_m L_b) - \cos(\beta_m L_b)}{\sinh(\beta_m L_b) - \sin(\beta_m x)} \tag{90}
$$

$$
\cos(\beta_m L_b) \cosh(\beta_m L_b) = 1. \tag{91}
$$

The optimization is performed using the Optimization Toolbox in Matlab (MathWorks, 1991).

**OPTIMAL LOCATION OF DISTRIBUTED SENSOR**

Table 1 shows the results from the optimization procedure to find the optimal location of a piezoelectric sensor for the cases of a clamped-free, simply supported, and clamped-clamped beam. In the first column, the number in parentheses represents the highest vibration mode that is as-
Assumed to be present in the system. The remaining numbers in the first column correspond to those modes whose contribution to the vibration of the system is assumed to be important. The numbers in parentheses in the second through fourth columns denote the normalized sensor positions, $x_{1i}/L_b$ and $x_{2i}/L_b$, corresponding to a clamped-free, simply supported, and clamped-clamped beam. The method through which the sensors observe the modes is by measuring the slope at given points. Thus an optimal sensor position occurs when the difference between the

Table 1. Normalized Optimal Locations of a Piezoelectric Sensor

<table>
<thead>
<tr>
<th>Modes</th>
<th>Clamped-Free Beam</th>
<th>Simply Supported Beam</th>
<th>Clamped-Clamped Beam</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (2)</td>
<td>(0.047082)</td>
<td>(0,1)</td>
<td>(0.22415,0.77585)</td>
</tr>
<tr>
<td>1 (3)</td>
<td>N/A</td>
<td>(0.16667,0.8333)</td>
<td>(0.20774,0.79226)</td>
</tr>
<tr>
<td>1, 2 (3)</td>
<td>(0.18342,0.76865)</td>
<td>(0.08334,0.58333) or</td>
<td>(0.16208,0.53957)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.41667,0.91667)</td>
<td></td>
</tr>
<tr>
<td>1, 3 (3)</td>
<td>(0.0505,0.4025)</td>
<td>(0,1)</td>
<td>(0.35008,0.64992)</td>
</tr>
<tr>
<td>1, 2, 3 (3)</td>
<td>(0.49645,1)</td>
<td>(0,0.3333) or</td>
<td>(0.13020,0.41322) or</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.6667,1)</td>
<td>(0.58678,0.86980)</td>
</tr>
<tr>
<td>1 (4)</td>
<td>N/A</td>
<td>(0.16667,0.8333)</td>
<td>(0.20774,0.79226)</td>
</tr>
<tr>
<td>1, 2 (4)</td>
<td>(0.039156,0.73629)</td>
<td>(0.41667,0.91667) or</td>
<td>(0.11187,0.56941)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.08333,0.58333)</td>
<td></td>
</tr>
<tr>
<td>1, 3 (4)</td>
<td>N/A</td>
<td>(0,1)</td>
<td>(0.35008,0.64992)</td>
</tr>
<tr>
<td>1, 2, 3 (4)</td>
<td>(0.38562,0.87001)</td>
<td>(0.083168,0.41683)</td>
<td>(0.23272,0.67944)</td>
</tr>
<tr>
<td>1, 2, 4 (4)</td>
<td>(0.35101,0.63812)</td>
<td>(0.041662,0.70833)</td>
<td>(0.22602,0.48296)</td>
</tr>
</tbody>
</table>
The slopes at the two sensor positions is maximized for the important modes, and minimized for the unimportant modes. In the case of the clamped-free beam, it was impossible to maximize the difference in slope for the first mode while minimizing the difference in slope for the other modes. In fact, when the first mode is assumed to be the important mode, and the second and third modes were labeled unimportant, the minimum eigenvalue of the observability grammian matrix $W_{x_1}$ for the unimportant modes was much greater than the maximum eigenvalue of the matrix $W_{x_2}$ for the important mode. This failure illustrates the limitation in using a uniform distributed piezoelectric sensor to measure specific vibration modes.

Table 2 represents the minimum eigenvalue in subsystem 1 and the maximum eigenvalue in subsystem 2. The numbers in parentheses in the second column through the fourth column denote the minimum eigenvalue in subsystem 1 and the maximum eigenvalue in subsystem 2. As shown in Table 1, the minimum eigenvalues in subsystem 1 are maximized while the maximum eigenvalues in subsystem 2 are minimized. When the highest mode is the third mode and the important modes are the first, second, and third modes the minimum eigenvalues of $W_{x_1}$ are plotted for several types of beam (Fig. 11–13). These figures illustrate the objective function $-J_0$ when the normalized locations, $x_{1s}/L_b$ and $x_{2s}/L_b$, are varied from 0 to 1. The peak notes the maximum

![FIGURE 11 Minimum eigenvalue of $W_{x_1}$ in clamped-free beam.](image)
value of the objective function $-J_0$ and its position indicates the optimal locations of a distributed sensor. Because $x_{2s}$ is always greater than $x_{1s}$, the value of $-J_0$ is set to zero in the region that $x_{2s}$ is less than $x_{1s}$. From the results, we can use the optimization criterion to find the optimal location of a uniform piezoelectric sensor to effectively sense the specific vibration modes.

**OPTIMAL LOCATION OF DISTRIBUTED ACTUATOR**

The optimal locations of a distributed actuator in a clamped-free, simply supported, and clamped-clamped beam are identical to those of a distributed sensor when the important modes and unimportant modes are the same for either case. The
### Table 3. \( \min \{ \lambda_i(W_{1w}) \} \) and \( \max \{ \lambda_i(W_{2w}) \} \)

<table>
<thead>
<tr>
<th>Modes</th>
<th>Clamped-Free Beam</th>
<th>Simply Supported Beam</th>
<th>Clamped-Clamped Beam</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (2)</td>
<td>(9.048e-5, 5.7137e-14)</td>
<td>(5.0937e-5, 3.8e-32)</td>
<td>(5.3041e-6, 4.1512e-20)</td>
</tr>
<tr>
<td>1 (3)</td>
<td>N/A</td>
<td>(3.82035e-5, 3.2005e-11)</td>
<td>(5.252e-6, 4.7105e-12)</td>
</tr>
<tr>
<td>1, 2 (3)</td>
<td>(2.098e-5, 2.8804e-11)</td>
<td>(2.3866e-6, 7.023e-15)</td>
<td>(9.0491e-7, 1.3401e-12)</td>
</tr>
<tr>
<td>1, 3 (3)</td>
<td>(1.6712e-6, 2.8667e-14)</td>
<td>(6.2874e-7, 1.32e-28)</td>
<td>(3.4716e-7, 8.8645e-15)</td>
</tr>
<tr>
<td>1, 2, 3 (3)</td>
<td>(3.3895e-6, 0)</td>
<td>(6.2805e-7, 0)</td>
<td>(1.7844e-7, 0)</td>
</tr>
<tr>
<td>1 (4)</td>
<td>N/A</td>
<td>(3.8203-e5, 5.3831e-10)</td>
<td>(5.252e-6, 4.0569e-12)</td>
</tr>
<tr>
<td>1, 2 (4)</td>
<td>(1.0773e-5, 1.8143e-10)</td>
<td>(2.3866e-6, 8.3217e-15)</td>
<td>(8.2284e-7, 1.4615e-10)</td>
</tr>
<tr>
<td>1, 3 (4)</td>
<td>N/A</td>
<td>(6.2874e-7, 2.1843e-27)</td>
<td>(3.4716e-7, 8.8645e-15)</td>
</tr>
<tr>
<td>1, 2, 3 (4)</td>
<td>(4.078e-6, 2.3138e-11)</td>
<td>(3.1402e-7, 2.1492e-14)</td>
<td>(1.2121e-7, 1.1157e-15)</td>
</tr>
<tr>
<td>1, 2, 4 (4)</td>
<td>(1.1456e-6, 1.0609e-13)</td>
<td>(1.4915e-7, 5.7283e-15)</td>
<td>(9.4257e-8, 3.026e-14)</td>
</tr>
</tbody>
</table>

Actuator position is chosen to ensure that important modes are highly controllable, and to ensure that unimportant modes are not excited. The method through which the actuators control the modes is by exciting the slope at given points. Therefore, optimal actuator position occurs when the difference between the slopes at the two actuator positions is maximized for the important modes, and minimized for the unimportant modes. The failures for finding an optimal location of a distributed actuator in the case of the clamped-free beam is the same as the failures in the case of a distributed sensor. Table 3 represents the minimum eigenvalue in subsystem 1 and the maximum eigenvalue in subsystem 2. The numbers in parentheses in the second column through the fourth column denote the minimum eigenvalue in subsystem 1 and the maximum eigenvalue in subsystem 2. As shown in the preceding table, the minimum eigenvalues in subsystem 1 are maximized while the maximum eigenvalues in subsystem 2 are minimized.

When the highest mode is the third mode and the important modes are the first, second, and third modes the minimum eigenvalues of \( W_{1w} \) are plotted in the case of several types of beam (Fig. 14–16). It is interesting to note that the optimal locations of the piezoceramic actuator and the piezoelectric sensor are exactly the same when the important and unimportant modes are the same for either case. The reason is that the matrices \( B \) and \( C \) are related to the difference of the

**FIGURE 14** Minimum eigenvalue of \( W_{1w} \) in clamped-free beam.
Simply Supported Beam (the highest mode = 3, significant modes = (1,2,3))

![Graph showing minimum eigenvalue of W in simply supported beam.](image1)

**FIGURE 15** Minimum eigenvalue of $W^c$ in simply supported beam.

slopes at two end points as follows:

$$c_m = \left( \frac{d\phi_m}{dx} |_{x=x_b} - \frac{d\phi_m}{dx} |_{x=x_1} \right)$$

$$b_m = \left( \frac{d\phi_m}{dx} |_{x=x_2} - \frac{d\phi_m}{dx} |_{x=x_1} \right)$$

where the subscript $m$ denotes the $m$th vibration mode. Thus it is reasonable that the optimal locations of the distributed sensor and actuator are identical. From these results, we can systematically find the optimal location of a uniform piezoelectric actuator to effectively control specific vibration modes.

Clamped-Clamped Beam (the highest mode = 3, significant modes = (1,2,3))

![Graph showing minimum eigenvalue of W in clamped-clamped beam.](image2)

**FIGURE 16** Minimum eigenvalue of $W^c$ in clamped-clamped beam.
CONCLUSION

In this article we have formulated a design method to find the optimal location of a uniform distributed sensor/actuator pair for a beam with several types of boundary conditions. The observability grammian matrix was chosen to provide a performance index for the optimal location of distributed sensors and the controllability grammian matrix was chosen to provide a performance index for the optimal location of distributed actuators. From the examples, we can see that using the minimum eigenvalues of the observability/controllability grammian matrices as performance indices is appropriate for determining the optimal location of a distributed sensor/actuator pair. The optimal locations of distributed sensor/actuator pairs are identical in the case of a beam when the important and unimportant vibration modes are the same for both the sensor and actuator.

REFERENCES


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