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Vibrations of a Beam Moving Over Supports with Clearance

The transverse vibration of a beam moving over two supports with clearance is analyzed using Euler beam theory. The equations of motion are formulated based on a Lagrangian approach and the assumed mode method. The supports with clearance are modeled as frictionless supports with piecewise-linear stiffness. A feature of the present formulation is that its complexity does not increase with increased number of supports. Results of numerical simulations are presented for various prescribed motions of the beam. The effect of support clearance on the stability of the beam is investigated. © 1994 John Wiley & Sons, Inc.

INTRODUCTION

The mechanics of a stationary beam acted upon by moving loads have been studied in connection with machining processes and applications in the behavior of railway tracks and bridges under moving loads. The earliest work on the vibration of a beam subjected to a constant moving load was presented by Timoshenko (1922). The effects of elastic foundation, moving mass, and deflection dependent moving loads were included in subsequent studies by Nelson and Conover (1971), Benedetti (1974), Steele (1967), Florence (1965), and Katz et al. (1987). A related problem on the dynamics of a beam of finite length moving over supports was first presented by Buffinton and Kane (1985) using the method of Kane's dynamics. The support was assumed to be perfect without any clearance. The dynamics of a beam moving over supports with clearance has not been reported in the literature.

It is difficult to eliminate clearance at the supports during the manufacturing or construction

process. There are numerous studies on the dynamics of mechanical systems with clearances using the model of an impact pair. A list of references can be found in the reported works by Comparin and Singh (1989, 1990). The characteristic of a clearance is often modeled as a spring with piecewise-linear stiffness (Comparin and Singh, 1990; Murakami and Sato, 1990; Dubowsky and Freudenstein, 1971), or a spring with hysteretic stiffness (Maezawa and Watanabe, 1975). The present analysis is an extension of an earlier work (Lee, 1992) for analyzing the dynamics of a beam moving over multiple supports without clearance. The supports in that article were modeled as linear springs with very large stiffness. For the present study, each support with clearance was modeled as a spring with piecewise-linear stiffness. This assumption removes the necessity to compute the normal forces at the supports. Moreover, it will be shown in the formulation that the complexity of the formulation does not increase with an increased number of supports.

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THEORY AND FORMULATIONS

The beam considered is assumed to be a uniform beam of length L moving horizontally over two supports shown in Fig. 1. Horizontal forces are applied at the left end of the beam to pull or push the beam over these supports. An assumption made in the following formulation is that transverse deflections are small so that the dynamic behavior of the beam is governed by Euler beam theory. Moreover, all the transverse deflections and the axial motion of the beam occur in the plane defined by $\mathbf{i}$ and $\mathbf{j}$ unit vectors fixed in the beam. The unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ form a set of mutually perpendicular unit vectors. The origin is located at the left end of the beam. Flexibility of the beam in the axial direction $\mathbf{i}$ is assumed to be negligible compared to the lateral direction $\mathbf{j}$.

The position vector of a general point P on the deformed beam is given by

$$\mathbf{p} = x\mathbf{i} + w\mathbf{j}. \quad (1)$$

The velocity at the point is

$$\mathbf{v}_p = U\mathbf{i} + \dot{w}\mathbf{j} \quad (2)$$

where $\dot{w} = dw/dt$ and $U\mathbf{i}$ is the prescribed velocity of the beam at $x = 0$ as a result of the external applied forces at the left end of the beam. Due to the assumption of axial rigidity and small transverse deflection, U is also the $\mathbf{i}$ component of the velocity for every point along the beam.

The kinetic energy T of the beam is

$$T = \frac{1}{2} m \int_0^L (U^2 + \dot{w}^2) dx \quad (3)$$

where m is the mass of the beam per unit length.

Assuming Euler's beam theory, the elastic strain energy of the beam due to bending is

$$V_e = \frac{1}{2} \int_0^L EI \left(\frac{\partial^2 w}{\partial x^2} \right)^2 dx \quad (4)$$

FIGURE 1 A beam moving over two frictionless supports with clearance.

FIGURE 2 Characteristic model of a support with clearance.

where E and I are the Young's modulus and the central principal second moment of area of the cross section of the beam about the $\mathbf{k}$ axis, respectively.

The locations of the two supports are given by $x = a$ and $x = b$. The deflections of the beam at these supports are denoted by $w(a)$ and $w(b)$, respectively. These relative locations of the supports change with the motion of the beam. The characteristic of a clearance at the support at $x = a$ is modeled as a piecewise-linear stiffness shown in Fig. 2. As shown in the figure, the clearance is equal to $2\delta_a$. A similar model is also assumed for the support at $x = b$ with a clearance of $2\delta_b$.

For $|w(a)| < \delta_a$, the potential energy due to the support at $x = a$ is zero. For $|w(a)| > \delta_a$, the potential energy due to this support is

$$V_{sa} = \frac{1}{2} k_a (|w(a)| - \delta_a)^2. \quad (5)$$

Similarly, the potential energy due to the support at $x = b$ is equal to zero for $|w(b)| < \delta_b$. For $|w(b)| > \delta_b$, the potential energy due to this support is given by

$$V_{sb} = \frac{1}{2} k_b (|w(b)| - \delta_b)^2. \quad (6)$$

It can be seen from the above discussion that increasing the number of supports will only increase the number of terms for the potential energy contributions. It will not directly affect the expressions for kinetic energy, strain energy, and the following expression for the potential energy due to the axial forces of the beam.

The potential energy due to the axial forces, f_x , is

$$V_a = \frac{1}{2} \int_0^L f_x \left(\frac{\partial w}{\partial x} \right)^2 dx. \quad (7)$$

The right end of the beam is stress free as the beam is being pulled or pushed over the supports by external forces applied at the left end of the beam. The axial forces for the remaining part of the theorem beam are

$$\begin{aligned} f_x &= -m \int_x^L \dot{U} dr \\ &= -m \dot{U}(L - x). \end{aligned} \quad (8)$$

Using the assumed mode method, the quantity w is expressed as

$$w = \sum_{i=1}^n q_i(t) \phi_i(x) \quad (9)$$

where ϕ_i are spatial functions that satisfy the geometric boundary conditions at the two ends of the beam. For the present study, ϕ_i are assumed to be the normalized model functions for the vibration of a uniform unrestrained beam. The assumed functions are

$$\phi_1(x) = 1 \quad (10)$$

$$\phi_2(x) = \sqrt{3} \left(1 - \frac{2x}{L} \right) \quad (11)$$

$$\begin{aligned} \phi_i(x) &= \cos \frac{\lambda_{i-2}x}{L} + \cosh \frac{\lambda_{i-2}x}{L} \\ &\quad - \gamma_{i-2} \left[\sin \frac{\lambda_{i-2}x}{L} + \sinh \frac{\lambda_{i-2}x}{L} \right] \\ &\quad (i = 3, \dots, n) \end{aligned} \quad (12)$$

where

$$\gamma_j = \frac{\cos \lambda_j - \cosh \lambda_j}{\sin \lambda_j - \sinh \lambda_j} \quad (13)$$

and $\lambda_1, \dots, \lambda_n$ are the consecutive roots of the transcendental equation

$$1 - \cos \lambda \cosh \lambda = 0. \quad (14)$$

The modal functions ϕ_1 and ϕ_2 correspond to the rigid body translation and rotation of an unrestrained beam. The assumed form of w enables the kinetic energy, the strain energy, and the potential energy to be expressed in matrix form as follows:

$$T = \frac{1}{2} m \dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}} \quad (15)$$

$$V_e = \frac{1}{2} EI \mathbf{q}^T \mathbf{H} \mathbf{q} \quad (16)$$

$$V_{sa} = \frac{1}{2} k_a \mathbf{q}^T \Phi_a \mathbf{q} - \gamma_a k_a \mathbf{q}^T \Phi_a \mathbf{q}_a + \frac{1}{2} k_a \mathbf{q}_a^T \Phi_a \mathbf{q}_a \quad (17)$$

$$V_{sb} = \frac{1}{2} k_b \mathbf{q}^T \Phi_b \mathbf{q} - \gamma_b k_b \mathbf{q}^T \Phi_b \mathbf{q}_b + \frac{1}{2} k_b \mathbf{q}_b^T \Phi_b \mathbf{q}_b \quad (18)$$

$$V_a = \frac{1}{2} m \dot{\mathbf{U}} \mathbf{q}^T \mathbf{Y} \mathbf{q} - \frac{1}{2} m \dot{\mathbf{U}} L \mathbf{q}^T \mathbf{Q} \mathbf{q}. \quad (19)$$

The quantity γ_a is equal to 1 for $w(a) > 0$ and is equal to -1 for $w_a < 0$. Similarly, γ_b is set to be 1 for $w(b) > 0$ and is equal to -1 for $w_b < 0$. Moreover, k_a is set to be zero when $|w(a)| < \delta_a$ in compliance with $V_{sa} = 0$. The support stiffness k_b is defined in a similar manner. The matrices $\mathbf{M}$, $\mathbf{H}$, $\mathbf{Y}$, $\mathbf{Q}$, Φ_a , and Φ_b are matrices defined as

$$(\mathbf{M})_{ij} = \int_0^L \phi_i \phi_j dx \quad (20)$$

$$(\mathbf{H})_{ij} = \int_0^L \phi_i'' \phi_j'' dx \quad (21)$$

$$(\mathbf{Y})_{ij} = \int_0^L x \phi_i' \phi_j' dx \quad (22)$$

$$(\mathbf{Q})_{ij} = \int_0^L \phi_i' \phi_j' dx \quad (23)$$

$$(\Phi_a)_{ij} = \phi_i(a) \phi_j(a) \quad (24)$$

$$(\Phi_b)_{ij} = \phi_i(b) \phi_j(b). \quad (25)$$

The quantities $\mathbf{q}$ and $\dot{\mathbf{q}}$ are $n \times 1$ column vectors consisting of q_i and $\dot{q}_i$, respectively. The column vectors $\mathbf{q}_a$ and $\mathbf{q}_b$ are defined by

$$\delta_a = \sum_{i=1}^n q_{ai} \phi_i(a) \quad (26)$$

$$\delta_b = \sum_{i=1}^n q_{bi} \phi_i(b) \quad (27)$$

where q_{ai} and q_{bi} are the i components of $\mathbf{q}_a$ and $\mathbf{q}_b$, respectively.

The matrices $\mathbf{M}$, $\mathbf{H}$, $\mathbf{Y}$, $\mathbf{Q}$, Φ_a , and Φ_b are symmetric matrices. All the matrices except Φ_a and Φ_b are independent of time. The matrices Φ_a and Φ_b need to be updated as the beam is moving over the supports.

The Lagrangian of the beam involving w can be expressed as

$$L = T - V_\varepsilon - V_a - V_{sa} - V_{sb}. \quad (28)$$

The linearized Euler–Lagrange equation is

$$m\mathbf{M}\ddot{\mathbf{q}} + (EI\mathbf{H} + m\dot{\mathbf{U}}\mathbf{Y} + k_a\Phi_a + k_b\Phi_b - m\dot{\mathbf{U}}\mathbf{L}\mathbf{Q})\mathbf{q} = \gamma_a k_a \Phi_a \mathbf{q}_a + \gamma_b k_b \Phi_b \mathbf{q}_b. \quad (29)$$

This equation of motion can be included in numerical integration programs using a fourth-order, variable time step Runge–Kutter method for investigating the response of a beam undergoing various prescribed motions. The magnitudes and signs of $w(a)$ and $w(b)$ are determined at the beginning of integration for each new time step. The quantities γ_a , γ_b , k_a , and k_b are then assigned suitable values for the subsequent numerical integration.

RESULTS AND SIMULATIONS

Sinusoidal Longitudinal Motions

For the present numerical simulations, $L = 1$ m, $m = 1$ kg/m, and $EI = 1$ N-m². The beam is assumed to be moving over two supports with a separation of $D = 0.25$ m. The initial configuration of the beam is such that the two supports are located symmetrically about the center of the beam at $a = 0.375$ m and $b = 0.625$ m. The initial shape of the beam is described by

$$w = \phi_1 q_1 + \phi_3 q_3. \quad (30)$$

The values of q_1 and q_2 are determined from the conditions that $w = 0$ m for both supports at $a = 0.375$ m and $b = 0.625$ m, and $w = 0.01$ m at $x = 0$ m and at $x = 1$ m. Because ϕ_1 corresponds

to the rigid body translation, the initial shape of the beam is the first symmetric flexural mode shape of an unrestrained beam passing through the two supports.

For a prescribed longitudinal sinusoidal motion of the beam, the variation of a is assumed to be

$$a = 0.375 - A \sin \Omega t \text{ m.} \quad (31)$$

It was found that the tip displacements for a six-term approximation for w are approximately the same as the tip displacements for a seven-term approximation for w . Consequently, n in Eqs. (26) and (27) is set to be 6 for the following numerical simulations.

The two supports are assumed to be identical having the same clearance with $\delta_a = \delta_b = \delta$ and $k_a = k_b = k$. The clearances δ_a and δ_b are expressed as

$$\delta_a = q_{a1} \phi_1 \quad (32)$$

$$\delta_b = q_{b1} \phi_1 \quad (33)$$

because $\phi_1 = 1$, $q_{a1} = q_{b1} = \delta$.

The effect of varying the value of k is examined in Fig. 3 with $\delta = 100$ μm , $A = 0.05$, and $\Omega = 20$ rad/s. It can be seen in Fig. 3 that the tip displacements at the two ends of the beam are relatively insensitive to the variation in k for $k > 10^6$ N m⁻¹. The displacements at the supports vary dramatically with the variation in k . For $k = 10^5$ N m⁻¹, the maximum displacement at the supports are about twice the magnitude of δ . For $k \geq 10^6$ N m⁻¹, the maximum tip displacement at the supports is close to the value of δ . The support stiffness k is set to be 10^6 N m⁻¹ for the subsequent numerical simulations.

The tip displacement at the left end of the beam and the displacement at the left support for $\Omega = 10$, 20, and 22 rad/s for $A = 0.05$ are shown in Figs. 4–6, respectively. The beam shows a stable behavior for $\Omega = 10$ rad/s and unstable behaviors for $\Omega = 20$ and 22 rad/s, which is the same as the behaviors of the beam undergoing the same prescribed longitudinal motion presented by Buffinton and Kane (1985) for supports without any clearance. For $\Omega = 10$ rad/s, shown in Fig. 4, a larger clearance at the supports does not result in a larger amplitude of oscillations at the tip. A slightly longer period of tip oscillation is observed with increased magnitude of clearance. For $\Omega = 20$ rad/s shown in Fig. 5, the

FIGURE 3 Displacements for a beam moving over supports: $\Omega = 20$ rad/s; $A = 0.05$; (—) $k = 10^7$ N m $^{-1}$; (---) $k = 10^6$ N m $^{-1}$; ($\cdots$) $k = 10^5$ N m $^{-1}$.

variation in the magnitude of the clearance does not change the unstable behavior of the beam. A larger clearance is found to result in a slower rate of increase of the amplitudes of tip displacements. However, for $\Omega = 22$ rad/s shown in Fig. 6, a larger clearance seems to lead to an increase in the amplitudes of the tip displacements. Numerical results for $\Omega = 20$ rad/s and $A = 0.025$ is shown in Fig. 7. The stable behaviors are consistent with the reported results by Buffinton and Kane (1985).

As has been shown, longitudinal excitation can lead to either stable or unstable transverse motions of the beam. A detailed discussion of this phenomenon can be found in Buffinton and Kane (1985). As explained in their work, the main cause of the transverse motions is that longitudinal motions give rise to transverse inertia forces that grow in magnitude as longitudinal acceleration increases. A detailed stability analysis can be performed by the method presented by Buffinton and Kane (1985) using Floquet's the-

FIGURE 4 Displacements for a beam moving over supports: $\Omega = 10$ rad/s; $A = 0.05$; (—) $\delta = 100 \mu\text{m}$; (---) $\delta = 500 \mu\text{m}$; ($\cdots$) $\delta = 1000 \mu\text{m}$.

ory. Such a laborious analysis will not be performed for the present study. However, the behavior for a particular combination of support clearances and amplitude and period of longitudinal excitation can be easily verified using the present algorithm.

Repositioning Maneuver

For a prescribed repositioning maneuver of the beam, the variation of a is assumed to be

$$a = C_1 - \frac{C_2}{T_d} \left[t - \frac{T}{2\pi} \sin \frac{2\pi t}{T_d} \right] \text{ m} \quad (34)$$

FIGURE 5 Displacements for a beam moving over supports: $\Omega = 20$ rad/s; $A = 0.05$; (—) $\delta = 100 \mu\text{m}$; (---) $\delta = 500 \mu\text{m}$; ($\cdots$) $\delta = 1000 \mu\text{m}$.

FIGURE 6 Displacements for a beam moving over supports: $\Omega = 22 \text{ rad/s}$; $A = 0.05$; (—) $\delta = 100 \mu\text{m}$; (---) $\delta = 500 \mu\text{m}$; ($\cdots$) $\delta = 1000 \mu\text{m}$.

where C_1 specifies the initial position of the beam, C_2 is the horizontal distance traversed by any point on the beam due to the assumption of axial rigidity and small transverse deflection, and T_d is the duration of the prescribed motion. The value $C_1 = 0.375 \text{ m}$ gives the same prescribed initial position of the beam as in the longitudinal sinusoidal motion.

The tip displacements of the beam for a repositioning maneuver with $C_2 = 0.35 \text{ m}$ are shown in Fig. 8 for a fast prescribed motion with $T_d = 0.5 \text{ s}$, and in Fig. 9 for a relatively slow prescribed motion with $T_d = 3.5 \text{ s}$. For both prescribed motions, the behaviors of the beam for various magnitudes of clearances remain almost identical for the initial period of the motion and

FIGURE 7 Displacements for a beam moving over supports: $\Omega = 20 \text{ rad/s}$; $A = 0.025$; (—) $\delta = 100 \mu\text{m}$; (---) $\delta = 500 \mu\text{m}$; ($\cdots$) $\delta = 1000 \mu\text{m}$.

FIGURE 8 Tip displacements for a beam undergoing a “fast” repositioning maneuvering: (—) $\delta = 100 \mu\text{m}$; (---) $\delta = 500 \mu\text{m}$; ($\cdots$) $\delta = 1000 \mu\text{m}$.

differ significantly toward the end of the prescribed motion. For a small clearance with fast prescribed motion shown in Fig. 8, the tip displacement at the left end of the beam decreases due to the decreasing distance between the left end of the beam and the left support. A large clearance at the supports causes an oscillatory motion at the left end of the beam with slightly larger amplitude toward the end of the prescribed motion. The motion at the right end of the beam

also appears to be more oscillatory. A similar behavior is observed for slow prescribed motion shown in Fig. 9.

CONCLUSION

Approximate equations of motion in matrix form are derived for the motion of a beam moving over supports with clearance. A support with clear-

FIGURE 9 Tip displacements for a beam undergoing a “slow” repositioning maneuvering: (—) $\delta = 100 \mu\text{m}$; (---) $\delta = 500 \mu\text{m}$; ($\cdots$) $\delta = 1000 \mu\text{m}$.

ance is modeled as a spring with piecewise-linear stiffness. Numerical simulations are carried out for both longitudinal sinusoidal motion and repositioning maneuvers of the beam. For longitudinal sinusoidal motion, the variation of the magnitude of the clearance does not change the stable behavior for $\Omega = 10$ rad/s and the unstable behavior at $\Omega = 20$ and 22 rad/s. A larger clearance however changes the periods of oscillation of the tip displacements for $\Omega = 10$ rad/s and the rate of increase of the amplitudes of the tip displacements for $\Omega = 20$ and 22 rad/s. For repositioning maneuvers with the beam being pushed over the supports, a larger clearance generally results in a more oscillatory transverse motion toward the end of the prescribed motion.

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