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Normal Form Analysis of Hopf Bifurcation Exemplified by Duffing's Equation

A method is proposed for calculating the normal form coefficients of the degenerate Hopf bifurcation system and the steady periodic solutions of a nonlinear vibration system. The results obtained by this method are the same as those obtained by the classical one. The present method is much simpler and can easily be implemented: that is, given the coefficients of the governing equations, the response is obtained directly by substitution. © 1994 John Wiley & Sons, Inc.

INTRODUCTION

The idea of the normal transformation came from Poincaré (1889). It has been taken up by many authors (Birkhoff, 1966; Arnold, 1978; Moser, 1973; Ioos and Joseph, 1980). It consists of carrying out a near identity change of variable allowing one to transform one system of nonlinear ordinary differential equations to a simpler one. The simplest forms are called normal forms. We can study the bifurcation behavior and the characteristics of the solution of the original equations more easily by normal form equations due to the reduced number of essential parameters. Mathematicians are interested in the kinds of normal form different differential equations can have. So many kinds of normal forms have been given at present, such as the Hopf, pitchfork, saddle-node bifurcation for the generate and degenerate cases. Engineers and scientists are more interested in how to get the normal form for a given system of equations, that is, what is the relationship of the coefficients between the origi-

nal equations and the normal form equations. To this end, we give a simple method to calculate the coefficients of the normal form of a Hopf bifurcation system for both the nondegenerate and the degenerate case. Furthermore, we use this method to get the steady periodic solution of nonlinear free vibrations. We explain the process by solving Duffing's equation.

NORMAL FORM THEORY

The vector normal form (or Poincaré normal form, standard form) is a simplified analytic expression. Such a simplified expression is obtained by a series of successive near identity nonlinear transformations of variables. The qualitative geometric structure can be acquired for every bifurcation of flows through the analysis of dynamical character of normal forms.

Consider the nonlinear differential equations

$$\dot{X} = f(X), \quad f \in C^K(\mathbf{R}^n), \quad K \text{ large integer.} \quad (1)$$

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We expand the vector field $f(x)$ into the Taylor series

$$f(X) = AX + F^2(X) + \cdots + F^r(X) + O(|X|^{r+1}), \quad X \in \mathbf{R}^n \quad (2)$$

where $A = D_x f(0)$ can be changed to a diagonal Jordan block form by a linear transformation, $F^k(X) \in H_n^k$ is a homogeneous polynomial of order k with n variables. We may suppose, without loss of generality, that matrix A is in Jordan standard form. Then Eq. (1) can be written as:

$$\dot{X} = AX + F^2(X) + \cdots + F^r(X) + O(|X|^{r+1}), \quad X \in \mathbf{R}^n. \quad (3)$$

We introduce a near identity nonlinear transformation of variables to simplify the nonlinear part of Eq. (3). We take

$$X = Y + P^k(Y), \quad P^k(Y) \in H_n^k, \quad k \geq 2. \quad (4)$$

If we substitute (4) into (3), the left side is

$$\dot{X} = \frac{d}{dt} [Y + P^k(Y)] = [I + D_Y P^k(Y)] \dot{Y} \quad (5)$$

and the right side is

$$f(X) = \frac{d}{dt} [Y + P^k(Y)] = A[Y + P^k(Y)] + \cdots + F^k[Y + P^k(Y)] + O(|Y|^{k+1}) \quad (6)$$

where $[I + D_Y P^k(Y)]$ is invertible for sufficient small Y . According to the binomial theorem for matrices, we get

$$[I + D_Y P^k(Y)]^{-1} = I - D_Y P^k(Y) + [D_Y P^k(Y)]^2 + \cdots$$

From (5) and (6)

$$\dot{Y} = AY + F^2(Y) + \cdots + F^{k-1}(Y) + F^k(Y) - ad_A^k P^k(Y) + O(|Y|^{r+1}), \quad Y \in \mathbf{R}^n \quad (7)$$

where $ad_A^k P^k(Y)$ is a linear operator, defined as: $ad_A^k: H_n^k \rightarrow H_n^k$

$$ad_A^k P^k(Y) = D_Y P^k(Y)AY - AP^k(Y).$$

We find that the term whose order is less than k is not changed, but the terms whose order equal or are greater than k are changed after the trans-

formation. If

$$F^k(Y) - ad_A^k P^k(Y) = 0 \quad (8)$$

the k th order term is the simplest form. Because ad_A^k is a linear operator, Eq. (8) is solvable if ad_A^k is invertible or $F^k(Y)$ is the range of ad_A^k . Otherwise choose a complement G^k for $ad_A^k(H_n^k)$ in H_n^k , so that

$$H_n^k = R(ad_A^k) \oplus G^k$$

then $F^k(Y)$ can be written as:

$$F^k(Y) = r^k(Y) + g^k(Y)$$

where $r^k(Y) \in R(ad_A^k)$ and $g^k(Y) \in G^k$.

Then we can find a $P^k(Y)$ such that $ad_A^k P^k(Y) = r^k(Y)$. Equation (7) becomes:

$$\dot{Y} = AY + F^2(Y) + \cdots + F^{k-1}(Y) + g^k(Y) + O(|Y|^{k+1}). \quad (9)$$

THEOREM 1. Let $\dot{X} = f(X)$ be a C^r system of differential equations with $f(0) = 0$ and $Df(0) = A$. Choose a complement G_k for $ad_A^k(H_n^k)$ in H_n^k , so that $H_n^k = ad_A^k(H_n^k) + G_k$. Then there is an analytic change of coordinates in a neighborhood of the origin that transforms the system

$$\dot{X} = f(X) \text{ to } \dot{Y} = g(Y) = AY + g^{(2)}(Y) + \cdots + g^{(r)}(Y) + R_r$$

with $g^{(k)}(Y) \in G_k$ for $2 \leq k \leq r$ and $R_r = o(|Y|^r)$.

Refer to Wang (1990) for the proof of this theorem.

DEFINITION 1. Intercept system

$$\dot{Y} = AY + g^2(Y) + \cdots + g^r(Y) \quad (10)$$

is called an A -normal form up to order r .

HOPF BIFURCATION

Consider the system:

$$\dot{x} = f(x, \mu), \quad x \in \mathbf{R}^2, \quad \mu \in \mathbf{R} \quad (11)$$

Suppose

$$f(0, 0) = 0, \quad D_x f(0, 0) = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix}.$$

Let

$$\begin{cases} z = x_1 + ix_2 \\ \bar{z} = x_1 - ix_2 \end{cases}$$

and substitute these transformations into Eq. (11). We have:

$$\begin{bmatrix} \dot{z} \\ \dot{\bar{z}} \end{bmatrix} = A_0 \begin{bmatrix} z \\ \bar{z} \end{bmatrix} + \begin{bmatrix} g(z, \bar{z}) \\ \bar{g}(z, \bar{z}, \mu) \end{bmatrix} \quad (12)$$

where

$$g = o(|z|^2), \quad A_0 = \begin{bmatrix} \omega_0 i & 0 \\ 0 & -\omega_0 i \end{bmatrix}.$$

The eigenvalues corresponding to A_0 are $\lambda_1 = \omega_0 i$ and $\lambda_2 = -\omega_0 i$. According to the Poincaré resonance condition: $\langle m, \lambda \rangle - \lambda_s = 0$, where $\langle \cdot, \cdot \rangle$ is the scalar product, we know that it holds if $m_1 - m_2 = 1$ and $m_1 + m_2 = m$, that is, m is an odd number, otherwise it fails. So there are only odd order terms in its normal form. The normal form of Eq. (12) in complex form is:

$$\dot{\bar{z}} = i\omega_0 z + c_3 z^2 \bar{z} + c_5 z^3 \bar{z}^2 + \text{h.o.t.} \quad (13)$$

where h.o.t. denotes higher order terms.

The real normal form in rectangular coordinates is:

$$\dot{U} = AU + W_3(U) + W_5(U) + \text{h.o.t.} \quad (14)$$

where

$$\begin{aligned} A &= \begin{bmatrix} d\mu & -(c\mu + \omega_0) \\ c\mu + \omega_0 & d\mu \end{bmatrix}, \\ W_3(U) &= (u_1^2 + u_2^2) \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} U, \\ U &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \\ W_5(U) &= (u_1^2 + u_2^2)^2 \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} U. \end{aligned}$$

In polar coordinates, it is:

$$\begin{cases} \dot{r} = (d\mu + a_1 r^2 + a_2 r^4)r + \text{h.o.t.} \\ \dot{\theta} = \omega_0 + c\mu + b_1 r^2 + b_2 r^4 + \text{h.o.t.} \end{cases} \quad (15)$$

According to the following theorem of Hopf bifurcation, we know that Eq. (12) is a Hopf bifurcation system.

THEOREM 2. (Hopf, 1942). Suppose that the system $\dot{x} = f(x, \mu)$, $x \in \mathbf{R}^n$, $\mu \in \mathbf{R}$ has an equilibrium point (x_0, μ_0) at which the following properties are satisfied: (H1) $D_x f(x_0, \mu_0)$ has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts.

Then (H1) implies that there is a smooth curve of equilibrium $[x(\mu), \mu]$ with $x(\mu_0) = x_0$. The eigenvalues $\lambda(\mu)$, $\bar{\lambda}(\mu)$ of $D_x f(x_0, \mu_0)$ that are imaginary at $\mu = \mu_0$ vary smoothly with μ . If, moreover, (H2)

$$\frac{d}{d\mu} \operatorname{Re} \lambda(\mu)|_{\mu=\mu_0} = d \neq 0$$

then there is a unique three-dimensional center manifold passing through (x_0, μ_0) in $\mathbf{R}^n \times \mathbf{R}$ and there is a smooth system of coordinates (preserving the planes $\mu = \text{const.}$) for which the Taylor expansion of degree 3 on the center manifold is given by Eq. (14). If $a_1 \neq 0$, there is a surface of periodic solutions in the center manifold that has quadratic tangency with the eigenspace of $\lambda(\mu_0)$, $\bar{\lambda}(\mu_0)$ agreeing to the second order with the paraboloid $\mu = -(a_1/d)(x^2 + y^2)$. If $a_1 < 0$, then these periodic solutions are stable limit cycles, but if $a_1 > 0$, the periodic solutions are repelling.

Note 1. It is called the first kind of degenerate case if $d = 0$.

Note 2. It is called the second kind of degenerate case if $a_1 = 0$. The stability of the system is determined by the coefficient of high order terms of its A-normal form.

Method of Calculating Coefficients of Normal Form

Consider the system [Eq. (2)]:

$$\begin{aligned} f(X) &= AX + F^2(X) + \cdots + F^r(X) \\ &\quad + O(|X|^{r+1}), \quad X \in \mathbf{R}^n \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}; \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$F^k(X) = \begin{bmatrix} F^{k(1)}(X) \\ F^{k(2)}(X) \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^k a_{ij} x_1^i x_2^j \\ \sum_{j=0}^k b_{ij} x_1^i x_2^j \end{bmatrix};$$

$$k = 2, \dots, 5; \quad i = k - j.$$

We have given the expressions of a_1 and b_1 of Eq. (15) in Chen (1990) for the nondegenerate case:

$$\begin{cases} a_1 = (b_{21}\omega_0 + 3a_{30}\omega_0 + 3b_{03}\omega_0 + a_{12}\omega_0 \\ \quad - b_{02}b_{11} + 2a_{02}b_{02} + a_{11}a_{02} + a_{11}a_{20} \\ \quad - b_{20}b_{11} - 2b_{20}a_{20})/(8\omega_0) \\ b_1 = (3b_{12}\omega_0 - 9a_{03}\omega_0 - 3a_{21}\omega_0 + 9b_{30}\omega_0 \\ \quad - 4b_{02}^2 + 5b_{02}a_{11} - 10b_{02}b_{20} - a_{11}^2 \\ \quad + a_{11}b_{20} - 10b_{20}^2 - 10a_{02}^2 + a_{02}b_{11} \\ \quad - 10a_{02}a_{20} - b_{11}^2 + 5b_{11}a_{20} - 4a_{20}^2)/(24\omega_0) \end{cases} \quad (16)$$

Now we extend the method to calculate a_2 and b_2 for the second kind of degenerate case.

In order to simplify the second order terms, we take the transformation:

$$X = X + P^2(X).$$

Note. The same variable value X is used for the purpose of computer implementation. The following second order transformation is to be determined,

$$P^2 = \begin{bmatrix} P_1^2 \\ P_2^2 \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^2 c_{ij}x_1^i x_2^j \\ \sum_{j=0}^2 d_{ij}x_1^i x_2^j \end{bmatrix}; \quad i = 2 - j.$$

Substituting into Eq. (2), we get

$$\begin{aligned} \dot{X} &= \sum_{N=0}^5 (-1)^N [DP^2(X)]^N \cdot \{AX + AP^2X \\ &\quad + \sum_{n=2}^5 \sum_{m=0}^n \frac{D^m F^n(X)}{m!} [P^2(X)]^m\} \\ &= AX + F_1^2(X) + F_1^3(X) + F_1^4(X) + F_1^5(X) \end{aligned} \quad (17)$$

where

$$F_1^k(X) = \begin{bmatrix} F_1^{k(1)}(X) \\ F_1^{k(2)}(X) \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^k a_{ij}^1 x_1^i x_2^j \\ \sum_{j=0}^k b_{ij}^1 x_1^i x_2^j \end{bmatrix};$$

$$k = 2, \dots, 5; \quad i = k - j.$$

Note. The explicit forms of $D^m F^n(X)$ are given in the Appendix.

By the normal form equation of Hopf bifurcation (14), we know that

$$F_1^2 = F^2 + AP^2 - DP^2AX = 0. \quad (18)$$

This means that the second order normal form is required to be identically zero because the Hopf bifurcation has only odd order terms. We obtain $P^2(X)$ by the above equation with results given below:

$$\begin{bmatrix} d_{20} \\ d_{11} \\ d_{02} \\ c_{20} \\ c_{11} \\ c_{02} \end{bmatrix} = \frac{1}{3\omega_0} \begin{bmatrix} a_{20} - b_{11} + 2a_{02} \\ 2b_{20} - a_{11} - 2b_{02} \\ 2a_{20} + b_{11} + a_{02} \\ -b_{20} - a_{11} - 2b_{02} \\ 2a_{20} + b_{11} - 2a_{02} \\ -2b_{20} + a_{11} - b_{02} \end{bmatrix} \quad (19)$$

$$F_1^3 = F^3 + DF^2P^2 - DP^2 \cdot F_1^2$$

$$F_1^4 = F^4 + \frac{1}{2} D^2 F^2 (P^2)^2 + DF^3P^2 - DP^2 \cdot F_1^3$$

$$F_1^5 = F^5 + \frac{1}{2} D^2 F^3 (P^2)^2 + DF^4P^2 - DP^2 \cdot F_1^4. \quad (20)$$

In order to simplify the third order term in Eq. (17), we take the transformation, $X = X + P^3(X)$. Then we have

$$\begin{aligned} \dot{X} &= \sum_{N=0}^5 (-1)^N [DP^3(X)]^N \cdot \{AX + AP^3(X) \\ &\quad + \sum_{n=3}^5 \sum_{m=0}^n \frac{D^m F^n(X)}{m!} [P^3(X)]^m\} \\ &= AX + F_2^3(X) + F_2^4(X) + F_2^5(X) \end{aligned} \quad (21)$$

where

$$P^3 = \begin{bmatrix} P_1^3 \\ P_2^3 \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^3 c_{ij}x_1^i x_2^j \\ \sum_{j=0}^3 d_{ij}x_1^i x_2^j \end{bmatrix}; \quad i = 3 - j$$

$$F_2^k(X) = \begin{bmatrix} F_2^{k(1)}(X) \\ F_2^{k(2)}(X) \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^k a_{ij}^2 x_1^i x_2^j \\ \sum_{j=0}^k b_{ij}^2 x_1^i x_2^j \end{bmatrix};$$

$$k = 3, 4, 5; \quad i = k - j.$$

By the normal form equation of Hopf bifurcation (14), it is required that:

$$F_2^3 = F_1^3 + A \cdot P^3 - DP^3 \cdot A \cdot X = W^3(X). \quad (22)$$

We obtain $P^3(X)$ by the above equation with results given below (c_{30}, c_{03} can be any constant):

$$\begin{aligned} c_{21} &= (8 \cdot \omega_0 \cdot c_{03} + b_{21}^1 - b_{03}^1 \\ &\quad + 3 \cdot a_{30}^1 - 3 \cdot a_{12}^1)/(8 \cdot \omega_0) \\ c_{12} &= (8 \cdot \omega_0 \cdot c_{30} - b_{30}^1 + b_{12}^1 \\ &\quad + 3 \cdot a_{21}^1 - 3 \cdot a_{03}^1)/(8 \cdot \omega_0) \\ d_{30} &= (-4 \cdot \omega_0 \cdot c_{03} - b_{21}^1 - b_{03}^1 + a_{30}^1 + a_{12}^1)/ \\ &\quad (4 \cdot \omega_0) \\ d_{21} &= (8 \cdot \omega_0 \cdot c_{30} + 5 \cdot b_{30}^1 - b_{12}^1 + a_{21}^1 \\ &\quad + 3 \cdot a_{03}^1)/(8 \cdot \omega_0) \end{aligned} \quad (23)$$

$$\begin{aligned} d_{12} &= (-8 \cdot \omega_0 \cdot c_{03} + b_{21}^1 - 5 \cdot b_{03}^1 \\ &\quad + 3 \cdot a_{30}^1 + a_{12}^1)/(8 \cdot \omega_0) \\ d_{03} &= (4 \cdot \omega_0 \cdot c_{30} + b_{30}^1 + b_{12}^1 + a_{21}^1 + a_{03}^1)/ \\ &\quad (4 \cdot \omega_0) \end{aligned}$$

$$F_2^4 = F_1^4 \quad (24)$$

$$F_2^5 = F_1^5 + DF_1^3 P^3 - DP^3 \cdot F_2^3. \quad (25)$$

In order to simplify the fourth order term in Eq. (20), we take the transformation, $X = X + P^4(X)$. Then we have

$$\begin{aligned} \dot{X} &= \sum_{N=0}^5 (-1)^N [DP^4(X)]^N \cdot \left\{ AX + AP^3X \right. \\ &\quad \left. + \sum_{n=4}^5 \sum_{m=0}^n \frac{D^m F_3^n(X)}{m!} [P^3(X)]^m \right\} \\ &= AX + F_2^3(X) + F_3^4(X) + F_3^5(X) \end{aligned} \quad (26)$$

where

$$\begin{aligned} P^4 &= \begin{bmatrix} P_1^4 \\ P_2^4 \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^4 c_{ij} x_1^i x_2^j \\ \sum_{j=0}^4 d_{ij} x_1^i x_2^j \end{bmatrix}; \quad i = 4 - j \\ F_3^k(X) &= \begin{bmatrix} F_3^{k(1)}(X) \\ F_3^{k(2)}(X) \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^k a_{ij}^3 x_1^i x_2^j \\ \sum_{j=0}^k b_{ij}^3 x_1^i x_2^j \end{bmatrix}; \\ k &= 4, 5; \quad i = k - j. \end{aligned}$$

By the normal form equation of Hopf bifurcation (14), it is required that:

$$F_3^4 = F_2^4 + A \cdot P^4 - DP^4 \cdot A \cdot X = 0. \quad (27)$$

We obtain $P^4(X)$ by the above equation with results given below:

$$\begin{aligned} d_{40} &= (-3 \cdot b_{31}^2 - 2 \cdot b_{13}^2 + 3 \cdot a_{40}^2 + 2 \cdot a_{22}^2 + 8 \\ &\quad \cdot a_{04}^2)/(15 \cdot \omega_0) \\ d_{31} &= (12 \cdot b_{40}^2 - 2 \cdot b_{22}^2 - 8 \cdot b_{04}^2 - 3 \cdot a_{31}^2 - 2 \cdot \\ &\quad a_{13}^2)/(15 \cdot \omega_0) \\ d_{22} &= (b_{31}^2 - b_{13}^2 + 4 \cdot a_{40}^2 + a_{22}^2 + 4 \cdot a_{04}^2)/(5 \cdot \\ &\quad \omega_0) \\ d_{13} &= (8 \cdot b_{40}^2 + 2 \cdot b_{22}^2 - 12 \cdot b_{04}^2 - 2 \cdot a_{31}^2 - 3 \cdot \\ &\quad a_{13}^2)/(15 \cdot \omega_0) \\ d_{04} &= (2 \cdot b_{31}^2 + 3 \cdot b_{13}^2 + 8 \cdot a_{40}^2 + 2 \cdot a_{22}^2 + 3 \cdot \\ &\quad a_{04}^2)/(15 \cdot \omega_0) \\ c_{40} &= (-3 \cdot b_{40}^2 - 2 \cdot b_{22}^2 - 8 \cdot b_{04}^2 - 3 \cdot a_{31}^2 - 2 \cdot \\ &\quad a_{13}^2)/(15 \cdot \omega_0) \\ c_{31} &= (3 \cdot b_{31}^2 + 2 \cdot b_{13}^2 + 12 \cdot a_{40}^2 - 2 \cdot a_{22}^2 - 8 \cdot \\ &\quad a_{04}^2)/(15 \cdot \omega_0) \\ c_{22} &= (-4 \cdot b_{40}^2 - b_{22}^2 - 4 \cdot b_{04}^2 + a_{31}^2 - a_{13}^2)/(5 \cdot \\ &\quad \omega_0) \\ c_{13} &= (2 \cdot b_{31}^2 + 3 \cdot b_{13}^2 + 8 \cdot a_{40}^2 + 2 \cdot a_{22}^2 - 12 \cdot \\ &\quad a_{04}^2)/(15 \cdot \omega_0) \\ c_{04} &= (-8 \cdot b_{40}^2 - 2 \cdot b_{22}^2 - 3 \cdot b_{04}^2 + 2 \cdot a_{31}^2 + 3 \cdot \\ &\quad a_{13}^2)/(15 \cdot \omega_0) \end{aligned} \quad (28)$$

$$F_3^5 = F_2^5 \quad (29)$$

In order to simplify the fifth order term in Eq. (23), we take the transformation: $X = X + P^5(X)$. Then we have

$$\begin{aligned} \dot{X} &= \sum_{N=0}^5 (-1)^N [DP^5(X)]^N \cdot \left\{ AX + AP^5X \right. \\ &\quad \left. + \sum_{n=4}^5 \sum_{m=0}^n \frac{D^m F_3^n(X)}{m!} [P^5(X)]^m \right\} \\ &= AX + F_2^3(X) + F_3^4(X) + F_3^5(X) \end{aligned} \quad (30)$$

where

$$P^5 = \begin{bmatrix} P_1^5 \\ P_2^5 \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^5 c_{ij}^5 x_1^i x_2^j \\ \sum_{j=0}^5 d_{ij}^5 x_1^i x_2^j \end{bmatrix}; \quad i = 5 - j$$

$$F_4^k(X) = \begin{bmatrix} F_4^{k(1)}(X) \\ F_4^{k(2)}(X) \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^k a_{ij}^4 x_1^i x_2^j \\ \sum_{j=0}^k b_{ij}^4 x_1^i x_2^j \end{bmatrix};$$

$$k = 5; \quad i = k - j.$$

By the normal form equation of the Hopf bifurcation (14), we know that:

$$F_4^5 = F_3^5 + A \cdot P^5 - DP^5AX = W^5(X) \quad (31)$$

We calculate a_2 , b_2 , and $P^5(X)$ by the above equation with results given below:

$$\begin{cases} a_2 = -(-b_{41}^3 - b_{23}^3 - 5 \cdot b_{05}^3 - 5 \cdot a_{50}^3 - a_{32}^3 - a_{14}^3)/16 \\ b_2 = -(-5 \cdot b_{50}^3 - b_{32}^3 - b_{14}^3 + a_{14}^3 + a_{41}^3 + a_{23}^3 + 5 \cdot a_{05}^3)/16. \end{cases} \quad (32)$$

This is the coefficients of normal form of the second kind degenerate case. c_{50} , c_{05} can be any constant.

$$\begin{aligned} c_{41} &= (48 \cdot \omega_0 \cdot c_{05} + 5 \cdot b_{41}^3 + b_{23}^3 - 7 \cdot b_{05}^3 + 25 \cdot a_{50}^3 - 7 \cdot a_{32}^3 - 11 \cdot a_{14}^3)/(48 \cdot \omega_0) \\ c_{32} &= (32 \cdot \omega_0 \cdot c_{50} - 3 \cdot b_{50}^3 + b_{32}^3 + b_{14}^3 + 7 \cdot a_{41}^3 - a_{23}^3 - 5 \cdot a_{05}^3)/(16 \cdot \omega_0) \\ c_{23} &= (32 \cdot \omega_0 \cdot c_{05} + b_{41}^3 + b_{23}^3 - 3 \cdot b_{05}^3 + 5 \cdot a_{50}^3 + a_{32}^3 - 7 \cdot a_{14}^3)/(16 \cdot \omega_0) \\ c_{14} &= (48 \cdot \omega_0 \cdot c_{50} - 7 \cdot b_{50}^3 + b_{32}^3 + 5 \cdot b_{14}^3 + 11 \cdot a_{41}^3 + 7 \cdot a_{23}^3 - 25 \cdot a_{05}^3)/(48 \cdot \omega_0) \\ d_{50} &= (-12 \cdot \omega_0 \cdot c_{05} - 2 \cdot b_{41}^3 - b_{23}^3 - 2 \cdot b_{05}^3 + 2 \cdot a_{32}^3 + 2 \cdot a_{14}^3)/(12 \cdot \omega_0) \\ d_{41} &= (16 \cdot \omega_0 \cdot c_{50} + 11 \cdot b_{50}^3 - b_{32}^3 - b_{14}^3 + a_{41}^3 + a_{23}^3 + 5 \cdot a_{05}^3)/(16 \cdot \omega_0) \\ d_{32} &= (-96 \cdot \omega_0 \cdot c_{05} + 5 \cdot b_{41}^3 - 11 \cdot b_{23}^3 - 31 \cdot b_{05}^3 + 25 \cdot a_{50}^3 + 5 \cdot a_{32}^3 + 13 \cdot a_{14}^3)/(48 \cdot \omega_0) \\ d_{23} &= (96 \cdot \omega_0 \cdot c_{50} + 31 \cdot b_{50}^3 + 11 \cdot b_{32}^3 - 5 \cdot b_{14}^3 + 13 \cdot a_{41}^3 + 5 \cdot a_{23}^3 + 25 \cdot a_{05}^3)/(48 \cdot \omega_0) \\ d_{14} &= (-16 \cdot \omega_0 \cdot c_{05} + b_{41}^3 + b_{23}^3 - 11 \cdot b_{05}^3 + 5 \cdot a_{50}^3 + a_{32}^3 + a_{14}^3)/(16 \cdot \omega_0) \\ d_{05} &= (12 \cdot \omega_0 \cdot c_{50} + 2 \cdot b_{50}^3 + b_{32}^3 + 2 \cdot b_{14}^3 + 2 \cdot a_{41}^3 + a_{23}^3 + 2 \cdot a_{05}^3)/(12 \cdot \omega_0) \end{aligned} \quad (33)$$

In conclusion, if the coefficients of the governing Eq. (2) are given, the normal form (15) is calculated by Eqs. (16) and (32).

Nonlinear Free Vibration Problems

Consider the Duffing oscillator with no damping term for illustration:

$$\frac{d^2x}{dt^2} + \omega_1^2 = cx^3. \quad (34)$$

Let $x = x_1$, $\dot{x} = -\omega_1 x_2$,

$$\text{then } \dot{x}_2 = \omega_1 x_1 - \frac{c}{\omega_1} x_1^3 \quad (35)$$

To solve the equations, we can either repeat the process step by step or calculate the normal form (15) directly by Eqs. (16) and (32) using the given coefficients. We proceed the former.

Substituting the above transformation into Eq. (31), we have:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_1 \\ \omega_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{c}{\omega_1} x_1^3 \end{bmatrix} \quad (36)$$

or simply write it as,

$$\{\dot{X}\} = \{X\} + F^3(X). \quad (37)$$

Comparing this equation with Eq. (17), we find that $b_{30} = -(c/\omega_1)$ and all the other coefficients are zero. So if we substitute these coefficients into Eq. (16), we get $a_1 = 0$, $b_1 = -(3c/8\omega_1)$. Because there are no second order terms in this system, we directly take the third order transformation of coordinates:

$$\{X\} = [A]\{Y\} + P^3(Y). \quad (38)$$

We can obtain $P^3(Y)$ by Eq. (23).

Let $c_{30} = c_{03} = 0$, then

$$c_{21} = d_{30} = d_{12} = 0 \quad (39)$$

$$c_{12} = \frac{c}{8\omega_1^2}, \quad d_{21} = -\frac{5c}{8\omega_1^2}, \quad d_{03} = \frac{c}{4\omega_1^2}.$$

Then we can obtain F_2^4 by equation (24) and F_2^5 by equation (25). For $F_2^4 = F_1^4 = \{0\}$, i.e., the fourth order terms are already in its simplest form, we directly take the change of variable:

$$\{Y\} = \{U\} + P^5(U). \quad (40)$$

We can obtain $P^5(U)$ by Eq. (33).

Let $c_{50} = c_{05} = 0$, then

$$\begin{aligned} c_{32} &= (21 * c^2)/(256 * \omega_1^4) \\ c_{23} &= c_{41} = d_{50} = d_{14} = d_{32} = 0 \\ c_{14} &= (25 * c^2)/(256 * \omega_1^4) \\ d_{41} &= (-39 * c^2)/(256 * \omega_1^4) \\ d_{23} &= (-67 * c^2)/(256 * \omega_1^4) \\ d_{05} &= -c^2/(32 * \omega_1^4). \end{aligned} \quad (41)$$

The coefficients of normal form a_2 and b_2 can be obtained by Eq. (32).

$$\begin{cases} a_2 = 0 \\ b_2 = -\frac{21c^2}{256\omega_1^3}. \end{cases} \quad (42)$$

The normal form of Eq. (36) in polar coordinate system is:

$$\begin{cases} \dot{r} = 0 \\ \dot{\theta} = \omega_1 - \frac{3c}{8\omega_1} r^2 - \frac{21c^2}{256\omega_1^3} r^4. \end{cases} \quad (43)$$

The first order asymptotic solution of this normal form equation is:

$$\begin{cases} u_1 = r \cos \theta \\ \frac{dr}{dt} = 0 \\ \frac{d\theta}{dt} = \omega_1 - \frac{3c}{8\omega_1} r^2 - \frac{21c^2}{256\omega_1^3} r^4. \end{cases} \quad (44)$$

The steady period solution of equation (44) is:

$$\begin{cases} u_1 = x_0 \cos \theta, \quad u_2 = x_0 \sin \theta \\ \theta = \left(\omega_1 - \frac{3c}{8\omega_1} x_0^2 - \frac{21c^2}{256\omega_1^3} x_0^4 \right) t. \end{cases} \quad (45)$$

In order to get the steady periodic solution of Duffing equation (34), we should utilize the transformation equation (38) and (40).

$$\begin{aligned} \{X\} &= \{Y\} + P^3(Y) \\ &= \{U\} + P^5(U) + P^3[U + P^5(U)] \\ &\approx \{U\} + P^3(U) + P^5(U). \end{aligned} \quad (46)$$

The steady periodic solution of the Duffing equation is:

$$\begin{cases} x = \left(x_0 + \frac{c}{32\omega_1^2} x_0^3 + \frac{23c^2}{1024\omega_1^4} x_0^5 \right) \cos \theta \\ \quad - \left(\frac{c}{32\omega_1^2} x_0^3 + \frac{3c^2}{128\omega_1^4} x_0^5 \right) \cos 3\theta \\ \quad + \frac{c^2}{1024\omega_1^4} x_0^5 \cos 5\theta \\ \theta = \left(\omega_1 - \frac{3c}{8\omega_1} x_0^2 - \frac{21c^2}{256\omega_1^3} x_0^4 \right) t. \end{cases} \quad (47)$$

In order to prove this new result, we have studied this equation by the Lindstedt–Poincaré method and found that they are the same.

CONCLUSION

Applying near identity change of variable through matrix representation, a simple method for calculating the coefficients of normal form of nondegenerate and degenerate Hopf bifurcation systems has been introduced. We use this method to study the steady-state period solution of nonlinear free vibration systems. This method is simple and easy for computer implementation.

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APPENDIX

The definitions of $D^m F(X)$. Suppose

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad F(X) = \begin{bmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{bmatrix}, \quad \text{then}$$

$$DF(X) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix}$$

$$D^2 F(X) = \begin{bmatrix} \frac{\partial^2 F_1}{\partial x_1^2} & \frac{\partial^2 F_1}{\partial x_1 \partial x_2} & \frac{\partial^2 F_1}{\partial x_2^2} \\ \frac{\partial^2 F_2}{\partial x_1^2} & \frac{\partial^2 F_2}{\partial x_1 \partial x_2} & \frac{\partial^2 F_2}{\partial x_2^2} \end{bmatrix}$$

$$D^3 F(X) = \begin{bmatrix} \frac{\partial^3 F_1}{\partial x_1^3} & \frac{\partial^3 F_1}{\partial x_1^2 \partial x_2} & \frac{\partial^3 F_1}{\partial x_1 \partial x_2^2} & \frac{\partial^3 F_1}{\partial x_2^3} \\ \frac{\partial^3 F_2}{\partial x_1^3} & \frac{\partial^3 F_2}{\partial x_1^2 \partial x_2} & \frac{\partial^3 F_2}{\partial x_1 \partial x_2^2} & \frac{\partial^3 F_2}{\partial x_2^3} \end{bmatrix}.$$