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On the Effectiveness of Using Only Eigenvalues in Structural Model Updating Problems

In this article the subject of model updating using the inverse eigensensitivity method that uses only eigenvalues in the calculation is examined. The problem of creating an underdetermined set of equations is studied and ways of preventing it are addressed.

INTRODUCTION

Using only eigenvalues (and not the eigenvectors) in model updating by the inverse eigensensitivity (IES) method has many attractive features and, as reported by Dascotte and Vanhonacker (1989), is used in practice by some of the commercially available model updating packages. The most obvious advantage of this approach is that, among all the nonunique solutions to the problem, it is much easier to find an updated model that reproduces only the experimental eigenvalues. Other advantages are:

1. the accuracy of experimental natural frequencies is generally much higher than that of the related mode shapes (Luber and Lotze, 1990), and thus using only these in the calculations will increase the accuracy of the results;
2. calculation of the sensitivity elements related to the eigenvalues requires less effort on the part of the eigensolution; and
3. because eigenvector derivatives are not required in the calculations, one can avoid the following computational problems:
   (a) calculation of the effect of higher modes on the eigenvector sensitivities is not required, thus there is no need to calculate flexibility matrix $[K]^{-1}$; and
   (b) close modes that make the sensitivity matrix ill-conditioned will cause no problem.

Despite the above advantages, the question remains as to how will an updating based only on reducing $||\Delta f||$ will affect $||\Delta\phi||$, where:

$$||\Delta f|| = \sqrt{\sum^{m} ((f_{xi} - f_{ai})/f_{xi})^2}$$

$$||\Delta\phi|| = \sqrt{\sum^{m}\sum^{n} ((\phi_{xij} - \phi_{aij})^2}$$

i and j are designated as mode and coordinates indexes, respectively.

EFFECT OF UPDATING ON $||\Delta\phi||$

The question asked here is "to what extent can one ignore the eigenvectors in identifying/updating a structure?" The answer to this question

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depends on the degree of accuracy of the initial analytical model in representing the real structure. For example, one can update a simple 10 degrees-of-freedom (DOF) mass-spring system so that it reproduces the first 10 eigenvalues of a 1000 DOF model of a continuous structure. It is obvious that for this case neither the original model nor the updated one is truly representative of the real structure, a fact that will reveal itself in substantial differences between the eigenvectors of the model and those of the real structure.

The authors' experience shows that even with a reasonable initial finite element (FE) model of a structure, for which there is good agreement between the eigenvectors of both model and real structure, the eigenvalues can still be substantially different. This can be explained by considering derivatives of the eigenvalues and eigenvectors of a linear system with simple (i.e., nonrepeated) eigenvalues as:

$$\frac{\partial \lambda_r}{\partial \rho} = \{\phi_r\}^T \left(\frac{\partial [K]}{\partial \rho} - \lambda_r \frac{\partial [M]}{\partial \rho}\right) \{\phi_r\} \tag{1}$$

and

$$\frac{\partial \{\phi_r\}}{\partial \rho} = \sum_{k=1}^{P} \gamma_{rk} \{\phi_k\} \tag{2}$$

where (λ_r, ϕ_r) are modal parameters of mode r and $[K]$ and $[M]$ are system stiffness and mass matrices, respectively. Matrices $[K]$ and $[M]$ are functions of variable ρ, and,

$$\begin{aligned} \gamma_{rk} &= \frac{\{\phi_k\}^T \left(\frac{\partial [K]}{\partial \rho} - \lambda_r \frac{\partial [M]}{\partial \rho}\right)\{\phi_r\}}{\lambda_r - \lambda_k} \quad \textit{for} \quad r \neq k \\ &= -\tfrac{1}{2}\{\phi_r\}T \frac{\partial [M]}{\partial \rho} \{\phi_r\} \quad r = k. \end{aligned} \tag{3}$$

Comparing Eq. (1) with Eqs. (2) and (3), it is evident that

$$\frac{\partial \lambda_r}{\partial \rho} = O(\lambda) \frac{\partial \phi_r}{\partial \rho}. \tag{4}$$

Equation (4) means that, generally, the sensitivities of the eigenvectors are much smaller than those related to the eigenvalues and this, in turn, implies that errors in the analytical model will affect the eigenvalues more significantly than the eigenvectors.

Thus, if an initial FE model represents a real structure reasonably well, it should be possible to update the FE model by using only the eigenvalues in the updating process. According to Eq. (4), the effect of such an updating process on $\|\Delta\phi\|$ will be small.

PROBLEM OF AN UNDERDETERMINED SET OF EQUATIONS

Having defined $[S]$ as sensitivity matrix and $\{\Delta p\}$ as modifications to original mass and stiffness matrices, when using a least-squares (LS) method to solve the sensitivity-based updating equation:

$$\{\delta\} = [S]\{\Delta p\} \tag{5}$$

the following condition must hold in order to have an overdetermined set of equations:

$$m \times (n + 1) \geq P \tag{6}$$

where in Eq. (6): m is the number of modes used in $\{\delta\}$ (usually the number of measured modes); n is the number of coordinates used in each modal vector (usually equal to the number of measured coordinates); and P is the total number of modification factors, Δp, involved in the updating. For local updating, that is, element-by-element updating, P is equal to at least the number of elements involved in the FE model.

Due to the limited number of measured eigenvalues that are usually available in practice, it is not possible to satisfy Eq. (6) using only eigenvalues in the updating calculations unless P is reduced dramatically by resorting to some sort of global updating scheme.

In the general case, there are two ways of tackling the problem of Eq. (5) being underdetermined: using the singular value decomposition (SVD) and by applying additional constraint equations, as explained below.

Application of SVD

The underdetermined set of equations in Eq. (5) has an infinite number of solutions. The SVD technique (explained below) results in a solution for $\{\Delta p\}$ that is unique in the sense that it has a minimum second normal length, (Nobari, 1992; Golub and Van Loan, 1983), that is,

$$\begin{aligned} &\|\{\Delta p\}\|_2 \quad \text{(for the solution achieved from SVD)} \\ &< \|\{\Delta p\}\|_2 \quad \text{(of all other possible solutions).} \end{aligned} \tag{7}$$

This solution is quite convenient because among all possible solutions, it provides the minimum-length change to the FE model. The procedure of applying the SVD to solve Eq. (5) is as follows.

Consider the SVD of a general matrix $[S]$ as

$$[S] = [U][\Sigma][V]^H \tag{8}$$

where $[U]$ and $[V]$ are left and right singular matrices, respectively, and $[\Sigma]$ is the matrix of singular values.

Now, the SVD-based inverse of $[S]$ is

$$[S]^+ = \{v\}_1\{u\}_1^H(1/\sigma_1) + \{v\}_2\{u\}_2^H(1/\sigma_2) + \cdots \quad \text{for} \quad \sigma_i \neq 0. \tag{9}$$

Using the pseudoinverse in Eq. (9), the minimum-norm solution to Eq. (5), explained in Eq. (7), can be achieved. Although the solution represented by Eq. (9) seems to have only mathematical significance, one can attach some physical meaning to it, that is, the initial FE model is a reasonable mathematical model of the real structure and it needs the smallest of modifications to give better correlation with the experimental results.

Application of Constraint Equations

Although the SVD-based technique explained in the previous section is quite robust, it is computationally lengthy and relatively expensive. Another way of getting around the problem presented by the underdeterminedness of Eq. (5) is to introduce the following constraints into Eq. (5);

$$[\alpha]\{\Delta p\} = 0 \tag{10}$$

where diagonal matrix $[\alpha]$ is a matrix of weighting (or penalty) coefficients.

Adding the system of equations in (10) to (5) not only makes the resulting set overdetermined and suitable for an LS solution, but also expresses the desire that the change to the FE model must be as small as possible. The same physical significance given to the SVD solution applies here as well, that is, a reasonable FE model that needs the least modification, with the slight difference that here the $\|\{\Delta p\}\|_2$ is not an absolute minimum and its value depends on the magnitude of α_i given to each individual Δp_i (note that $\|\Delta p\}\|_{2\text{SVD}}$ can be smaller or larger than $\|\{\Delta p\}\|_{2\text{constraint}}$). The matter of choosing proper values for $[\alpha]$ is analyst-dependent and a reasonable value can usually be found by a few trial-and-error steps. However, this method has the advantage that it it much cheaper than the one based on the SVD and it provides a means of expressing different levels of confidence in different elements of the FE model, should this information be available.

CASE STUDIES

To demonstrate the efficiency of the constraint equations method mentioned in the preceding section, two practical case studies were undertaken.

Case Study 1

Structure and Test Data. The structure used for this case study is a three-bay truss shown in Fig. 1. The structure is made of Dural (L105) and consists of 18 cylindrical side-members and three

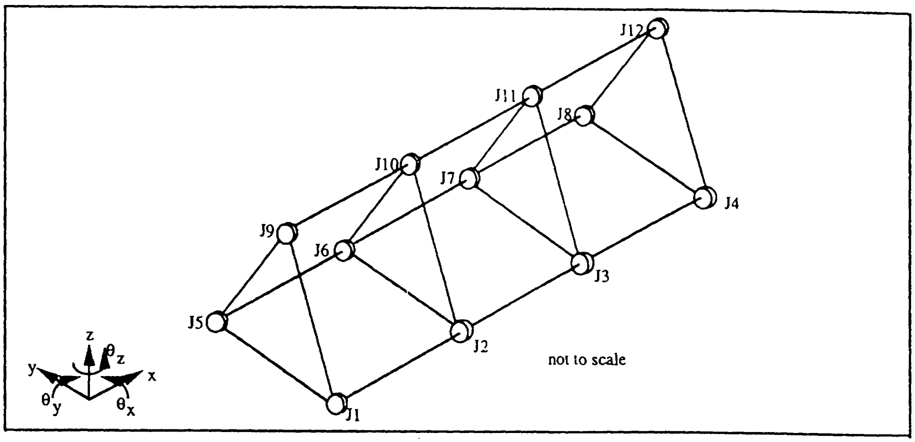


FIGURE 1 The truss structure used in case study 1.

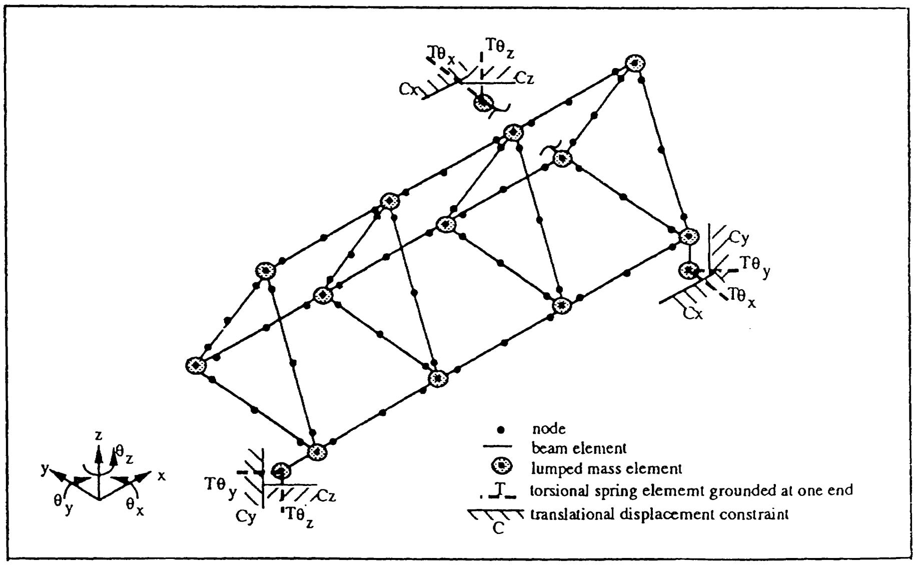


FIGURE 2 The FE model of truss structure.

ridge-members connected by eight side-joints and four ridge-joints. The structure was freely suspended for testing and data were acquired using three-point simultaneous excitation in the form of band-limited noise.

A total of 36 response coordinates, namely three translational DOF at each of the 12 joints, were measured. Each of the 108 Frequency Response Function (FRF) curves (36 responses related to three excitations) was measured at 2048 frequencies for a baseband frequency range of 0–128 Hz.

FE Model. The initial FE model of the structure is shown in Fig. 2. Two small beam elements represented the joint area of the cylindrical trusses and the 12 joints of the three-bay truss structure were modelled using lumped masses. Each continuous member of the truss was modelled using two beam elements between joint elements. The effect of the three excitation devices was also included in the FE model. The final FE model consisted of 105 mass elements and 90 stiffness elements, resulting in a total of 462 DOFs.

Correlation of FE and Experimental Results. The degree of correlation between the FE and the experimental models was investigated in three different ways:

1. comparison of natural frequencies in Table 1;
2. Modal Assurance Criterion (MAC) plot in Fig. 3; and
3. FRF overlays in Fig. 4.

Table 1. Measured and Predicted Natural Frequencies of Truss Structure

Exp. Mode	Exp. (Hz)	FE Mode	FE (Hz)	Error (%)
1	25.8	1	23.5	8.0
2	34.6	2	30.8	11.0
3	38.0	3	32.8	13.7
4	39.8	4	33.2	17.0
5	41.25	5	37.8	8.0
6	54.8	7	53.0	3.0
7	56.5	6	50.8	10.0
8	86.7	8	80.9	7.0
12	115.2	9	113.2	1.7

Modes are paired according to the MAC plot in Fig. 3. (See also Table 3).

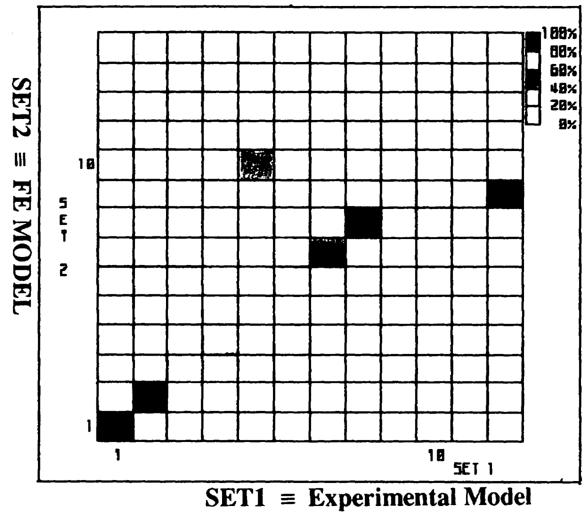


FIGURE 3 MAC plot before updating.

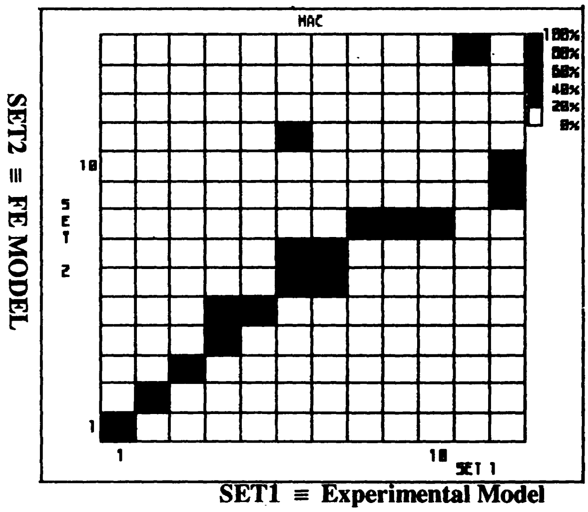


FIGURE 5 MAC plot after updating.

Updating of the FE Model. The constraint equations method was incorporated into program MODULATE (Nobari and Sanliturk, 1993) and, using this program, the FE model of the structure was updated using the first 12 measured eigenvalues in the calculation and ignoring experimental modes 9, 10, and 11 from the calculations because of their poor correlation with FE modes. A value of 0.5 was used for α. Again, the results of the updated FE model were compared with the experimental data and the original FE model at three levels, as follows:

1. comparison of natural frequencies in Table 2;
2. comparison of MAC values before and after updating in Table 3 and Fig. 5; and
3. overlaying FRFs before and after updating in Fig. 6.

As is evident from Table 3 and Fig. 5, the updating not only improved the mode pairing characteristics but also provided more even and generally better MAC values, a fact that can also be deduced from Table 4.

Examining the FRF overlays in Fig. 6 also reveals a significant improvement in correlation between experimental and updated model FRFs.

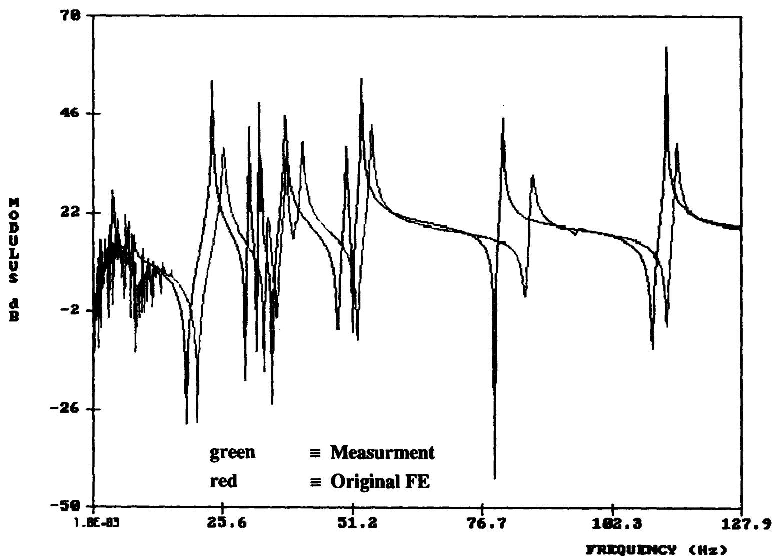


FIGURE 4 Measured and original FE point FRFs overlay.

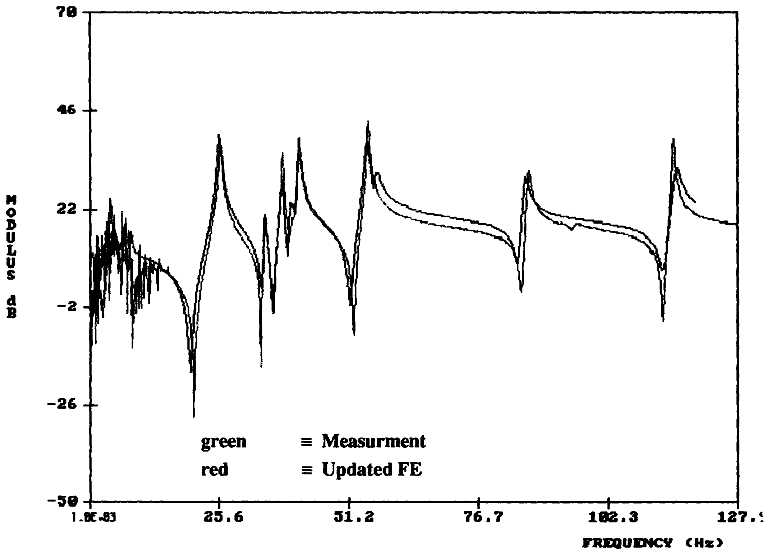


FIGURE 6 Measured and updated FE point FRFs overlay.

Table 2. Measured and Updated Natural Frequencies of Truss Structure

Exp. Mode	Exp. (Hz)	Updated FE Mode	Updated FE (Hz)	Error (%)
1	25.8	1	25.8	0.0
2	34.6	2	34.6	0.0
3	38.0	3	38.0	0.0
4	39.8	4	39.8	0.0
5	41.25	5	41.25	0.0
6	54.8	6	54.8	0.0
7	56.5	7	56.5	0.0
8	86.7	8	86.7	0.0
12	115.2	9	115.2	0.0

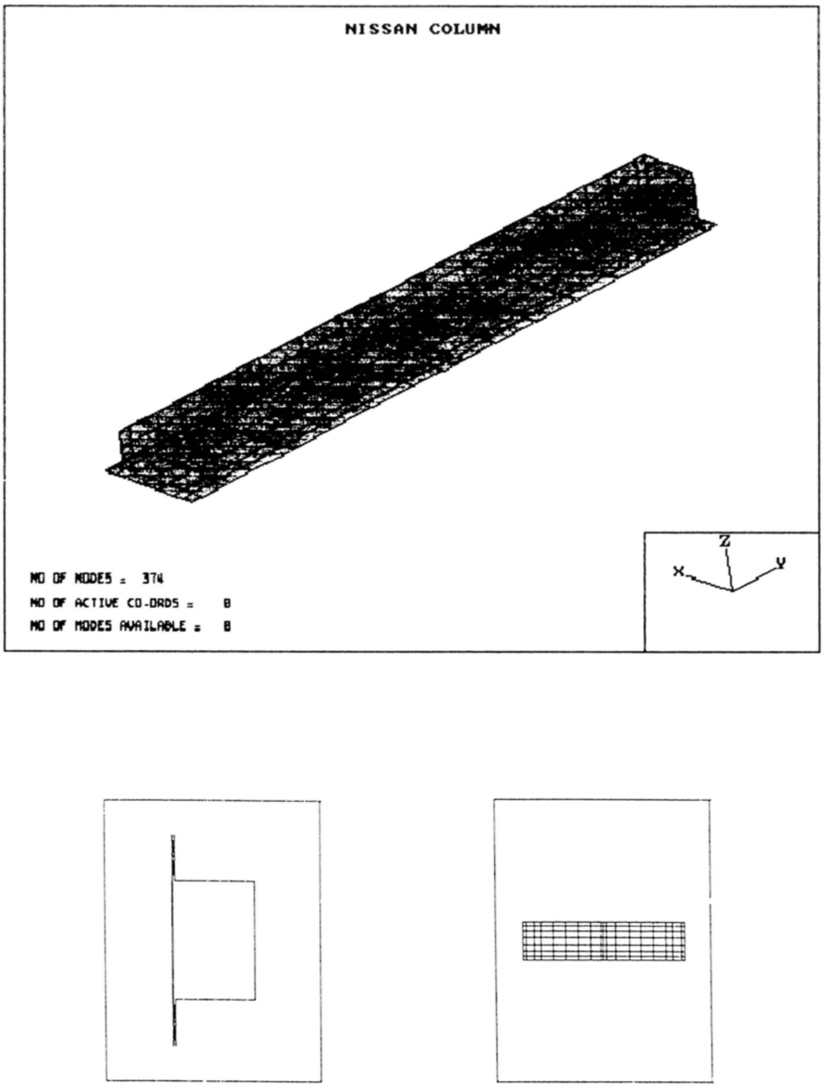
Modes are paired according to the MAC plot in Fig. 5. (See also Table 3).

Table 3. MAC Values and Mode Pairs Before and After Updating

Initial Mode Pairs	MAC Before	Updated Mode Pairs	MAC after
1, 1	94.9	1, 1	94.1
2, 2	84.3	2, 2	83.0
3, 3	75.7	3, 3	95.4
4, 4	60.2	4, 4	83.4
5, 5	55.3	5, 5	72.0
6, 7	58.5	6, 6	68.5
7, 6	32.7	7, 7	44.0
8, 8	90.9	8, 8	89.8
12, 9	92	12, 9	91.5

Table 4. Initial and After Updating Norms of Differences in Modal Parameters

	Initial	Updated	Improvement
$\|\|\Delta f\|\|$	0.3	0.01	97%
$\|\|\Delta \phi\|\|$	7.3	6.2	15%



cross-section and bottom views

FIGURE 7 The channel structure used in case study.

Case Study 2

The structure used for this case study is a channel structure with overall dimensions of $570 \times 90 \times 1$ mm and with material properties of $E = 206$ GN/m^2 and $\rho = 7860$ kg/m^3. The structure is made of several overlapping plates joined together by 35 spot welds (Fig. 7).

The structure was tested using a hammer test technique in the range of 0–1600 Hz with 1 Hz frequency increment and, in total, 120 points were measured in the out-of-plane direction. The measured FRFs were analyzed and modal parameters were identified. Some of the experimental modal vectors were quite complex, a feature believed to be due to the significant influence of the joints in those particular modes.

FE Model. An FE model of the channel was created with the ANSYS package using four-noded quadratic elements (STIF63). Joints were ignored in modelling but overlaps between plates were represented. A total of 341 elements and 374 nodes were used in the model of the channel, which had a total of 2066 DOF.

Correlation of FE and Experimental Results. The degree of correlation between the FE and the experimental models was investigated in the same three ways as before:

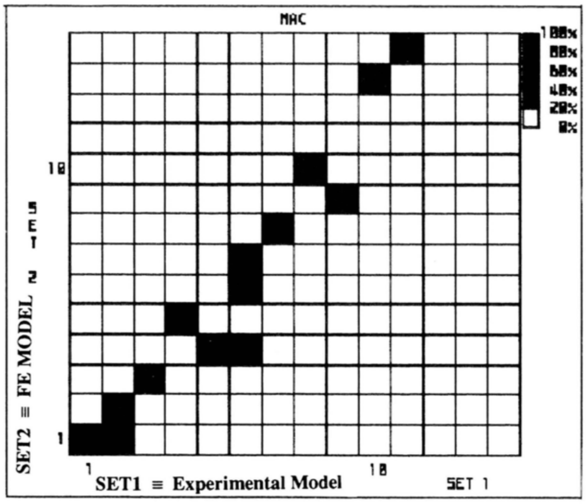


FIGURE 8 MAC plot before updating.

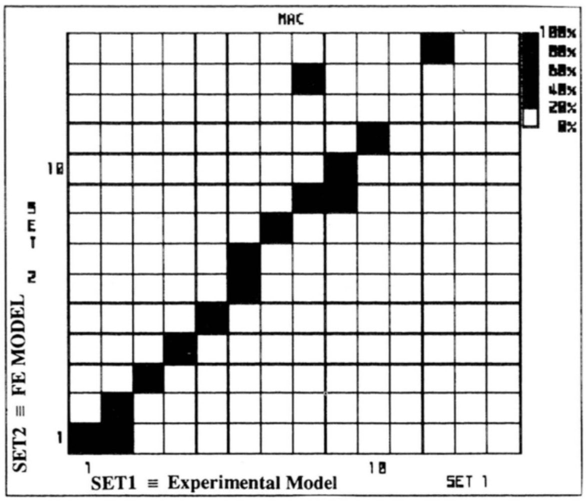


FIGURE 10 MAC plot after updating.

1. comparison of natural frequencies in Table 5;
2. MAC plot in Fig. 8; and
3. FRF overlays in Fig. 9.

Updating of the FE Model. Again, using the constraint equations method and the program MODULATE, the finite element model of the channel was updated using only the first 10 measured eigenvalues in the calculation. A value of 0.5 was used for α. The results of the updated FE model were compared with the experimental data and the original FE model at three levels, as follows:

1. comparison of natural frequencies in Table 6;
2. comparison of MAC values before and after updating in Table 7 and Fig. 10; and
3. overlaying FRFs before and after updating in Fig. 11.

Again, as is evident from Table 7 and Fig. 10, the updating has improved both the mode pairing characteristics and some of the MAC Values presenting a more unified MAC table, a fact which can also be deduced from Table 8.

Examining the FRF overlays in Fig. 11 also

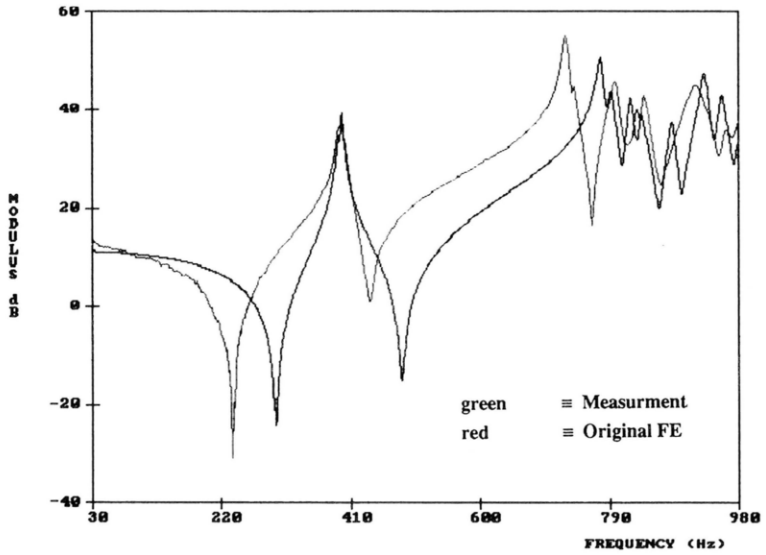


FIGURE 9 Measured and original FE point FRFs overlay.

Table 5. Measured and Predicted Natural Frequencies of Channel Structure

Exp. Mode	Exp. (Hz)	FE Mode	FE (Hz)	Error (%)
1	393.2	1	392.1	0.2
2	397.9	2	396.8	0.3
3	725.9	4	785.7	−8.2
4	736.2	5	791.8	−7.6
5	800.1	3	777.5	−2.8
6	828.9	6	820.8	1.0
7	841.4	8	882.2	−5.0
8	919.9	9	929.6	−1.0
9	960.9	10	954.7	0.6
10	979.5	11	984.4	−0.5

Modes are paired according to the MAC plot in Fig. 8. See also Table 7.

Table 6. Measured and Updated Natural Frequencies of Channel Structure

Exp. Mode	Exp (Hz)	Updated FE Mode	Updated FE (Hz)	Error (%)
1	393.2	1	393.2	0.0
2	397.9	2	397.9	0.0
3	725.9	3	725.6	0.0
4	736.2	4	736.1	0.0
5	800.1	5	799.9	0.0
6	828.9	7	829.0	0.0
7	841.4	8	841.6	0.0
8	919.9	9	919.9	0.0
9	960.9	10	960.5	0.0
10	979.5	11	978.9	0.0

Modes are paired according to the MAC plot in Fig. 10. See also Table 7.

Table 7. MAC Values and Mode Pairs Before and After Updating

Initial Mode Pairs	MAC	Updated Mode Pairs	MAC
1, 1	92	1, 1	80
2, 2	94	2, 2	83
3, 4	43	3, 3	78
4, 5	64	4, 4	83
5, 3	10	5, 5	45
6, 6	49	6, 7	72
7, 8	86	7, 8	74
8, 9	43	8, 9	28
9, 10	76	9, 10	61
10, 11	72	10, 11	65

Table 8. Initial and After Updating Norms of Differences in Modal Parameters

	Initial	Updated	Improvement
$\|\|\Delta f\|\|$	0.13	0.0	99%
$\|\|\Delta\phi\|\|$	26.2	22.0	14%

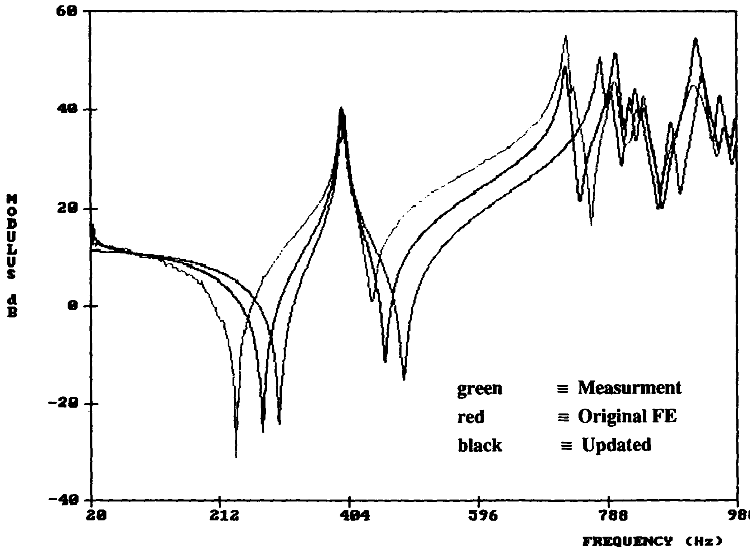


FIGURE 11 Measured original and updated FE point FRFs overlay.

reveals a marked improvement in correlation between experimental and updated FRFs.

CONCLUSIONS AND REMARKS

The following conclusions can be drawn:

1. model updating using eigenvalues only can be an efficient means of improving FE models of structures;
2. the success of this method depends on having a reasonable initial analytical model;
3. two methods have been proposed to overcome the problems associated with the underdetermined nature of the set of equations: the method using additional constraints is very efficient from both a time and cost point of view;
4. taking 2 into account, the correlation between the eigenvectors can be improved even though they are not involved in the updating process.

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