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Structural Synthesis in the Frequency Domain: A General Formulation

A general formulation for frequency domain structural synthesis is presented. The formulation is based on a physical coordinate transformation of the presynthesis frequency response system model and addresses modification, general coupling, and constraint imposition. Notable features of the formulation includes the ability to directly synthesize response quantities such as displacements, stresses, and strains, and the accommodation of Boolean matrices that organize the connectivity in the synthesis of complex systems. The theory is shown to be a highly efficient and exact means of doing static and complex dynamic reanalysis.

INTRODUCTION

Frequency domain structural synthesis is concerned with substructure coupling and structural modification using frequency response function data exclusively; no modal information is required whatsoever. The origins of the theory can be found as far back as 1939 (Kron). Various contributions to the literature in frequency domain synthesis have appeared since (see Gordis et al., 1989, for a survey), and all contributions develop an operative equation of synthesis identical in form. It is the intent of this article to present a general formulation that encompasses all prior contributions to the theory, and to place the various extensions and variants of the theory on a unified theoretical footing, based on first principles.

The general formulation for structural synthesis developed herein results in an operative equation that is applicable to the following static and dynamic structural analyses:

1. direct coupling of substructures;
2. indirect coupling of substructures;
3. modification of structures;
4. constraint application.

The theory is cast in physical coordinates and in the frequency domain, thereby providing for an exact and arbitrary order model reduction in conjunction with any and all of the above analysis options. The theory makes use of frequency response functions, and therefore treats dynamic problems that can have nonproportional linear damping, and any type of frequency dependency in any of the system parameters. Complex modes are easily accommodated. Static problems are also treated as the zero-frequency case. The theory allows for the direct synthesis of response information of any kind, a capability originally put forth in Jetmundsen (1986). Using a generalized definition of frequency response, displacement, velocity, acceleration, stress, and strain information may be directly synthesized. Based on this generalization, the theory will be shown to be an ideal means for doing static and dynamic design reanalysis.

We will examine two alternative physical coordinate systems in which to develop the govern-

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ing equation of synthesis. The first consists of the standard physical coordinates associated with either a finite element model or an experimentally derived frequency response model. The second coordinate system to be employed is a differential response coordinate system. This coordinate system naturally arises in the direct coupling of substructures and in constraint imposition, specifically in the equilibrium and compatibility relations between coordinates being joined or constrained. The problems of indirect substructure coupling and structural modification are developed entirely in this coordinate system as well, leading to an alternative form of impedance and frequency response consistent with the differential response coordinate system.

An attractive feature of the differential response coordinate system is its direct correlation with the directed graph. The directed graph provides a graphical means of representing the connectivity to be established in a synthesis. We will examine the application of directed graphs to the synthesis problem, originally pursued in Jetmundsen et al. (1988). However, here will develop the relationship between structural synthesis and directed graphs in an analytic manner, starting from first principles. The application of graphs to the synthesis problem gives rise to matrix coefficients, referred to here as mapping matrices, in the operative equation of synthesis. As will be shown, the mapping matrices are Boolean for a restricted class of synthesis problems, and these matrices are well suited to the accommodation of information pertaining to connectivity, that is, what is connected to what. The connectivity information as accommodated by the mapping matrices conveniently corresponds to the organization of the signs of the coupling forces and their reactions generated in a synthesis. The Boolean mapping matrices therefore provide an "automated" way of establishing and computationally handling a sign convention for the coupling forces and reactions, and the associated graph provides a graphical means of establishing and portraying the connectivity. It will be seen that the differential coordinates are completely consistent with the Boolean formulation, due to the correspondence with the equilibrium of synthesis mentioned above.

From the analytic development of the mapping matrix, we will identify the restricted class of synthesis problem for which the associated mapping matrix is Boolean; it is for these synthesis problems that the directed graph provides an effective graphical means of representing the connectivity to be established. Note that the use of Boolean mapping matrices in conjunction with frequency domain structural synthesis was applied to the analysis of stress due to fastener tolerance in assembled components (Gordis and Flannelly, 1993).

GENERALIZED FREQUENCY RESPONSE

A frequency response function structural model is indicated in general as

$$\{x(\omega)\} = [H(\omega)]\{f(\omega)\}. \tag{1}$$

Here, $\{x(\omega)\}$ and $\{f(\omega)\}$ are vectors of complex-valued generalized response and excitation coordinates, respectively, at a specific frequency ω, and $[H(\omega)]$ is an appropriately sized frequency response function matrix, evaluated at the frequency ω. In general, an element of the frequency response function matrix is defined as,

$$H_{ij} = \partial x_i / \partial f_j \tag{2}$$

the partial derivative of the ith generalized response coordinate with respect to the jth generalized excitation coordinate. The displacement-force frequency response at a frequency ω can be found from the impedance matrix $[Z]$,

$$[H(\omega)] = [Z(\omega)]^{-1} \quad \text{where} \quad [Z(\omega)] = [K - \omega^2 M + j\omega C]$$

and $[K]$, $[M]$, and $[C]$ are the stiffness, mass, and damping (if available) matrices that result from the finite element assembly process, and $j = \sqrt{-1}$. Of course, $[H]$ can also be found from a vibration test. The flexibility, which is the frequency response evaluated at zero frequency, is found analytically from a nonsingular stiffness matrix,

$$[H(\omega = 0)] = [K]^{-1}.$$

We will employ other types of frequency response, classified by the type of coordinates involved. For example, strain-force and stress-force frequency response are defined as,

$$H_{ij}^{\varepsilon} = \partial \varepsilon_i / \partial f_j \quad H_{ij}^{\sigma} = \partial \sigma_i / \partial f_j \tag{2a,b}$$

and the associated frequency response relations are

$$\{\varepsilon\} = [H^{\varepsilon}]\{f\} \quad \{\sigma\} = [H^{\sigma}]\{f\} \tag{3a,b}$$

where ε_i and σ_i are complex-valued strains and stresses, respectively, at coordinate i, at a specific frequency ω, and $\{\sigma\}$ and $\{\varepsilon\}$ are sets of stress and strain response coordinates, respectively.

MATRIX PARTITIONING

Consider a structural system comprised of either a single structure for which a structural modification(s) is to be made, or two or more substructures to be coupled. The set of all physical coordinates that describe the structure(s) are denoted as coordinate set e. The set e is comprised of two subsets. The first subset of coordinates are those at which the modifications are to be installed or substructures are to be coupled, and this set is denoted as set c (connection coordinates). The second subset is the complement to set c, and is comprised of all the coordinates not associated with the modifications or couplings. This coordinate set is denoted as i (internal coordinates). Therefore, $e = i \cup c$.

The structural system is described in the frequency domain, and at a specific frequency, by

$$\begin{Bmatrix} x_i \\ x_c \end{Bmatrix} = \begin{bmatrix} H_{ii} & H_{ic} \\ H_{ci} & H_{cc} \end{bmatrix} \begin{Bmatrix} f_i \\ f_c \end{Bmatrix}. \tag{4a}$$

We will construct a transformation, referred to as the structural synthesis transformation, which operates on Eq. (4a) producing the synthesized system analog, which reflects the new and/or redundant load paths installed as the result of a substructure coupling and/or a structural modification.

Returning to Eq. (4a), we may append, for example, a set of stress coordinates, then Eq. (4a) becomes

$$\begin{Bmatrix} \sigma \\ x_i \\ x_c \end{Bmatrix} = \begin{bmatrix} H_{\sigma i} & H_{\sigma c} \\ H_{ii} & H_{ic} \\ H_{ci} & H_{cc} \end{bmatrix} \begin{Bmatrix} f_i \\ f_c \end{Bmatrix}. \tag{4b}$$

The stress coordinates will allow the direct calculation of synthesized system stress.

In general, the connection coordinates (set c) may experience both externally applied forces and coupling/modification/constraint forces (to be established through synthesis),

$$f_c = f_c^{\text{ext}} + f_c \tag{5a}$$

and by definition of the subscript i, we may have only

$$f_i = f_i^{\text{ext}}. \tag{5b}$$

Introducing Eqs. (5a,b) into Eq. (4b) yields the expanded equation

$$\begin{Bmatrix} \sigma \\ x_e \\ x_c \end{Bmatrix}^* = \begin{bmatrix} H_{\sigma i} & H_{\sigma c} & H_{\sigma c} \\ H_{ei} & H_{ic} & H_{ic} \\ H_{ci} & H_{cc} & H_{cc} \end{bmatrix} \begin{Bmatrix} f_i^{\text{ext}} \\ f_c^{\text{ext}} \\ f_c \end{Bmatrix} \tag{6}$$

where the asterisk denotes a synthesized quantity due to the fact that we have introduced (symbolically as yet) the forces of synthesis, $\{f_c\}$. Using the set union $e = i \cup c$, we can repartition Eq. (4b) as

$$\begin{Bmatrix} \sigma \\ x_e \\ x_c \end{Bmatrix}^* = \begin{bmatrix} H_{\sigma e} & H_{\sigma c} \\ H_{ee} & H_{ec} \\ H_{ce} & H_{cc} \end{bmatrix} \begin{Bmatrix} f_e \\ f_c \end{Bmatrix} \tag{7}$$

where $\{f_e\} = \lfloor f_i^{\text{ext}^T} \quad f_c^{\text{ext}^T} \rfloor^T$, the vector of externally applied forces which may exist at all physical coordinates.

MODIFICATION AND INDIRECT COUPLING

We will develop here the governing equation for structural modification and indirect substructure coupling. Structural modification is concerned with the creation of redundant load paths in a structure; indirect substructure coupling is concerned with the creation of new load paths between uncoupled structures, with an interconnecting structural element.

The theory allows for the modification to be comprised of any number and spatial distribution of linear, frequency-dependent impedances. The modifications are to be installed at the connection coordinates, that is, coordinate set c. The only restriction on the modifications to be made are that they be described by

$$\{f_c\} = -[K(\Omega) - \Omega^2 M(\Omega) + jC(\Omega)]\{x_c\} \quad \text{or} \quad \{f_c\} = -[Z]\{x_c\} \quad (8)$$

where the minus sign indicates that we are considering the reactions imposed by the modifications on the host structure.

We will construct the transformation of forces based on Eq. (8) to be introduced into Eq. (7). The resulting relationship is the "modified system" version of Eq. (7). The transformation which operates on Eq. (7) is

$$\begin{Bmatrix} f_e \\ f_c \end{Bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -Z \end{bmatrix} \begin{Bmatrix} f_e \\ x_c^* \end{Bmatrix}. \quad (9)$$

The transformed version of Eq. (7) is

$$\begin{Bmatrix} \sigma \\ x_e \\ x_c \end{Bmatrix}^* = \begin{bmatrix} H_{\sigma e} - H_{\sigma c} Z \\ H_{ee} - H_{ec} Z \\ H_{ce} - H_{cc} Z \end{bmatrix} \begin{Bmatrix} f_e \\ x_c^* \end{Bmatrix} \quad (10)$$

The third row of Eq. (10) provides

$$\{x_c\}^* = [I + H_{cc} Z]^{-1} [H_{ce}]\{f_e\} \quad (11)$$

and introducing Eq. (11) into the upper two rows of Eq. (10) yields

$$\begin{bmatrix} H_{\sigma e} \\ H_{ee} \end{bmatrix}^* = \begin{bmatrix} H_{\sigma e} \\ H_{ee} \end{bmatrix} - \begin{bmatrix} H_{\sigma c} \\ H_{ec} \end{bmatrix} [Z^{-1} + H_{cc}]^{-1} [H_{ce}]. \quad (12)$$

Equation (12) is the operative equation for structural modification. All terms on the right-hand side are frequency response quantities for the presynthesis structure, and [**Z**] describes the structural modifications to be installed. The quantities on the left-hand side are frequency response values for the synthesized structure. Note that the quantity $[H_{\sigma e}]$ makes possible the direct calculation of stress due to externally applied loads for the synthesized structure.

COUPLING AND CONSTRAINT IMPOSITION

We now develop the analogous theory substructure coupling. It will be clear that the developments for substructure coupling apply to constraint imposition as well, the only difference being that coupling involves two substructures and constraint imposition involves one. We will consider here only direct coupling, interconnection impedances (indirect coupling) having been treated in the previous section.

We begin again with Eq. (7), where the existence of two uncoupled substructures is manifest in the appropriate off-diagonal elements in each partition being 0. Coupling is to be established between pairs of coordinates in the c coordinate set, where the two coordinates in each pair are each from distinct substructures in a coupling, and from the same structure in the imposition of a constraint. The coupling of coordinates results in their merging into a single coordinate.

We extract the third row of Eq. (7),

$$\{x_c\} = [H_{ce}]\{f_e\} + [H_{cc}]\{f_c\} \quad (13)$$

and construct the conditions for equilibrium and compatibility to be imposed on the c coordinates. For the purpose of example, we will focus on a single pair of coordinates to be coupled, say x_c^{A} and x_c^{B}, where the superscripts denote that the first connection coordinate is from substructure A and the second from substructure B. The equilibrium and compatibility relations associated with this pair of coordinates are

$$f_c^{\mathrm{A}} + f_c^{\mathrm{B}} = 0 \quad x_c^{\mathrm{A}} - x_c^{\mathrm{B}} = 0. \quad (14a,b)$$

Considering now all pairs of coordinates to be coupled, we can rewrite Eq. (14a,b) to define the mapping matrices as

$$\{f_c\} = [M]\{\tilde{f}_c\} \quad \{\tilde{x}_c\} = [M]^T\{x_c\} \quad (15a,b)$$

where the tilde indicates the arbitrarily selected independent subset of the c coordinates.

We now assemble the transformation, analogous to Eq. (9), which operates on Eq. (7), and produces the synthesized version of Eq. (7) reflecting the coupled system. The transformation is assembled from Eq. (15a,b), and is

$$\begin{Bmatrix} f_e \\ f_c \end{Bmatrix} = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} f_e \\ \tilde{f}_c \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma \\ x_e \\ \tilde{x}_c \end{Bmatrix}^* = \begin{bmatrix} I & 0 \\ 0 & M^T \end{bmatrix} \begin{Bmatrix} \sigma \\ x_e \\ x_c \end{Bmatrix}^*. \quad (16a,b)$$

Operating on Eq. (7) these transformations produce

$$\begin{Bmatrix} \sigma \\ x_e \\ \tilde{x}_c \end{Bmatrix}^* = \begin{bmatrix} H_{\sigma e} & H_{\sigma c}M \\ H_{ee} & H_{ec}M \\ M^T H_{ce} & H_{cc}M \end{bmatrix} \begin{Bmatrix} f_e \\ \tilde{f}_c \end{Bmatrix}. \quad (17)$$

Extracting the third row of Eq. (17) and imposing the compatibility condition between the c coordinates, $\{\tilde{x}_c\} = \{0\}$ yields the following

$$\{\tilde{f}_c\} = -[\tilde{H}_{cc}]^{-1}[M]^T[H_{ce}]\{f_e\}. \quad (18)$$

Extracting the first two rows of Eq. (17) and substituting Eq. (18) leads to the operative equation for direct substructure coupling

$$\begin{bmatrix} H_{\sigma e} \\ H_{ee} \end{bmatrix}^* = \begin{bmatrix} H_{\sigma e} \\ H_{ee} \end{bmatrix} - \begin{bmatrix} H_{\sigma c} \\ H_{ec} \end{bmatrix} [M][\tilde{H}_{cc}]^{-1}[M]^T[H_{ce}] \quad (19)$$

where $[\tilde{H}_{cc}] = [M]^T[H_{cc}][M]$. In Eq. (19), all terms in the right-hand side are frequency response values calculated or measured from the uncoupled substructures. The term on the left-hand side reflects the coupled system response. Note again that the synthesis provides coupled system stress response directly. Also note that Eq. (15a,b) defines the differential coordinate system, which is a reduced-order system.

DIRECTED GRAPHS AND MAPPING MATRICES

The use of Eq. (19) to perform substructure coupling requires the construction of the mapping matrices, $[M]$. As was developed in the preceding section, each column of $[M]$ represents a statement of the equilibrium and compatibility enforced for each pair of connection coordinates being coupled. We now demonstrate that $[M]$ can be constructed from a graph drawn to represent the connectivity to be established through the synthesis.

Consider the coupling depicted on the left in Fig. 1. Substructure A is being coupled to substructure B, through, say, a single pair of connection coordinates, x_c^A and x_c^B. The coupling of this pair of coordinates creates load path I. To construct the mapping matrix for this connection I, we arbitrarily assign a value of 1 to the connection coordinate of substructure A and a value -1 to the connection coordinate of substructure B. The mapping matrix for this connection is

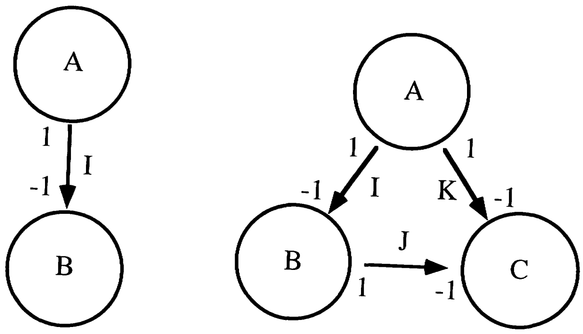

FIGURE 1 Substructure couplings and directed graphs.

$$[M] = \begin{array}{c} I \\ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \end{array}. \quad (20)$$

Considering now the more complicated coupling on the right of Fig. 1, and also acknowledging that in general two substructures are coupled using more than one pair of connection coordinates, we may construct the mapping matrix. Here, the connections I, J, and K consist of more than one pair of connection coordinates each; these are, in general, sets of connection coordinate pairs. The mapping matrix

$$[M] = \begin{array}{c} \begin{array}{ccc} I & J & K \end{array} \\ \begin{bmatrix} I & 0 & I \\ -I & I & 0 \\ 0 & -I & -I \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \end{array} \quad (21)$$

where each column contains plus/minus identity matrices whose elements correspond to the coupling to be established between each pair of connection coordinates. For example, in column 2 of the above mapping matrix, all connection coordinates associated with substructure A are assigned a 1 (i.e. $[I]$) and they are to be coupled to their counterparts in substructure C that have been assigned a -1 (i.e. $-[I]$). The coupling of these coordinates constitutes the set of load paths denoted as J.

The directed graphs and their Boolean mapping matrices provide a means of organizing complex couplings, and also provide a framework for the computational implementation of the synthesis, that is, Eq. (19). Of course, care must be exercised to insure that all matrices in Eq. (19) are appropriately partitioned.

MODIFICATION AND INDIRECT COUPLING USING MAPPING MATRICES

We now repeat the development of the operative equation of synthesis for indirect substructure coupling and structural modification. We will make use of the mapping matrices with the intent of bringing to bear the organization they provide.

Two classes of synthesis can be pursued using the mapping matrices and their directed graphs. The first is direct coupling, as discussed above. Here the mapping matrix is a Boolean matrix that represents the connectivity to be established between the connection coordinates. The mapping matrix is applicable to this problem because it conveniently contains information as to the equilibrium and compatibility between the substructures.

The second class of synthesis for which the mapping matrix approach is applicable is in indirect coupling and modification. Here, the connectivity information contained in the mapping matrix must correspond to the equilibrium of the interconnecting impedance (in the case of indirect coupling), or to the equilibrium of the modification. As will be seen, the mapping matrix is constructed from a statement of equilibrium for the impedance element.

We will find, however, that the use of the mapping matrices for these synthesis problems comes with some restrictions. It will be shown that only a limited class of structural element can be used as an interconnection impedance if its connectivity is to be described by a mapping matrix. Furthermore, it will be shown that only lumped elements yield Boolean mapping matrices, and therefore only indirect couplings that involve lumped interconnection impedances (e.g., springs, dampers) can make use of the directed graph as a means of organizing the synthesis. Finally, it will be shown that only zero mass interconnection impedances can be used in conjunction with the mapping matrices.

Analytic Determination of Mapping Matrices

We now determine the mapping matrices for an arbitrary interconnecting impedance or modification. Consider a structural element described by its impedance matrix (which does not contain mass terms, a restriction to be explained below):

$$\{f_c\} = [Z]\{x_c\} \tag{22}$$

where $\{x_c\}$ is the set of connection coordinates at which a structural modification is to be installed, or in the case of an indirect coupling, $\{x_c\} = [x_c^{\mathrm{A}} \quad x_c^{\mathrm{B}}]$ and $\{x_c^{\mathrm{A}}\}$ and $\{x_c^{\mathrm{B}}\}$ are the sets of connection coordinates to be coupled via the interconnection impedance. Henceforth, we will refer to both an interconnection impedance and a modification as simply the impedance. The connection coordinates communicate elastodynamic information between each other by virtue of the elastic modes of the impedance. Alternatively, we expect that for a nonmass impedance rigid body modes are superfluous to this process. Therefore, we write equilibrium for the impedance:

$$[R]\{f_c\} = \{0\}. \tag{23}$$

Selecting a subset of independent coordinates from the set c, we rewrite Eq. (23) as

$$[R_{\mathrm{A}} \quad R_{\mathrm{B}}]\begin{Bmatrix} f_c^{\mathrm{A}} \\ f_c^{\mathrm{B}} \end{Bmatrix} = \{0\} \tag{24}$$

where the forces $\{f_c^{\mathrm{A}}\}$ are at the arbitrarily selected independent coordinate set whose number equals the number of independent equilibrium equations in Eqs. (23) and (24). We construct the mapping matrix from Eq. (24),

$$\begin{Bmatrix} f_c^{\mathrm{A}} \\ f_c^{\mathrm{B}} \end{Bmatrix} = [M]\{f_c^{\mathrm{A}}\} \quad \text{where} \quad [M] = \begin{bmatrix} I \\ -R_{\mathrm{B}}^{-1}R_{\mathrm{A}} \end{bmatrix}. \tag{25}$$

This expression for $[M]$ will produce the identical result as Eq. (20) in the case of a lumped interconnection impedance, such as a simple spring. From Eq. (24), it is seen that for nonlumped impedances, such as a beam, the mapping matrix will be non-Boolean due to the inclusion of element dimensions in the moment equilibrium equations.

In order to use the mapping matrices in the indirect coupling and modification problems, the impedance matrix must take on a new form consistent with the differential coordinate system used to describe the coupling forces and responses. We will transform Eq. (8) using Eq.

(15a,b). The transformed impedance, referred to as the reduced impedance is

$$\{\tilde{f}_c\} = -[\tilde{Z}]\{\tilde{x}_c\} \quad \text{where} \quad [\tilde{Z}] = [M]^+[Z]([M]^T)^+. \quad (26)$$

Note that $\frac{1}{2}[x_c]^T[Z]\{x_c\} = \frac{1}{2}\{\tilde{x}_c\}^T[\tilde{Z}]\{\tilde{x}_c\}$, as shown in Eq. (2).

Transformation for Reduced Impedance

For a direct connection, we have shown that $[M]$ can be found analytically by considering the equilibrium for the connection coordinates. In the case of an indirect connection, $[M]$ is found by considering the equilibrium imposed on the interconnecting impedance element. The utility of the mapping matrix approach arises from its relation to the directed graph with its graphical means of representing the connectivity of the synthesis.

The reduced impedance is defined from the original impedance via a transformation involving the pseudoinverse of the mapping matrix, $[M]$. We will show the conditions for which the resulting reduced impedance $[\tilde{Z}]$ is full rank and contains an exact representation of the elastic mode information contained in the original impedance, $[\tilde{Z}]$. We restrict our consideration to impedance elements with no mass. The necessity of this restriction will be explained based on the result of what follows.

The impedance relation for the impedance element is $\{f_c\} = [Z]\{x_c\}$ and equilibrium for the impedance can be written

$$[R]\{f_c\} = [R][Z]\{x_c\} = \{0\} \quad (27)$$

where $\{f_c\} = \lfloor f_c^{a^T} \quad f_c^{b^T} \rfloor^T$. Selecting a subset of the connection coordinates as independent allows the following partitioning,

$$[R_a \quad R_b]\begin{Bmatrix} f_c^a \\ f_c^b \end{Bmatrix} = \{0\}. \quad (28)$$

The mapping matrix is found by rearranging Eq. (28), and is

$$\begin{Bmatrix} f_c^a \\ f_c^b \end{Bmatrix} = [M]\{\tilde{f}_c\} \quad (29)$$

where $\{\tilde{f}_c\}$ are the generalized forces at the independent subset of the connection coordinates, for example, $\{f_a^c\}$. We see from the above equations that $[R][M] = [0]$. The impedance matrix $[Z]$ has a modal decomposition,

$$[Z] = [\Phi_r \quad \Phi_e][\Lambda][\Phi_r \quad \Phi_e]^T = [\Phi_e][\Lambda_e][\Phi_e]^T \quad (30)$$

where $[\Lambda]$ is the spectral matrix of the impedance matrix $[Z]$, $[\Phi r]$ are the eigenvectors associated with the eigenvalues $\Lambda_i = 0, i = 1, 2, 3, \ldots, n_r$ where n_r is the number of zero eigenvalues (rigid body modes) possessed by $[Z]$, and $[\Phi e]$ are the eigenvectors associated with the eigenvalues $\Lambda_i \neq 0$, $i = 1, 2, 3, \ldots, n_e$ where n_e is the number of nonzero eigenvalues (elastic/dissipative modes) possessed by $[Z]$. Equation (30) makes clear that $\mathfrak{R}(Z) = \text{span}\{\Phi_e\}$, that is, the elastic modes provide a basis for the range of $[Z]$. We will also make use of the fact that $C^n = \text{span}\{\Phi_r\} \oplus \text{span}\{\Phi_e\}$, the rigid body modes $[\Phi r]$ define a subspace C^r which is the orthogonal complement to the subspace C^e defined by the elastic modes $[\Phi e]$. Given these facts, we see that $\mathfrak{R}(M) = \mathfrak{R}(\Phi^e)$. The transformation of the impedance relation begins by substituting $f_c = M\tilde{f}_c$ which yields

$$M\tilde{f}_c = Zx_c \quad (31)$$

We premultiply by $[M]^T$,

$$M^TM\tilde{f}_c = M^TZx_c \quad (32)$$

and solve for the reduced coupling forces,

$$\tilde{f}_c = (M^TM)^{-1}M^TZx_c. \quad (33)$$

We must solve the contragradient transformation for the displacements,

$$\tilde{x}_c = M^Tx_c. \quad (34)$$

Requiring that the connection coordinate responses be a linear combination of the columns of $[M]$ guarantees that we retain all elastic mode information, then $\tilde{x}_c = M^TM\alpha$ and $\alpha = (M^TM)^{-1}\tilde{x}_c$. Therefore, the reduced impedance matrix is

$$\tilde{f}_c = (M^TM)^{-1}M^TZM(M^TM)^{-1}\tilde{x}_c \quad (35)$$

or $\tilde{f}_c = \tilde{Z}\tilde{x}_c$ where $\tilde{Z} = M^+Z(M^T)^+$ and $(\ )^+$ indicates pseudoinverse.

From this analysis, we see that:

1. the mapping transformation is Boolean only for direct coupling and for lumped interconnection impedances;
2. $[M]$ is always full rank due to the fact that it is constructed from the linearly independent equilibrium equations;
3. the number of independent connection coordinates is determined by the connectivity to be established, and therefore so is the dimension of the reduced dimension coordinate system;
4. the mapping transformation purges the impedance matrix of rigid body mode information. The new coordinates produced are differential response coordinates, which do not represent rigid body motion, hence the exclusion of mass terms in the impedance.
5. If the number of independent coordinates equals the number of elastic modes of the impedance element, then $\mathcal{R}(M) = \mathcal{R}(\Phi^e)$, the reduced impedance will be full rank, and will provide an exact representation of the elastic mode content of the original impedance.

If we are interested in accommodating the connectivity information for the connection in a mapping matrix (which requires the extraction of the connectivity information from the original element impedance matrix, $[Z]$), then we must insure that the coupling to be established be compatible with a (i.e., produce a full rank) nonsquare transformation that preserves the elastic mode information.

Transformation for Indirect Coupling and Modification

We now present the transformation matrices that operate on Eq. (7) and lead to the operative equation for indirect coupling and modification, using the mapping matrices.

$$\begin{Bmatrix} f_e \\ f_c \end{Bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -M\tilde{Z} \end{bmatrix} \begin{Bmatrix} f_e \\ \tilde{x}_c \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma \\ x_e \\ \tilde{x}_c \end{Bmatrix}^* = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & M^T \end{bmatrix} \begin{Bmatrix} \sigma \\ x_e \\ x_c \end{Bmatrix}^* . \quad \text{(36a,b)}$$

The transformed version of Eq. (7) is

$$\begin{Bmatrix} \sigma \\ x_e \\ \tilde{x}_c \end{Bmatrix}^* = \begin{bmatrix} H_{\sigma e} & H_{\sigma c}M\tilde{Z} \\ H_{ee} & H_{ec}M\tilde{Z} \\ M^T H_{ce} & M^T H_{cc} M\tilde{Z} \end{bmatrix} \begin{Bmatrix} f_e \\ \tilde{x}_c^* \end{Bmatrix} . \quad (37)$$

The lowest row of Eq. (37) provides

$$\{\tilde{x}_c^*\} = [I + \tilde{H}_{cc}\tilde{Z}]^{-1}[M]^T[H_{ce}]\{f_e\}. \quad (38)$$

The upper two rows of Eq. (37) are extracted, Eq. (38) is substituted, and this leads to the operative equation for indirect coupling or modification:

$$\begin{bmatrix} H_{\sigma e} \\ H_{ee} \end{bmatrix}^* = \begin{bmatrix} H_{\sigma e} \\ H_{ee} \end{bmatrix} - \begin{bmatrix} H_{\sigma c} \\ H_{ec} \end{bmatrix} [M][\tilde{Z}^{-1} + \tilde{H}_{cc}]^{-1}[M]^T[H_{ce}] \quad (39)$$

where $[\tilde{H}_{cc}] = [M]^T[H_{cc}][M]$.

SIMULTANEOUS DIRECT AND INDIRECT SYNTHESIS

The operative equation for simultaneous indirect substructure coupling and modification, using the mapping matrices follow. This equation presents, in concise form, the generality of the frequency domain theory and makes clear the similarities of substructure coupling and structural modification. As already discussed, indirect substructure coupling is identical to structural modification, with respect to the derivation and application of the theory. Direct coupling and constraint imposition are also identical, and these two operations differ from the former two operations in the merging of connection coordinates versus the preservation of independence of connection coordinates.

We expand Eq. (7) by distinguishing between two types of connection coordinates, a subset of connection coordinates involved in a direct synthesis, $\{x_d\}$, and a subset involved in indirect synthesis, $\{x_i\}$, that is, $c = d \cup i$.

$$\begin{Bmatrix} \sigma \\ x_e \\ x_d \\ x_i \end{Bmatrix}^* = \begin{bmatrix} H_{\sigma e} & H_{\sigma d} & H_{\sigma i} \\ H_{ee} & H_{ed} & H_{ei} \\ H_{de} & H_{dd} & H_{di} \\ H_{ie} & H_{id} & H_{ii} \end{bmatrix} \begin{Bmatrix} f_e \\ f_d \\ f_i \end{Bmatrix} . \quad (40)$$

The transformations that operate on Eq. (31) are

$$\begin{Bmatrix} f_e \\ f_d \\ f_i \end{Bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & M_d & 0 \\ 0 & 0 & -M_i\tilde{Z} \end{bmatrix} \begin{Bmatrix} f_e \\ \tilde{f}_d \\ \tilde{x}_i^* \end{Bmatrix} \qquad (41)$$

$$\begin{Bmatrix} \sigma \\ x_e \\ \tilde{x}_d \\ \tilde{x}_i \end{Bmatrix}^* = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & M_d^T & 0 \\ 0 & 0 & 0 & M_i^T \end{bmatrix} \begin{Bmatrix} \sigma \\ x_e \\ x_d \\ x_i \end{Bmatrix}^* \qquad (42)$$

where $[M_d]$ and $[M_i]$ are the mapping matrices constructed for the direct and indirect connections respectively. The transformed version of Eq. (31) is;

$$\begin{Bmatrix} \sigma \\ x_e \\ \tilde{x}_d \\ \tilde{x}_i \end{Bmatrix}^* = \begin{bmatrix} H_{\sigma e} & H_{\sigma d}M_d & -H_{\sigma i}M_i\tilde{Z} \\ H_{ee} & H_{ed}M_d & -H_{ei}M_i\tilde{Z} \\ M_d^T H_{de} & M_d^T H_{dd}M_d & -M_d^T H_{di}M_i\tilde{Z} \\ M_i^T H_{ie} & M_i^T H_{id}M_d & -M_i^T H_{ii}M_i\tilde{Z} \end{bmatrix} \begin{Bmatrix} f_e \\ \tilde{f}_d \\ \tilde{x}_i^* \end{Bmatrix}. \qquad (43)$$

The lower two rows provide

$$\begin{Bmatrix} \tilde{x}_d \\ \tilde{x}_i \end{Bmatrix} = \begin{bmatrix} M_d^T H_{de} \\ M_i^T H_{ie} \end{bmatrix} \{f_e\} + \begin{bmatrix} M_d^T H_{dd}M_d & -M_d^T H_{di}M_i\tilde{Z} \\ M_i^T H_{id}M_d & -M_i^T H_{ii}M_i\tilde{Z} \end{bmatrix} \begin{Bmatrix} \tilde{f}_d \\ \tilde{x}_i^* \end{Bmatrix}. \qquad (44)$$

Enforcing compatibility, that is, $\{\tilde{x}_d\} = \{0\}$, yields

$$\begin{bmatrix} H_{\sigma e} \\ H_{ee} \end{bmatrix}^* = \begin{bmatrix} H_{\sigma e} \\ H_{ee} \end{bmatrix} - \begin{bmatrix} H_{\sigma d} & H_{\sigma i} \\ H_{ed} & H_{ei} \end{bmatrix} \begin{bmatrix} M_d & 0 \\ 0 & M_i \end{bmatrix} * \ldots$$
$$\ldots \begin{bmatrix} M_d^T H_{dd}M_d & M_d^T H_{di}M_i \\ M_i^T H_{id}M_d & \tilde{Z}^{-1} + M_i^T H_{ii}M_i \end{bmatrix}^{-1}$$
$$\begin{bmatrix} M_d & 0 \\ 0 & M_i \end{bmatrix}^T \begin{bmatrix} H_{de} \\ H_{ie} \end{bmatrix}^T \qquad (45)$$

which is the operative equation for general synthesis using mapping matrices. This equation will directly synthesize displacement or stress information.

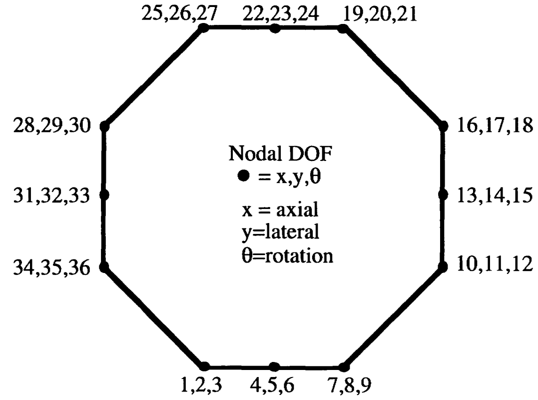

FIGURE 2 Idealized submarine hull model.

EXAMPLE OF SYNTHESIS: INDIRECT SUBSTRUCTURE COUPLING

This example presents the coupling of a machinery support platform to the hull of a submarine. It is desirable to minimize the transmissibility to the hull of machinery vibration. Therefore, the machinery support platform is indirectly coupled to the hull using viscoelastic isolators, and the frequency response between the machinery location and a point on the hull is calculated. This is an example of indirect substructure coupling. Note that the inclusion of the isolators defines a nonproportionally damped structure. The idealized submarine hull is shown on Fig. 2, and the idealized machinery platform and typical viscoelastic isolator is shown in Fig. 3. The synthesized system is shown in Fig. 4. All structures are modeled using two-noded beam elements with three DOF at each node. Therefore, each isolator shown will represent three isolators, one for each pair of nodal DOF being coupled.

The connection coordinates for the hull model are 10–15 and 31–36. The remaining coordinates are internal coordinates. Coordinates 1–3 and 13–21 of the platform model are connection coordinates and the remaining coordinates are internal. For this structural synthesis method, the

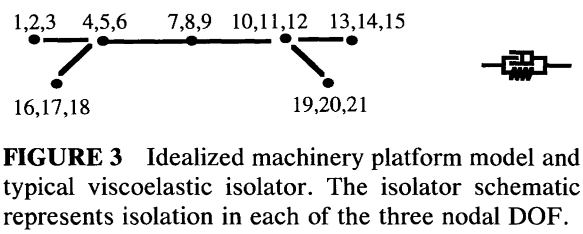

FIGURE 3 Idealized machinery platform model and typical viscoelastic isolator. The isolator schematic represents isolation in each of the three nodal DOF.

spring-damper isolators are treated as a lumped system (with no physical dimensions) installed at the connection coordinates. The connection coordinates do not merge into one but are joined by way of the isolators. The isolators are represented by the impedance matrix, $[Z(\Omega)] = [K] + j\Omega[C]$, where a simple viscous damping model for the isolator is assumed. The operative equation for indirect coupling with mapping matrices is

$$[H_{ee}]^* = [H_{ee}] - [H_{ec}][M][\tilde{Z}^{-1} + \tilde{H}_{ce}]^{-1}[M]^T[H_{ce}],$$

where $[\tilde{Z}] = [M]^{\dagger}[Z][M]^{T\dagger}$ and $[\tilde{H}_{cc}] = [M]^T[H_{cc}][M]$.

The Boolean mapping matrix $[M]$ is determined as follows. The connection coordinates for the hull and platform are listed above. We define the mapping matrix by $\{f_c\} = [M]\{\tilde{f}_c\}$ where $\{f_c\}$ is a vector of all the connection coordinates of both structures and $\{\tilde{f}_c\}$ is the arbitrarily selected independent subset of the connection coordinates relating to one of the substructures. We select arbitrarily the hull connection coordinates as the independent set of connection coordinates. The mapping matrix $[M]$ (shown below, where each I and 0 refer to a 3×3 partition associated with three nodal DOF) in this case is a matrix of size 24×12, where the rows correspond to the totality of connection coordinates, and the columns correspond to the selected set of independent connection coordinates. Alternatively, the columns can be seen to correspond to the load paths being established between pairs of coordinates.

$$[M] = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & -I & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & 0 & -I \\ -I & 0 & 0 & 0 \end{bmatrix}.$$

Note that $[\tilde{Z}]$ reduces to $[I](k + j\Omega\beta k)$ and its size is (12×12).

The FRF matrix $[H]$ for both substructures is required. First the $[K]$ and $[M]$ matrices are generated for each substructure. We next form the impedance matrix for each substructure. The impedance matrix is of the form $[Z] = [K] - \Omega^2[M]$. With the impedance matrix generated for each substructure, the FRF matrix H can be calculated by inverting the impedance matrix. Referring to the synthesis equation above, the matrices $[H_{ee}]$, $[H_{ec}]$, $[H_{cc}]$, and $[H_{ce}]$ are formed by combining $[H_1]$ and $[H_2]$ by appropriate partitioning. The partitioning is shown below:

$$[H_{ee}] = \begin{array}{c} \\ i_1 \\ i_2 \\ c_1 \\ c_2 \end{array} \begin{array}{c} \begin{array}{cccc} i_1 & i_2 & c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|cc} H_1(i_1, i_1) & [0] & H_1(i_1, c_1) & [0] \\ [0] & H_2(i_2, i_2) & [0] & H_2(i_2, c_2) \\ \hline H_1(c_1, i_1) & [0] & H_1(c_1, c_1) & [0] \\ [0] & H_2(c_2, i_2) & [0] & H_2(c_2, c_2) \end{array}\right] \end{array}$$

$$[H_{ce}] = \begin{array}{c} \\ c_1 \\ c_2 \end{array} \begin{array}{c} \begin{array}{cccc} i_1 & i_2 & c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|cc} H_1(c_1, i_1) & [0] & H_1(c_1, c_1) & [0] \\ [0] & H_2(c_2, i_2) & [0] & H_2(c_2, c_2) \end{array}\right] \end{array}$$

$$[H_{ce}] = \begin{array}{c} \\ c_1 \\ c_2 \end{array} \begin{array}{c} \begin{array}{cccc} i_1 & i_2 & c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|cc} H_1(c_1, i_1) & [0] & H_1(c_1, c_1) & [0] \\ [0] & H_2(c_2, i_2) & [0] & H_2(c_2, c_2) \end{array}\right] \end{array}$$

$$[H_{cc}] = \begin{array}{c} \\ c_1 \\ c_2 \end{array} \begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{bmatrix} H_1(c_1, c_1) & [0] \\ [0] & H_2(c_2, c_2) \end{bmatrix} \end{array}.$$

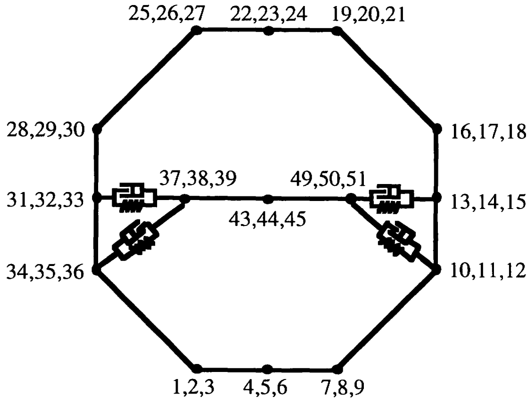

FIGURE 4 Combined hull-isolator-platform model.

Referring to the figures, i1 denotes the set of internal coordinates of the hull model that include DOF 1–9 and 16–30; c1 denotes the set of connection coordinates of the hull model which include 10–15 and 31–36; i2 denotes the set of internal coordinates of the platform model that include 4–12; c2 denotes the set of connection coordinates of the platform model that are 1–3 and 13–21. With the appropriate partitioning complete, the synthesis can be performed using the indirect coupling relation:

$$[H_{ee}]^* = [H_{ee}] - [H_{ec}][M][\tilde{Z}^{-1} + \tilde{H}_{cc}]^{-1}[M]^T[H_{ce}].$$

The synthesis is repeated over the frequency range of interest, and represents an exact solution for the nonproportionally damped combined system.

SUMMARY

A general formulation for frequency domain structural synthesis has been developed. The formulation addresses substructure coupling (both direct and indirect), constraint imposition, and structural modification. The formulation treats both static and complex dynamic analyses and accommodates any frequency dependencies in the system parameters. The formulation provides a highly efficient reanalysis methodology due to its exploitation of matrix partitioning available in frequency response function models, and due to the inclusion of the generalized frequency response function, allowing any type of response quantity to be directly synthesized. The formulation addresses the role of Boolean connectivity matrices in synthesis, and makes plain the relationship of these graph-based quantities with the mechanics of synthesis. The analysis of the reduced impedance presented delimits the extent to which a purely graph-based interpretation of the mapping matrices and reduced impedance can be made.

This work is dedicated to Julia Rebecca Sheed on the occasion of her first birthday.

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