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Dynamic Stiffness Matrix for a Beam Element with Shear Deformation

A method for calculating the dynamic transfer and stiffness matrices for a straight Timoshenko shear beam is presented. The method is applicable to beams with arbitrarily shaped cross sections and places no restrictions on the orientation of the element coordinate system axes in the plane of the cross section. These new matrices are needed because, for a Timoshenko beam with an arbitrarily shaped cross section, deflections due to shear in the two perpendicular planes are coupled even when the coordinate axes are chosen to be parallel to the principal axes of inertia. © *1995 John Wiley & Sons, Inc.*

INTRODUCTION

Two recent studies of the Timoshenko shear beam by Romano, Rosati, and Ferro (1992) and Schramm et al. (1994), have reconsidered some fundamental aspects of shear deformation coefficients. It is shown in Schramm et al. (1994) that the shear deformation coefficients α_{ij}, $i, j = y, z$, derived by an energy approach, form a symmetric tensor. The beam element and the right-handed Cartesian coordinate axes x, y, and z are shown in Fig. 1; the x axis is parallel to the axis of the beam and the orientation of the y and z axes may be chosen arbitrarily in the plane perpendicular to the x axis. For an unsymmetrical cross section, the principal axes of the shear deformation tensor, for which $\alpha_{xy} = 0$, are not parallel to the principal axes of inertia, for which the product of inertia $I_{xy} = 0$. Consequently, the deflections of the beam are coupled in general, even if the coordinate axes are chosen to be parallel to the principal axes of inertia. For a symmetrical cross section, the shear deformation coefficients form a diagonal tensor when referred to the principal axes of inertia, so that if the chosen coordinate axes are parallel to these axes, both α_{xy} and I_{xy} become zero and there is no coupling between the deflections v and w.

If deflection due to shear is to be considered in beams with unsymmetrical cross sections, element matrices that take the coupling between the deflections v and w into account must be used in static and dynamic analysis. The purpose of this article is to develop a method for determining two such element matrices for a straight Timoshenko shear beam in any Cartesian coordinate system. The first of these matrices is a dynamic transfer matrix and the second, which is expressed in terms of the entries of the first, is a dynamic stiffness matrix. A detailed review of the dynamic stiffness method is found in Fergusson and Pilkey (1993a,b).

In the development of the dynamic transfer matrix, Sylvester's theorems and the Cayley–Hamilton theorem from linear algebra are used. For a detailed presentation of these theorems in

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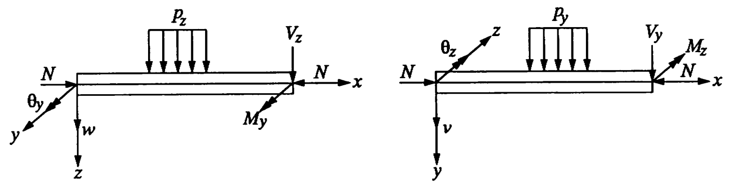

FIGURE 1 Beam element in two perpendicular planes.

the context of linear system theory, the reader is referred to Chen (1970), Lewis (1977), and Johnson and Johnson (1975).

BEAM ELEMENT EQUATIONS

The usual assumptions of beam theory are adopted. The material is linear elastic with elastic modulus E, Poisson's ratio ν, shear modulus $G = E/(2(1 + \nu))$, and mass per unit volume ρ. The loads on the beam may include distributed transverse forces $p_y(x, t)$ and $p_z(x, t)$, a temperature change distribution $T(x, y, z)$ and an axial compressive force N. The line of action of the transverse load resultant is assumed to pass through the shear center so that no torsion occurs. Axial deformation is not included. The internal section forces are the shear forces V_y, V_z, and the bending moments are M_y, M_z. The sign convention for the transfer matrix is shown in Fig. 2.

With these assumptions, the first-order differential equations for a beam element in an arbitrarily oriented coordinate system with the x axis parallel to the axis of the beam are (Pilkey and Wunderlich, 1993)

$$\frac{\partial v}{\partial x} = \theta_z + \alpha_{yy} \frac{V_y}{GA} + \alpha_{yz} \frac{V_z}{GA} \tag{1}$$

$$\frac{\partial \theta_z}{\partial x} = \frac{(M_z + M_{Tz})I_{zz}}{E(I_{yy}I_{zz} - I_{yz}^2)} + \frac{(M_y + M_{Ty})I_{yz}}{E(I_{yy}I_{zz} - I_{yz}^2)} \tag{2}$$

$$\frac{\partial w}{\partial x} = -\theta_y + \alpha_{yz} \frac{V_y}{GA} + \alpha_{zz} \frac{V_z}{GA} \tag{3}$$

$$\frac{\partial \theta_y}{\partial x} = \frac{(M_z + M_{Tz})I_{yz}}{E(I_{yy}I_{zz} - I_{yz}^2)} + \frac{(M_y + M_{Ty})I_{yy}}{E(I_{yy}I_{zz} - I_{yz}^2)} \tag{4}$$

$$\frac{\partial V_y}{\partial x} = \rho A \frac{\partial^2 v}{\partial t^2} - p_y(x, t) \tag{5}$$

$$\frac{\partial M_z}{\partial x} = -V_y - N\theta_z + \rho I_{yy} \frac{\partial^2 \theta_z}{\partial t^2} \tag{6}$$

$$\frac{\partial V_z}{\partial x} = \rho A \frac{\partial^2 w}{\partial t^2} - p_z(x, t) \tag{7}$$

$$\frac{\partial M_y}{\partial x} = V_z - N\theta_y + \rho I_{zz} \frac{\partial^2 \theta_y}{\partial t^2}. \tag{8}$$

In these equations, A is the cross-sectional area; I_{xx}, I_{yy} are area moments of inertia; I_{xy} is the area product of inertia; ρ is mass per unit volume; α_{ij}, $i, j = y, z$ are the shear deformation coefficients; and M_{Ty}, M_{Tz} are thermal moments.

To calculate the dynamic transfer matrix, each dependent variable $h(x, t)$ in the beam differential equations is replaced by

$$h(x, t) = h(x)e^{i\omega t}. \tag{9}$$

This eliminates the time dependence in the equations and a set of ordinary differential equations containing the frequency ω is obtained

$$\frac{d\mathbf{z}}{dx} = \mathbf{Az} + \bar{\mathbf{P}}. \tag{10}$$

In this equation, $\bar{\mathbf{P}}$ is the loading vector

$$\bar{\mathbf{P}} = [0 \quad b_1 M_{Tz} + b_2 M_{Ty} \quad 0 \quad b_2 M_{Tz} + b_3 M_{Ty} \quad -p_y \quad 0 \quad -p_z \quad 0]^T \tag{11}$$

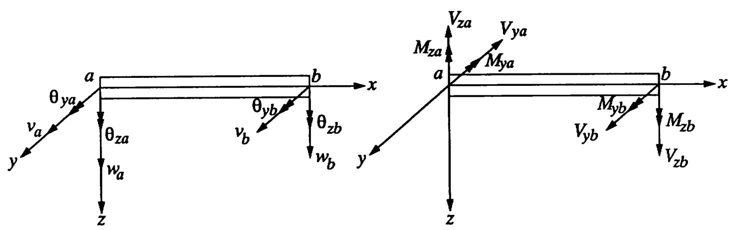

FIGURE 2 Sign convention for the transfer matrix.

and the matrix $\mathbf{A}$ is defined by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & a_1 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_1 & 0 & b_2 \\ 0 & 0 & 0 & -1 & a_2 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_2 & 0 & b_3 \\ -f_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -f_2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -f_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -f_3 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{12}$$

where

$$a_1 = \frac{\alpha_{yy}}{GA}$$
$$a_2 = \frac{\alpha_{yz}}{GA}$$
$$a_3 = \frac{\alpha_{zz}}{GA}$$
$$b_1 = \frac{I_{zz}}{E(I_{yy}I_{zz} - I_{yz}^2)}$$
$$b_2 = \frac{I_{yz}b_1}{I_{zz}}$$
$$b_3 = \frac{I_{yy}b_1}{I_{zz}}$$
$$f_1 = A\rho\omega^2$$
$$f_2 = N + I_{yy}\rho\omega^2$$
$$f_3 = N + I_{zz}\rho\omega^2.$$

The vector $\mathbf{z}$ contains the x-dependent parts of the deflection, slope, shear force, and bending moment variables in the following order

$$\mathbf{z} = [v \quad \theta_z \quad w \quad \theta_y \quad V_y \quad M_z \quad V_z \quad M_y]^T. \tag{13}$$

The integration of the beam differential equation in this form from one end of the beam element at $x = 0$ to the other end at $x = l$, gives

$$\mathbf{z}(l) = e^{\mathbf{A}l}\mathbf{z}(0) + \bar{\mathbf{z}} \tag{14}$$

where

$$\bar{\mathbf{z}} = e^{\mathbf{A}l} \int_0^l e^{-\mathbf{A}x}\bar{\mathbf{P}}(x)\, dx \tag{15}$$

from which it is seen that the dynamic transfer matrix $\mathbf{U}(\omega, l)$ is the matrix exponential $e^{\mathbf{A}l}$.

CALCULATION OF DYNAMIC STIFFNESS MATRIX

The characteristic polynomial of $\mathbf{A}$ is

$$\Delta(\lambda) = \lambda^8 + K_6\lambda^6 + K_4\lambda^4 + K_2\lambda^2 + K_0 \tag{16}$$

where

$$K_0 = (b_2^2 - b_1b_3)(-1 + a_1f_2 + a_3f_3 + (a_2^2 - a_1a_3)f_2f_3)f_1^2$$
$$K_2 = (-a^3b_1 - 2a_2b_2 - a_1b_3 + (a_1a_3 - a_2^2)(b_1f_2 + b_3f_3))f_1^2 + (b_2^2 - b_1b_3)(f_2 + f_3 - (a_1 + a_3)f_2f_3)f_1$$
$$K_4 = (a_1a_3 - a_2^2)f_1^2 + (b_1b_3 - b_2^2)f_2f_3 + (-b_1 - b_3 + (a_1 + a_3)(b_1f_2 + b_3f_3))f_1$$
$$K_6 = (a_1 + a_3)f_1 + b_1f_2 + b_3f_3.$$

Because the characteristic polynomial is an even polynomial in λ, its factors must also be even polynomials in λ. These may be of two types

$$P_1(\lambda) = \lambda^2 + a$$

and

$$P_2(\lambda) = \lambda^4 + b\lambda^2 + c$$

where a, b, and c are real. The factor $P_1(\lambda)$ is reducible over the field of real numbers for $a \leq 0$, in which case it has two real zeros, and irreducible for $a > 0$, in which case it has two imaginary zeros. The quartic factor $P_2(\lambda)$ is irreducible and has four zeros, two in the right half plane and two in the left half plane, symmetrically located with respect to the real and imaginary axes. Thus, the possible factorizations of the characteristic polynomial are

$$\Delta_1(\lambda) = (\lambda^4 + a\lambda^2 + b)(\lambda^4 + c\lambda^2 + d) \tag{17}$$
$$\Delta_2(\lambda) = (\lambda^2 + a)(\lambda^2 + b)(\lambda^4 + c\lambda^2 + d) \tag{18}$$
$$\Delta_3(\lambda) = (\lambda^2 + a)(\lambda^2 + b)(\lambda^2 + c)(\lambda^2 + d) \tag{19}$$

where a, b, c, and d are real and the quartic polynomial factors are irreducible over the field of real numbers.

If the characteristic polynomial $\Delta(\lambda)$ of $\mathbf{A}$ is expressible in the form $\Delta_1(\lambda)$, then the characteristic roots can be denoted by α, $-\alpha$, $\bar{\alpha}$, $-\bar{\alpha}$, β, $-\beta$, $\bar{\beta}$, $-\bar{\beta}$, where the overbar indicates complex conjugation. If $\alpha \neq \beta$, then the characteristic roots are distinct, and the matrix exponential $e^{\mathbf{A}l}$ can be obtained by formally expanding the determinant in the equation

$$\begin{vmatrix} e^{\alpha l} & e^{-\alpha l} & e^{\bar{\alpha} l} & e^{-\bar{\alpha} l} & e^{\beta l} & e^{-\beta l} & e^{\bar{\beta} l} & e^{-\bar{\beta} l} & e^{\mathbf{A} l} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \mathbf{I} \\ \alpha & -\alpha & \bar{\alpha} & -\bar{\alpha} & \beta & -\beta & \bar{\beta} & -\bar{\beta} & \mathbf{A} \\ \alpha^2 & \alpha^2 & \bar{\alpha}^2 & \bar{\alpha}^2 & \beta^2 & \beta^2 & \bar{\beta}^2 & \bar{\beta}^2 & \mathbf{A}^2 \\ \alpha^3 & -\alpha^3 & \bar{\alpha}^3 & -\bar{\alpha}^3 & \beta^3 & -\beta^3 & \bar{\beta}^3 & -\bar{\beta}^3 & \mathbf{A}^3 \\ \alpha^4 & \alpha^4 & \bar{\alpha}^4 & \bar{\alpha}^4 & \beta^4 & \beta^4 & \bar{\beta}^4 & \bar{\beta}^4 & \mathbf{A}^4 \\ \alpha^5 & -\alpha^5 & \bar{\alpha}^5 & -\bar{\alpha}^5 & \beta^5 & -\beta^5 & \bar{\beta}^5 & -\bar{\beta}^5 & \mathbf{A}^5 \\ \alpha^6 & \alpha^6 & \bar{\alpha}^6 & \bar{\alpha}^6 & \beta^6 & \beta^6 & \bar{\beta}^6 & \bar{\beta}^6 & \mathbf{A}^6 \\ \alpha^7 & -\alpha^7 & \bar{\alpha}^7 & -\bar{\alpha}^7 & \beta^7 & -\beta^7 & \bar{\beta}^7 & -\bar{\beta}^7 & \mathbf{A}^7 \end{vmatrix} = 0. \tag{20}$$

If this determinant is expanded about the first row, each minor to be calculated is a Vandermonde determinant, for which a closed form expression is known. This expansion is equivalent to computing the matrix exponential by Sylvester's theorem. The resulting expression for the dynamic transfer matrix $\mathbf{U}$ is expressible as a sum of four real matrices

$$\begin{aligned} \mathbf{U} &= \mathbf{C}(\alpha, \beta) + \mathbf{C}(-\alpha, \beta) \\ &\quad + \mathbf{C}(-\beta, \alpha) + \mathbf{C}(\alpha, \beta) \end{aligned} \tag{21}$$

where

$$\mathbf{C}(\alpha, \beta) = \frac{(\mathbf{A}^2 + 2 \operatorname{Re} \alpha \, \mathbf{A} + |\alpha|^2\mathbf{I})(A^4 - 2 \operatorname{Re} \beta^2\mathbf{A}^2 + |\beta|^4\mathbf{I})}{4 \operatorname{Im} \alpha \operatorname{Re} \alpha} \operatorname{Im}\left[\frac{e^{\alpha l}(\mathbf{A} - \bar{\alpha}\mathbf{I})}{\alpha(\alpha^2 - \beta^2)(\alpha^2 - \bar{\beta}^2)}\right]. \tag{22}$$

If the characteristic polynomial $\Delta(\lambda)$ has the factored form $\Delta_2(\lambda)$ with no repeated roots, then the roots may be denoted by α, $-\alpha$, β, $-\beta$, γ, $-\gamma$, $\bar{\gamma}$, and $-\bar{\gamma}$, where α and β are either real or imaginary, $\alpha \neq \beta$, and γ is complex. In this case an application of Sylvester's theorem gives the dynamic transfer matrix as a sum of four real matrices in the form

$$\mathbf{U} = \mathbf{D}(\alpha, \beta) + \mathbf{D}(\beta, \alpha) + \mathbf{E}(\gamma) + \mathbf{E}(-\gamma) \tag{23}$$

where

$$\mathbf{D}(\alpha, \beta) = \frac{(\mathrm{A}^2 - \beta^2\mathbf{I})(\mathbf{A}^4 - 2 \operatorname{Re} \gamma^2\mathbf{A}^2 + |\gamma|^4\mathbf{I})}{(\alpha^2 - \beta^2)|\alpha^2 - \gamma^2|^2} \left[\frac{\sinh \alpha l\mathbf{A}}{\alpha} + \cosh \alpha l\mathbf{I}\right] \tag{24}$$

and

$$\begin{aligned} \mathbf{E}(\gamma) &= \frac{(\mathrm{A}^2 - \alpha^2\mathbf{I})(\mathrm{A}^2 - \beta^2\mathbf{I})(\mathbf{A}^2 + 2 \operatorname{Re} \gamma\mathbf{A} + |\gamma|^2\mathbf{I})}{4 \mathit{Re}\, \gamma \operatorname{Im} \gamma} \\ &\quad \operatorname{Im}\left[\frac{e^{\gamma l}(\mathbf{A} - \bar{\gamma}\mathbf{I})}{\gamma(\gamma^2 - \alpha^2)(\gamma^2 - \beta^2)}\right]. \end{aligned} \tag{25}$$

If the characteristic polynomial can be factored in the form $\Delta_3(\lambda)$ and has distinct roots, the set of roots may be denoted by α, $-\alpha$, β, $-\beta$, γ, $-\gamma$, δ, and $-\delta$, where each root is either real or imaginary. In this case, the expression for the dynamic transfer matrix is

$$\begin{aligned} \mathbf{U} &= \mathbf{H}(\alpha, \beta, \gamma, \delta) + \mathbf{H}(\beta, \alpha, \gamma, \delta) \\ &\quad + \mathbf{H}(\gamma, \beta, \alpha, \delta) + \mathbf{H}(\delta, \beta, \gamma, \alpha) \end{aligned} \tag{26}$$

with

$$\mathbf{H}(\alpha, \beta, \gamma, \delta) = \frac{(\mathbf{A}^2 - \beta^2\mathbf{I})(\mathbf{A}^2 - \gamma^2\mathbf{I})(\mathbf{A}^2 - \delta^2\mathbf{I})}{(\alpha^2 - \beta^2)(\alpha^2 - \gamma^2)(\alpha^2 - \delta^2)} \left[\frac{\sinh \alpha l\mathbf{A}}{\alpha} + \cosh \alpha l\mathbf{I}\right]. \quad (27)$$

The foregoing algebraic procedure is not applicable when the characteristic polynomial has repeated roots. If, for example, the characteristic polynomial $\Delta(\lambda)$ has the factored form $\Delta_1(\lambda)$ with repeated roots, then

$$\Delta(\lambda) = (\lambda - \alpha)^2(\lambda - \bar{\alpha})^2(\lambda + \alpha)^2(\lambda + \bar{\alpha})^2$$

and the assumption that the quartic polynomial factors of $\Delta_1(\lambda)$ be irreducible over the real number field guarantees that $\alpha \neq 0$. Under these conditions, the matrix exponential $e^{\mathbf{A}l}$ can be found by formally expanding the determinant in the equation

$$\begin{vmatrix} e^{\alpha l} & e^{\bar{\alpha} l} & e^{-\alpha l} & e^{-\bar{\alpha} l} & le^{\alpha l} & le^{\bar{\alpha} l} & le^{-\alpha l} & le^{-\bar{\alpha} l} & e^{\mathbf{A}l} \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \mathbf{I} \\ \alpha & \bar{\alpha} & -\alpha & -\bar{\alpha} & 1 & 1 & 1 & 1 & \mathbf{A} \\ \alpha^2 & \bar{\alpha}^2 & \alpha^2 & \bar{\alpha}^2 & 2\alpha & 2\bar{\alpha} & -2\alpha & -2\bar{\alpha} & \mathbf{A}^2 \\ \alpha^3 & \bar{\alpha}^3 & -\alpha^3 & -\bar{\alpha}^3 & 3\alpha^2 & 3\bar{\alpha}^2 & 3\alpha^2 & 3\bar{\alpha}^2 & \mathbf{A}^3 \\ \alpha^4 & \bar{\alpha}^4 & \alpha^4 & \bar{\alpha}^4 & 4\alpha^3 & 4\bar{\alpha}^3 & -4\alpha^3 & -4\bar{\alpha}^3 & \mathbf{A}^4 \\ \alpha^5 & \bar{\alpha}^5 & -\alpha^5 & -\bar{\alpha}^5 & 5\alpha^4 & 5\bar{\alpha}^4 & 5\alpha^4 & 5\bar{\alpha}^4 & \mathbf{A}^5 \\ \alpha^6 & \bar{\alpha}^6 & \alpha^6 & \bar{\alpha}^6 & 6\alpha^6 & 6\bar{\alpha}^6 & -6\alpha^6 & -6\bar{\alpha}^6 & \mathbf{A}^6 \\ \alpha^7 & \bar{\alpha}^7 & -\alpha^7 & -\bar{\alpha}^7 & 7\alpha^6 & 7\bar{\alpha}^6 & 7\alpha^6 & 7\bar{\alpha}^6 & \mathbf{A}^7 \end{vmatrix} = 0. \quad (28)$$

The expansion of this determinant about the first row is again equivalent to applying a theorem due to Sylvester for finding an analytic function of a matrix with repeated characteristic values. The expression obtained for $\mathbf{U}$ can be written as

$$\mathbf{U} = \mathbf{G}(\alpha) + \mathbf{G}(-\alpha) \quad (29)$$

where

$$\mathbf{G}(\alpha) = \frac{(\mathbf{A}^2 + 2\,\mathrm{Re}\,\alpha\mathbf{A} + |\alpha|^2\mathbf{I})^2}{2(\alpha^2 - \bar{\alpha}^2)^2} \mathrm{Re}\left[\frac{e^{\alpha l}(\mathbf{A} - \bar{\alpha}\mathbf{I})^2}{\alpha^3}\left(\alpha\mathbf{I} - \left(1 - l\alpha + \frac{4\alpha^2}{\alpha^2 - \bar{\alpha}^2}\right)(\mathbf{A} - \alpha\mathbf{I})\right)\right]. \quad (30)$$

An alternative to using Sylvester's theorem in this calculation is to appeal directly to the Cayley–Hamilton theorem. The first step in this method is to solve the linear equation

$$\begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6 & \alpha^7 \\ 1 & \bar{\alpha} & \bar{\alpha}^2 & \bar{\alpha}^3 & \bar{\alpha}^4 & \bar{\alpha}^5 & \bar{\alpha}^6 & \bar{\alpha}^7 \\ 1 & -\alpha & -\alpha^2 & -\alpha^3 & -\alpha^4 & -\alpha^5 & -\alpha^6 & -\alpha^7 \\ 1 & -\bar{\alpha} & -\bar{\alpha}^2 & -\bar{\alpha}^3 & -\bar{\alpha}^4 & -\bar{\alpha}^5 & -\bar{\alpha}^6 & -\bar{\alpha}^7 \\ 0 & 1 & 2\alpha & 3\alpha^2 & 4\alpha^3 & 5\alpha^4 & 6\alpha^5 & 7\alpha^6 \\ 0 & 1 & 2\bar{\alpha} & 3\bar{\alpha}^2 & 4\bar{\alpha}^3 & 5\bar{\alpha}^4 & 6\bar{\alpha}^5 & 7\bar{\alpha}^6 \\ 0 & 1 & -2\alpha & -3\alpha^2 & -4\alpha^3 & -5\alpha^4 & -6\alpha^5 & -7\alpha^6 \\ 0 & 1 & -2\bar{\alpha} & -3\bar{\alpha}^2 & -4\bar{\alpha}^3 & -5\bar{\alpha}^4 & -6\bar{\alpha}^5 & -7\bar{\alpha}^6 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{bmatrix} = \begin{bmatrix} e^{\alpha l} \\ e^{\bar{\alpha} l} \\ e^{-\alpha l} \\ e^{-\bar{\alpha} l} \\ le^{\alpha l} \\ le^{\bar{\alpha} l} \\ le^{-\alpha l} \\ le^{-\bar{\alpha} l} \end{bmatrix} \quad (31)$$

for the scalars c_i, $0 \le i \le 7$. The transfer matrix $\mathbf{U}$ is then given as a weighted sum of powers of $\mathbf{A}$.

$$\mathbf{U} = \sum_{i=0}^{7} c_i \mathbf{A}^i. \tag{32}$$

A numerical method for calculating $\mathbf{U}$ can therefore be based on the Cayley–Hamilton theorem. The method is applicable to any square matrix $\mathbf{A}$ regardless of the multiplicities of its characteristic roots. The first step is to form the linear equation

$$\mathbf{Y}\mathbf{c} = \mathbf{f} \tag{33}$$

where $\mathbf{Y}$ is an 8×8 matrix and $\mathbf{c}$ and $\mathbf{f}$ are column vectors of length 8. If α is a root of multiplicity 1, then its contribution to the matrix $\mathbf{Y}$ is the row

$$\mathbf{r}(\alpha) = [1 \quad \alpha \quad \alpha^2 \quad \alpha^3 \quad \alpha^4 \quad \alpha^5 \quad \alpha^6 \quad \alpha^7] \tag{34}$$

and the corresponding row of $\mathbf{f}$ is $e^{\alpha l}$. If α is of multiplicity $n > 1$, it contributes $n - 1$ additional rows to $\mathbf{Y}$

$$\left.\frac{d^k \mathbf{r}(x)}{dx^k}\right|_{x=\alpha} \quad \text{for} \quad 1 \le k \le n - 1 \tag{35}$$

and the row of $\mathbf{f}$ corresponding to the kth contribution of α to $\mathbf{Y}$ is the kth derivative of e^{xl} evaluated at $x = \alpha$. The second step is to solve for $\mathbf{c}$ and the final step is to evaluate the sum appearing in Eq. (32) where c_i is the element in row $i + 1$ of $\mathbf{c}$.

When the characteristic polynomial of $\mathbf{A}$ has the factored form $\Delta_1(\lambda)$ or $\Delta_2(\lambda)$ with repeated roots, there are five possible root distributions, and $\Delta(\lambda)$ may assume any one of the following forms

$$\begin{aligned}
\psi_1(\lambda) &= (\lambda^2 + a)^2(\lambda^4 + c\lambda^2 + d) \\
\psi_2(\lambda) &= (\lambda^2 + a)^2(\lambda^2 + b)(\lambda^2 + c) \\
\psi_3(\lambda) &= (\lambda^2 + a)^2(\lambda^2 + b)^2 \\
\psi_4(\lambda) &= (\lambda^2 + a)(\lambda^2 + b)^3 \\
\psi_5(\lambda) &= (\lambda^2 + a)^4.
\end{aligned}$$

In the quadratic factors, the multiplicity of a root changes according to whether or not the constant term is zero. Thus, $\psi_5(\lambda)$ has two distinct roots α, $-\alpha$, each of multiplicity 4 when a is nonzero, but only one distinct root with multiplicity 8 when $a = 0$. Because a change in the multiplicity changes the form of the matrix $\mathbf{Y}$ in the Cayley–Hamilton method, the two cases $a = 0$ and $a \neq 0$ are fundamentally different.

If axial force $N = 0$ and the frequency $\omega = 0$, the static transfer matrix for a beam with no axial loading is obtained. The characteristic polynomial becomes

$$\Delta(\lambda) = \lambda^8 \tag{36}$$

and the Cayley–Hamilton equation is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3! & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4! & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5! & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6! & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7! \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{bmatrix} = \begin{bmatrix} 1 \\ l \\ l^2 \\ l^3 \\ l^4 \\ l^5 \\ l^6 \\ l \end{bmatrix} \tag{37}$$

which has the solution

$$c_i = \frac{l^i}{i!} \quad \text{for} \quad 0 \le i \le 7. \tag{38}$$

Because the powers of $\mathbf{A}$ greater than 3 are zero in this case, the expression for the transfer matrix is

$$\mathbf{U} = \sum_{i=0}^{3} \frac{l^i \mathbf{A}^i}{i!}. \tag{39}$$

Once the dynamic transfer matrix has been determined, it can be converted into a stiffness matrix. A stiffness matrix relates displacement variables to force variables, whereas in the transfer matrix the vector $\mathbf{z}$ of Eq. (14) contains both displacement and force variables. The conversion of a transfer matrix into a stiffness matrix is, therefore, a matter of rearranging Eq. (14)

$$\mathbf{z}(l) = \mathbf{U}\mathbf{z}(0) + \bar{\mathbf{z}}$$

such that force and displacement variables in $\mathbf{z}$ are collected into two separate vectors. In order to write an equation for the dynamic stiffness matrix, Eq. (14) is first rewritten in the partitioned form

$$\begin{bmatrix} \mathbf{v}_b \\ \mathbf{p}_b \end{bmatrix} = \begin{bmatrix} \mathbf{U}_{vv} & \mathbf{U}_{vp} \\ \mathbf{U}_{pv} & \mathbf{U}_{pp} \end{bmatrix} \begin{bmatrix} \mathbf{v}_a \\ \mathbf{p}_a \end{bmatrix} + \begin{bmatrix} \mathbf{f}_v \\ \mathbf{f}_p \end{bmatrix} \tag{40}$$

where the subscript a refers to the left end of the beam element at $x = 0$ and the subscript b to the right end at $x = l$. The definitions of the vectors $\mathbf{v}_a$, $\mathbf{p}_a$, $\mathbf{v}_b$, and $\mathbf{p}_b$ are

$$\begin{aligned} \mathbf{v}_a &= [v_a \quad \theta_{z_a} \quad w_a \quad \theta_{y_a}]^T \\ \mathbf{p}_a &= [V_{y_a} \quad M_{z_a} \quad V_{z_a} \quad M_{y_a}]^T \\ \mathbf{v}_b &= [v_b \quad \theta_{z_b} \quad w_b \quad \theta_{y_b}]^T \\ \mathbf{p}_b &= [V_{y_b} \quad M_{z_b} \quad V_{z_b} \quad M_{y_b}]^T. \end{aligned}$$

Each of the matrices $\mathbf{U}_{vv}$, $\mathbf{U}_{vp}$, $\mathbf{U}_{pv}$, and $\mathbf{U}_{pp}$ in Eq. (40) is a 4×4 submatrix of the transfer matrix $\mathbf{U}$. The vector $\mathbf{f}_v$ is made up of the first four elements of the loading vector $\bar{\mathbf{z}}$ and the vector $\mathbf{f}_p$ consists of the last four elements of $\bar{\mathbf{z}}$. Equation (40) gives

$$\mathbf{p}_a = \mathbf{U}_{vp}^{-1}(-\mathbf{U}_{vv}\mathbf{v}_a + \mathbf{v}_b - \mathbf{f}_v) \tag{41}$$

$$\mathbf{p}_b = (\mathbf{U}_{pv} - \mathbf{U}_{pp}\mathbf{U}_{vp}^{-1}\mathbf{U}_{vv})\mathbf{v}_a + \mathbf{U}_{pp}\mathbf{U}_{vp}^{-1}\mathbf{v}_b - \mathbf{U}_{pp}\mathbf{U}_{vp}^{-1}\mathbf{f}_v + \mathbf{f}_b \tag{42}$$

from which the expression for the stiffness matrix $\mathbf{K}$ is obtained as

$$\mathbf{K} = \begin{bmatrix} -\mathbf{U}_{vp}^{-1}\mathbf{U}_{vv} & \mathbf{U}_{vp}^{-1} \\ \mathbf{U}_{pv} - \mathbf{U}_{pp}\mathbf{U}_{vp}^{-1}\mathbf{U}_{vv} & \mathbf{U}_{pp}\mathbf{U}_{vp}^{-1} \end{bmatrix}. \tag{43}$$

Thus, the stiffness matrix equivalent of Eq. (14) is

$$\begin{bmatrix} \mathbf{p}_a \\ \mathbf{p}_b \end{bmatrix} = \mathbf{K} \begin{bmatrix} \mathbf{v}_a \\ \mathbf{v}_b \end{bmatrix} + \begin{bmatrix} -\mathbf{U}_{vp}^{-1} & 0 \\ -\mathbf{U}_{pp}\mathbf{U}_{vp}^{-1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{f}_v \\ \mathbf{f}_p \end{bmatrix}. \tag{44}$$

NUMERICAL EXAMPLE

The use of the dynamic stiffness matrix will be illustrated for a Timoshenko shear beam with the unsymmetrical cross section shown in Fig. 3. The details of the method used in calculating the shear coefficients are found in Schramm et al. (1994). For the example calculation, the cross-sectional dimensions shown in Fig. 3 are $t = 10$ mm, $h = 155$ mm, $L = 105$ mm, the cross-sectional area $A = 2500\ \text{mm}^2$, the moments of inertia $I_{yy} = 2350800\ \text{mm}^4$, $I_{zz} = 6207100\ \text{mm}^4$, $I_{yz} = -2241900\ \text{mm}^4$, Poisson's ratio $\nu = 0.3$, modulus of elasticity $E = 210$ GPa. The shear coefficients are given in Schramm et al. (1994) as $\alpha_{yy} = 3.061764$, $\alpha_{yz} = 0.04112366$, and $\alpha_{zz} = 1.899138$.

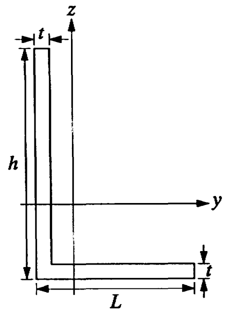

FIGURE 3 Cross section of the example beam.

The example beam, shown in Fig. 4, is subjected to a sinusoidally varying concentrated load $Ae^{i\omega t}$ at its free end. One beam element of length l is used to model the beam. The dynamic transfer matrix is

$$\mathbf{U}(\omega, l) = e^{\mathbf{A}l} \tag{45}$$

and

$$\mathbf{z}_b = \mathbf{U}(\omega, l)\mathbf{z}_a + \bar{\mathbf{z}} \tag{46}$$

where the forcing function in Eq. (46) is obtained from Eq. (15) by setting all forces equal to zero except $-p_z$, which is taken as a concentrated load of magnitude A at the free end of the beam, so that

$$\bar{\mathbf{z}} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -A \quad 0]^T. \tag{47}$$

The boundary conditions at the ends are enforced by taking

$$\mathbf{z}_a = [0 \quad 0 \quad 0 \quad 0 \quad V_{ya} \quad M_{za} \quad V_{za} \quad M_{ya}]^T \tag{48}$$

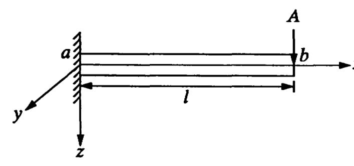

FIGURE 4 Cantilever beam with end load.

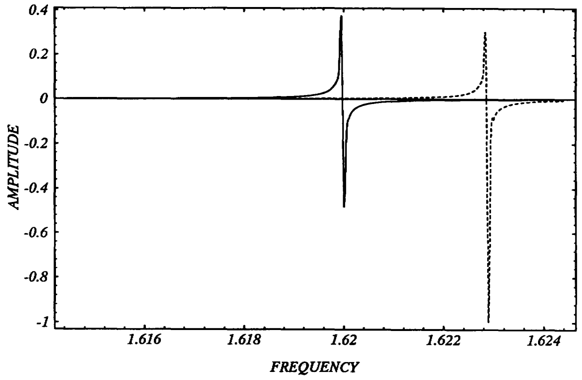

FIGURE 5 Frequency response near the first resonant frequency.

$$\mathbf{z}_b = [v_b \quad \theta_{zb} \quad w_b \quad \theta_{yb} \quad 0 \quad 0 \quad 0 \quad 0]^T. \tag{49}$$

For a given forcing function amplitude A and forcing frequency ω, Eq. (46) is a set of eight algebraic equations for the eight unknowns appearing on the right-hand sides of Eqs. (48) and (49).

Figures 5 and 6 show the frequency response of the beam near the first and second resonant frequencies. In each of these figures, the solid curve shows the response obtained when shear deformation effects are included and the dashed curve shows the response calculated by setting the shear deformation coefficients equal to zero. The ordinate in the plots is a normalized amplitude of the displacement response w_b at the free end of the beam. The abscissa is a normalized frequency variable Ω defined in terms of the excitation frequency ω by

$$\Omega = \omega^{1/2} l \left[\frac{\rho A}{EI_{yy}}\right]^{1/4}. \tag{50}$$

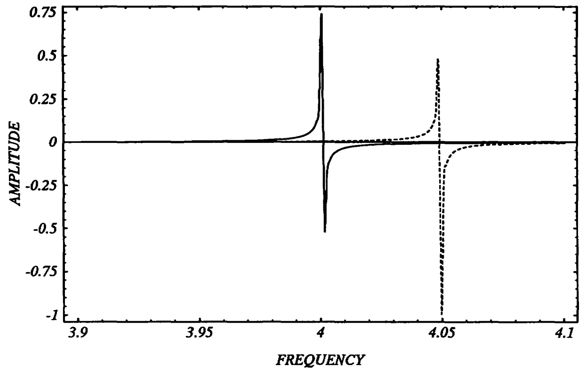

FIGURE 6 Frequency response near the second resonant frequency.

The length of the beam has been taken in this example as $l = 1.6$ m. The figures show that the natural frequencies are overestimated when shear effects are neglected.

CONCLUSION

A method for finding the dynamic transfer and stiffness matrices for a beam element with shear deformation has been described. Given numerical values for the elastic material properties, the mass density, the shear deformation coefficients, and the axial force, the matrix $\mathbf{A}$ for an element is computed for a desired frequency ω. The characteristic values of $\mathbf{A}$ are then calculated as the roots of an even polynomial $\Delta(\lambda)$ of degree 8. The transfer matrix $\mathbf{U}$ for a beam of length l is the matrix exponential $e^{\mathbf{A}l}$. A numerical method for calculating $e^{\mathbf{A}l}$ based on the Cayley–Hamilton theorem has been presented. This method is applicable to any matrix $\mathbf{A}$ regardless of the multiplicities of its eigenvalues. If the characteristic roots of $\mathbf{A}$ are distinct, $\mathbf{U}$ can be evaluated using Eqs. (21), (23), or (26), depending on the characteristic roots of $\mathbf{A}$. Finally, the dynamic stiffness matrix is given by Eq. (43) in terms of four 4×4 submatrices of the transfer matrix $\mathbf{U}$.

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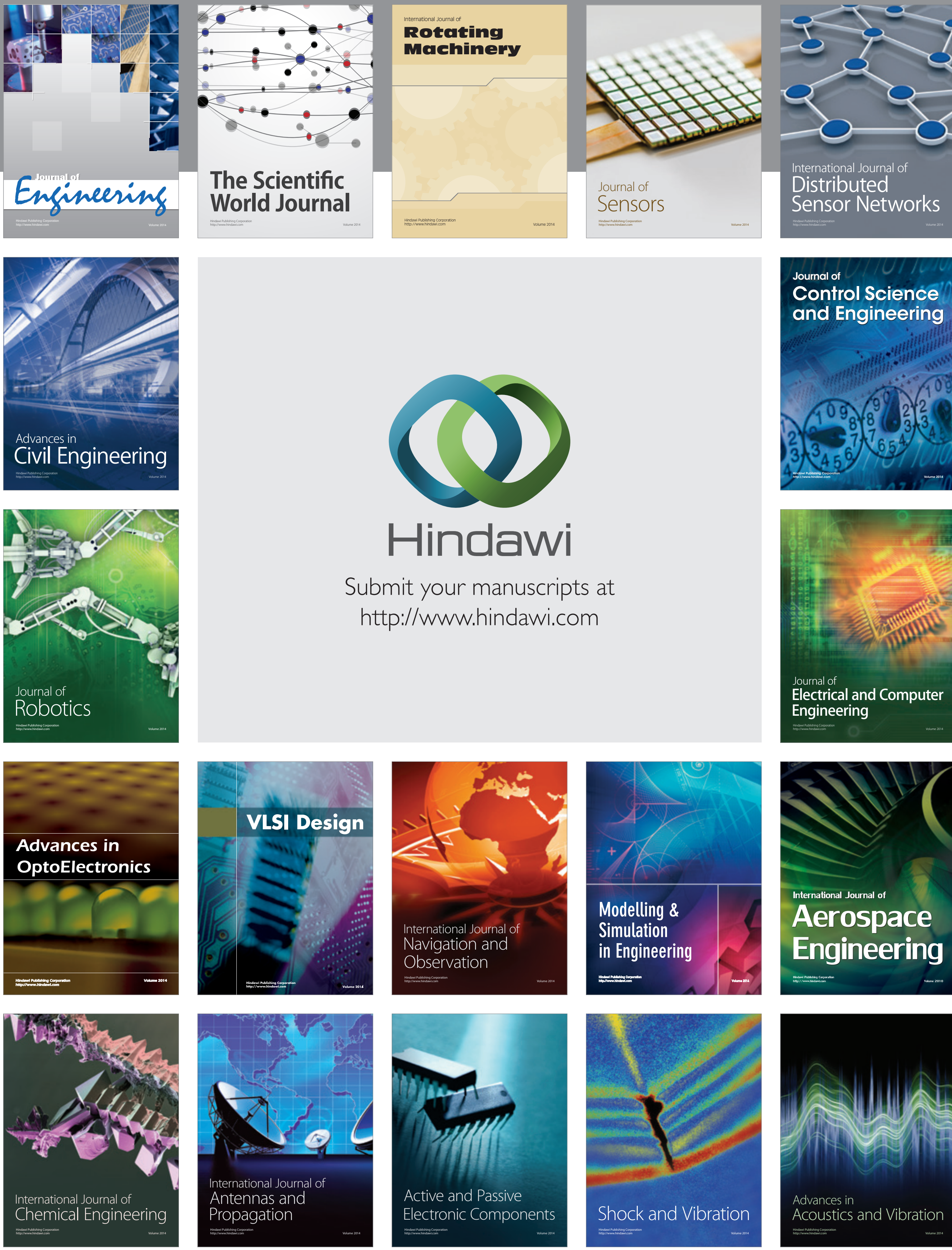
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