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Review

Waves, Solids, and Nonlinearities

In this article nonlinearity is taken as a basic property of continua or any other wave-bearing system. The analysis includes the conventional wave propagation problems and also the wave phenomena that are not described by traditional hyperbolic mathematical models. The basic concepts of continuum mechanics and the possible sources of nonlinearities are briefly discussed. It is shown that the technique of evolution equations leads to physically well-explained results provided the basic models are hyperbolic. Complicated constitutive behavior and complicated geometry lead to mathematical models of different character and, as shown by numerous examples, other methods are then used for the analysis. It is also shown that propagating instabilities possess wave properties and in this case the modeling of energy redistribution has a great importance. Finally, some new directions in the theory and applications are indicated.

INTRODUCTION

This is not a review article per se with the careful collection of all the references available on the topic, but rather a viewpoint of the author explained by numerous examples. Nonlinear wave motion as a physical phenomenon is a rich area of contemporary science where mechanics is spiced with mathematics, physics of materials interwoven with technology, and new problems form a driving force for developing new methods of analysis. From time to time, such complicated phenomena simply need an overall review.

A rather general statement, advocated in rational mechanics describes a wave as a state moving into another state with a finite speed. Some comments are needed here. First, in pure mathematical terms a finite speed means the existence of a real eigenvalue of the corresponding mathematical models. These models are called hyperbolic and they are of fundamental importance in wave motion (Whitham, 1974; Bland, 1969). On the other hand, the hyperbolic systems emphasize the principle of causality in its pure form: whatever a disturbance at a certain point of space, the neighboring points feel the disturbance after a finite time interval. Second, the state under consideration may be stress, deformation, displacement, etc. when using the conventional wave theory, but also a crack, a bulge, a buckle, etc. when taking a broader view. And third, apart from strict hyperbolicity, waves may be characterized just by their dispersive relations, i.e. the models should possess certain harmonic solutions with fixed wave numbers and frequencies. These waves are called dispersive (Whitham, 1974) and, as easily understood, not described by the definition above.

The mathematical models describing dynamics of continua are based on the conservation laws. Complemented by suitably chosen constitutive laws, the final set of equations is then de-

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rived that, generally speaking, should be of a hyperbolic character. However, the physical world is rich and the constitutive laws may reflect so many different properties that at this stage assumptions are often needed causing the loss of hyperbolicity in the strict mathematical sense. In many cases the hyperbolicity is still preserved in some asymptotical sense. Next, the existence of constraints (boundaries, supports, inclusions, etc.) may lead to additional complications, and again assumptions are introduced in order to obtain governing equations in forms suitable for analysis. Dealing with continua of complicated properties, besides the conventional observable variables, the notion of internal variables is useful (Kestin, 1992). This leads directly to nonhyperbolic governing equations that still may exhibit wave-type solutions.

Nonlinearity plays a decisive role in all these models. Its importance in contemporary physical sciences and engineering is hard to overestimate (West, 1985)—the existence of superposition as a trademark of a linear theory is an essential simplification that does not usually reflect the reality in a broad range. However, nonlinearity is seldom the only physical property to be taken into account in proper modeling. According to the principle of equipresence, all the effects of the same order must be taken into account simultaneously. Combined with other effects, nonlinearity leads to qualitative new results in reflecting reality. In this sense, the best example is a soliton-bearing system where nonlinearity is combined with dispersion.

Summing up this brief overview, the following must be pointed out:

1. the traditional mathematical models for wave motion are either strictly hyperbolic or asymptotically hyperbolic;
2. constitutive laws and constraints may lead to very complicated mathematical models with a wide range of properties;
3. hyperbolicity may be lost in deriving the governing equations but still certain wave phenomena could be described by these mathematical models.

This list gives the main idea of the study: starting from traditional mathematical models we would like to describe nonlinear waves in complicated situations characteristic to contemporary science and technology. We keep the definition given above and stress the notion of "a state" that should propagate whatever the mathematical model is. It is quite a good moment for writing such a review because in 1993 several meetings were organized worldwide: the IUTAM Symposium on Nonlinear Waves in Solids, Victoria (Wegner and Norwood, 1993); the 13th International Symposium in Nonlinear Acoustics, Bergen (Hobaek, 1993); and the CISM course on Nonlinear Waves in Solids, Udine (Jeffrey and Engelbrecht, 1994). That makes an excellent background for collecting the main ideas in this field.

The article is organized as follows. In the following section, the conceptual approach is envisaged according to conventional continuum mechanics. The steps of an effective analysis are presented with a special attention to the sources of nonlinearities. The next section deals with traditional problems on the basis of hyperbolic or asymptotically hyperbolic mathematical models. In this case the evolution equations for single waves are easily derived. The possibility to extract the model problems governed by model nonlinear evolution equations has great importance in contemporary science because parallels can be easily drawn on this basis between many physical phenomena. The following three sections are devoted to less traditional problems. The fourth section describes the complicated constitutive behavior (generalized potentials, coupled fields, existence of internal variables and inhomogeneities) with its consequences to wave motion. In the fifth section the waves in structures of complicated geometry are analyzed. The cases under inspection include surface waves and waves in wedges and elasticas. The sixth section deals with propagating instabilities. These instabilities, such as buckles, bulges, cracks, etc. are, as a rule, described by nonhyperbolic mathematical models. Still, they propagate as waves according to the definition given above. The final section summarizes the results. Some new directions in the theory and applications are indicated.

BASIC PRINCIPLES

The conceptual approach in constructing the mathematical models of wave motion is based on the following hierarchical sequence:

1. basic principles (initial assumptions and conservation laws);

2. constitutive theory (constitutive laws added together with auxiliary postulates in order to form a closed system);
3. mathematical models (auxiliary assumptions about the character of field variables and approximations of the constitutive laws).

The details of modeling can be found in monographs by Eringen (1962), Eringen and Suhubi (1974), Engelbrecht (1983), and others. Beside this sequence, certain physical and mathematical requirements should be obeyed in order to guarantee the best correspondence between the models and reality. Following Eringen and Maugin (1990), these are the following axioms: causality; determinism; equipresence; objectivity; time reversal; material invariance; neighborhood; memory; admissibility.

Two of this list should be stressed—equipresence and admissibility. When dealing with nonlinearities one should especially be aware of equipresence, i.e. all constitutive functionals are to be considered to depend on the same list of constitutive variables (Eringen and Maugin, 1990). This is a key for proper modeling of interaction of waves. In addition, as far as constitutive equations tend to be rather complicated, one should not forget that thermodynamic admissibility restricts the constitutive equations essentially.

Following these rules, a proper nonlinear mathematical model governing nonlinear waves should be derived, then solved, and the results analyzed. For an effective analysis, the following has to be understood (Engelbrecht, 1993):

1. the exact background of physical considerations and mathematical models;
2. the possible sources of nonlinearity(ies);
3. the accuracy and limits of mathematical tools;
4. the character of model problems and their possible generalizations;
5. the correspondence of mathematical results to physical realities.

A brief analysis of the possible sources of nonlinearities was given earlier by the author (Engelbrecht, 1993; Jeffrey and Engelbrecht, 1994). Restricting ourselves to observable variables only (stress, strain, displacement, etc.), the nonlinearities are (see also Moon, 1987):

1. material (physical), i.e. of the constitutive law;
2. geometrical, i.e. of deformation and displacements;
3. kinematical, i.e. of convectivity, compound motion, etc.;
4. structural, i.e. of constraints;
5. combined.

The last needs some explanation because the terminology is not established and combined nonlinearities could also be named coupling nonlinearities. They arise as a result of a combination of other nonlinearities listed above.

The importance of being nonlinear is nowadays widely accepted in physical sciences (West, 1985; Thompson and Stewart, 1986, a.o.) including wave motion (Whitham, 1974; Taniuti and Nishihara, 1983; Jeffrey and Kawahara, 1982; Engelbrecht, 1991). Distortion of wave profiles (spectral changes), amplitude-dependent velocities, interaction of waves, spatiotemporal chaos, and other physical effects are all appearances of nonlinear wave motion. In this broad field we restrict ourselves to processes where loading and unloading paths are not different. Therefore, the effects of plastic deformation where hysteresis, ratcheting, hardening, and other specific phenomena are of importance, are not discussed here, but are mentioned in the propagating instabilities section. Much on this topic can be found in Maugin (1992a).

TRADITIONAL PROBLEMS

The mathematical models of continua are based on conservation laws that can be found in many monographs, for example in Truesdell and Noll (1965) and Eringen (1962), among others. The more concise description with the view to nonlinear wave motion in solids is given by Bland (1969) and Engelbrecht (1983).

The main conservation laws are:

(i) the balance of momentum

$$(T^{KL}x^k_{,L})_{,K} + \left(\begin{Bmatrix} k \\ ml \end{Bmatrix} x^m_{,L}x^l_{,K} + \begin{Bmatrix} M \\ MK \end{Bmatrix} x^k_{,L}\right)T^{KL} + \rho_0(f^k - A^k) = 0; \tag{1}$$

(ii) conservation of energy

$$\rho\dot{\varepsilon} = T^{KL}\dot{E}_{KL} + Q^K_{,K} + \begin{Bmatrix} p \\ pl \end{Bmatrix} x^l_{,K}Q^K + \rho_0 h. \tag{2}$$

The notation used here is conventional (Eringen, 1962). Beside the kinematical relations that are exact "geometrical" expressions, we need constitutive laws

$$T^{KL} = T^{KL}(E_{KL}, T, \dot{E}_{KL}, \ldots), \tag{3}$$

$$f^k = f^k(E_{KL}, T, \ldots), \tag{4}$$

$$Q^K = Q^K(T, E_{KL}, \ldots), \tag{5}$$

obeying principles listed in the previous section.

As a result, a mathematical model for a wave process is derived that due to nonlinearity, equipresence, etc. may be rather complicated. The matrix representation has some advantages compared with tensorial or algebraic models, mainly because of the clear representation of coefficients and of the possible usage of matrix calculus. We shall use it also here.

Let $\mathbf{U}$ be the n-vector of field (observable) variables. Then the governing equations can be written as

$$I \frac{\partial \mathbf{U}}{\partial t} + A^K \frac{\partial \mathbf{U}}{\partial X^K} + \sum \varepsilon^{m(p)} B^{\alpha\beta}_{rs} \frac{\partial \mathbf{U}^p}{(\partial X^\alpha)^r \partial (X^\beta)^s} + \mathbf{H} = 0, \tag{6}$$

where $K = 1, 2, 3; \alpha, \beta = 0, 1, 2, 3; X^0 = t, r + s = p \geq 2, m \geq 1$ and

$$A^K = A^K(X^M, \mathbf{U}), \quad B^{\alpha\beta}_{rs} = B^{\alpha\beta}_{rs}(X^K, \mathbf{U}), \quad \mathbf{H} = \mathbf{H}(X^K, \mathbf{U}). \tag{7}$$

Vector $\mathbf{H}$ may also contain a small parameter ε and integral operators of $\mathbf{U}$. The solutions of Eq. (6) are waves $\mathbf{U}(X^k, t)$ and they are sought to satisfy initial and boundary conditions

$$\mathbf{U}(X^K, t)\,|_{t=0} = \Psi(X^K), \quad \mathbf{U}(X^K, t)\,|_S = \Phi(X^\alpha), \tag{8}$$

where S denotes a certain boundary. The detailed description of such a system is given, among the others, by Taniuti and Nishihara (1983) and Engelbrecht (1983).

The system (6) is quasilinear in leading terms but may possess stronger nonlinear properties because of $\mathbf{H}$. In other words, nonlinearities affect mostly A^K and $\mathbf{H}$, but also $B^{\alpha\beta}_{rs}$. The main and distinctive feature of (6) is that there is a possibility to find an associated linear, strictly hyperbolic system as needed for wave motion with finite velocity. We restrict ourselves here to this system but stress that in the case of internal variables the governing system may also contain the diffusion-reaction equations (Maugin, 1990).

There are no direct methods known to solve Eq. (6); therefore, in making a list of possible methods to solve (6), the exact solutions may be excluded at once. The approximate methods, however, may be divided into three main groups:

1. the approximate analysis of the exact solution;
2. the perturbative analysis of the solution with small (slow) derivation from a known one;
3. the simplification of mathematical models (equations) describing the process.

Referring to Whitham (1974), Van Dyke (1964), Jeffrey and Kawahara (1982) for details concerning the first two groups, we concentrate our attention upon the third group that has an explicit physical background. This is the notion of evolution equations governing just one single wave.

Physically it means the separation (if possible) of a multiwave process into separate waves. Each wave is then governed by a so-called evolution equation describing the distortion of the wave under consideration along a properly chosen characteristic (ray). The latter means that a certain velocity has been taken into account. However, the nonlinear coupling may affect this separation process (see below).

The main idea of this "taming" procedure of a nonlinear multiwave system is the following. A set of small parameters related either to the initial conditions or to the physical and/or geometrical parameters is introduced and the perturbation method together with the method of stretched coordinates involving a moving frame is then applied. Taniuti and his coworkers (see Taniuti and Nishihara, 1983) who have initiated such an approach called it the "reductive perturbation method." Now we know several methods used to simplify the initial multiwave system (Engelbrecht, Fridman, and Pelinovski, 1988).

The methods used to construct the evolution equations for single waves are the following:

1. asymptotic, i.e., reductive perturbation (Taniuti and Nishihara, 1983; Engelbrecht, 1983);

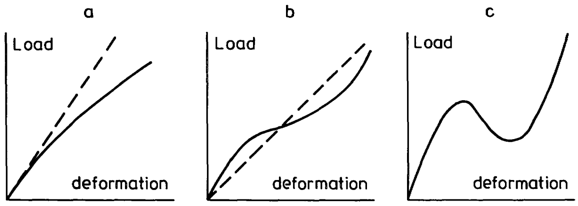

FIGURE 1 Load-deformation curves: (a) monotonous response; (b) convex–concave response; (c) local unstable response.

2. iterative (Gorschkov, Ostrovski, and Pelinovski, 1974);
3. spectral (Miropolski, 1981); and
4. approximate (etalon) evolution equations (Whitham, 1974; Engelbrecht, 1992).

Such an approach is possible if nonlinearities are weak (Taniuti and Nishihara, 1983). Schematically it means, that in physical terms the load-deformation curve is either monotonous [Fig. 1(a)] or convex–concave [Fig. 1(b)], where the conditions of weak nonlinearities are satisfied. The locally unstable response (Pence, 1992) as shown in Fig. 1(c), is seemingly beyond the limits of these conditions.

Using the methods described above, the evolution equations have been derived for many physical problems. The earlier articles are dated around the 1970s (Nariboli and Sedov, 1970; Sedov and Nariboli, 1971). Many examples are given by Engelbrecht (1983), Naugolnykh and Ostrovsky (1990), and Fusco and Jeffrey (1991). Here we give just one rather general representation of 1-D and 2-D evolution equations for deformation waves $v \sim \partial U_1/\partial t \sim \partial U_1/\partial X_1$. The moving frame is given by $\xi \sim ct - X_1$, $\tau \sim X_1$, $\eta \sim X_2$ with proper scaling. The 1-D case (dimensionless form) is

$$\begin{aligned} R(v) = \frac{\partial v}{\partial t} \pm v^m \frac{\partial v}{\partial \xi} + \Theta(\tau) \frac{\partial v}{\partial \xi} + \Lambda v \\ - \Gamma^{-1} \frac{\partial^2 v}{\partial \xi^2} + \Omega^{-2} \frac{\partial^3 v}{\partial \xi^3} \qquad (9) \\ - \Xi \frac{\partial}{\partial \xi} \int_0^\xi \frac{\partial v}{\partial z} K(\xi - z)\, dz + f(v) = 0. \end{aligned}$$

Here $m = 1, 2, \ldots$; Θ, Λ, Γ, Ω, Ξ are constants; $K(z)$ is a kernel function; and $f(v)$ denotes a nonlinear function. The derivation of such an equation is explained by Engelbrecht (1983) and it is applied for waves in soft tissues (Engelbrecht and Chivers, 1989). The 2-D case is then

$$\frac{\partial}{\partial \xi} [R(v) + F(v, \eta)] = \Delta \left[\frac{\partial^2 v}{\partial \eta^2} + \frac{n}{\eta} \frac{\partial v}{\partial \eta}\right], \qquad (10)$$

where Δ is a constant; $n = 0, 1$ for plane and cylindrical waves, respectively; and $F(v, \eta)$ describes the coupling effects (Peipman, Valdek, and Engelbrecht, 1992).

There are three types of nonlinearities involved in this level of analysis.

1. The second term in (9) reflects a character of constitutive laws and geometrical nonlinearities in a concise form; longitudinal waves are characterized by $m = 1$ and transverse waves by $m = 2$ caused by quadratic and cubic nonlinearities, respectively. The pulse distortions depend essentially on the values of m for most materials; physical and geometrical nonlinearities should be taken into account simultaneously.
2. The function $f(v)$ reflects the influence of the possible internal force (stress), as for example the body force.
3. Nonlinear coupling is described by the function $F(v, \eta)$ in (9), particularly nonlinear 2-D coupling of longitudinal and transverse waves [in linear model $F(v, \eta) \equiv 0$]; in the case of coupled fields like in elastoelectrodynamics of continua (Eringen and Maugin, 1990), coupling terms may also appear in the 1-D case.

The qualitative analysis of nonlinear waves in solids is given in the concise form by Engelbrecht (1993) but more can be found in Jeffrey and Engelbrecht (1994). Here we would like only to stress main effects arising due to nonlinearity. These are: distortion of wave profiles (spectral changes); amplitude dependent velocities; and interaction of waves.

COMPLICATED CONSTITUTIVE LAWS

Beside weak nonlinearity and other accompanying physical effects of the same accuracy, the constitutive laws may be of far more complicated character and then the direct derivation of evolution equations (the previous section) may become impossible. Here some examples are presented in order to demonstrate the arising problems.

Elastic Potentials

Material nonlinearity is usually described in terms of the Helmholtz free energy function F (potential). A series representation is the most common one:

$$\rho_0 F = \tfrac{1}{2}\lambda I_1^2 + \mu I_2 + \nu_1 I_1^3 + \nu_2 I_1 I_2 + \nu_3 I_3 + \kappa_1 I_1^4 + \kappa_2 I_1^2 I_2 + \kappa_3 I_1 I_3 + \kappa_4 I_2^2 + \ldots, \qquad (11)$$

where I_1, I_2, I_3 stand for the three independent algebraic invariants of the Green deformation tensor and the coefficients determine the second (λ, μ), third (ν_1, ν_2, ν_3), and fourth (κ_1, κ_2, κ_3, κ_4) order elastic constants while ρ_0 is the density. The values of the coefficients are determined experimentally (Breazeale and Philip, 1984; Cantrell, 1989, etc.). These energy functions usually lead to weak nonlinearities. Incompressible materials, however, like rubber need other type of potential (Eringen, 1962):

$$\rho_0 F = \sum_{m,n=0}^{\infty} A_{mn}(J_1 - 3)^m (J_2 - 3)^n, \quad A_{00} = 0, \qquad (12)$$

where A_{mn} are constants and

$$J_1 = 3 + 2I_1, \quad J_2 = 3 + 4I_1 + 2(I_1^2 - I_2). \qquad (13)$$

The widely used Mooney–Rivlin and Treloar potentials follow from (12) (see, for example, Haddow, Wegner, and Jiang, 1992). These potentials are applicable for large deformations and the conditions for deriving evolution equations must be checked carefully. There are interesting physical effects described by potentials like (12). Haddow (1993) has used a generalized Mooney–Rivlin potential with and without coupling to thermal effects. He has shown how the longitudinal and transverse waves are coupled demonstrating the dependence of the coupling on the value of $m = K/\mu$, where K and μ are the isothermal bulk and shear moduli, respectively. This result is obtained by using the similarity solution. One should compare these results with results by Peipman, et al. (1992) who derived the evolution equations of coupled longitudinal and transverse waves for materials obeying potentials like (11). Boulanger and Hayes (1993) have shown that for the isotropic incompressible Mooney–Rivlin material, the phenomenon of acoustical internal conical refraction is exhibited.

Coupled Fields

Nonlinear deformation waves in solids and particularly in crystals may be strongly affected by coupling effects with other fields. Electromechanical and magnetomechanical effects add new dimension to nonlinear wave motion and usually the direct approach to separate single waves (see the previous section) is not applicable. Nonlinear waves in coupled electromechanical systems are analyzed by Maugin et al. (1986, 1992). This analysis concerns piezoelectric, ionic, and ferroelectric crystals. It is shown that the method of slowly varying coupled amplitudes and the straightforward small parameter expansion techniques give good results. Particularly, anisochronism and intermodulation are described in resonators made of piezoelectric crystals. The first deals with the alteration of the vibration frequency of the fundamental wave due to nonlinearity and the second the production of frequency components as the sum and the difference of the basic two frequencies used as an excitation. In ferroelectric crystals, an important effect is the effect of polarization gradients affected by the nonlinear electric properties. We use this case to demonstrate that the separation of waves due to strong coupling is impossible. Maugin et al. (1986) gave the following final governing system

$$\frac{\partial^2 v}{\partial t^2} - v_T^2 \frac{\partial^2 v}{\partial X^2} = -\eta \frac{\partial}{\partial x}(\sin\phi), \qquad (14)$$

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \sin\phi = \eta \frac{\partial v}{\partial x} \cos\phi + F\cos\phi/2, \qquad (15)$$

where v is the elastic displacement orthogonal to the direction of propagation, v_T is the velocity of transverse elastic waves, ϕ is twice the angle of rotation of electric dipoles in the plane spanned by v and X, $\eta =$ const, and F is the magnitude of the applied elastic field (all variables dimensionless). Soliton-type solutions are found for this system. One should stress that a prospective approach to model coupled fields is to use the pseudomomentum description, in particular for systems that may exhibit solitons (Maugin, 1992b) like electroacoustic or magnetoelastic systems.

Certainly, there are many other interesting cases of coupled fields. In the solidification of dilute binary mixtures, nonlinear couplings among heat, mass, and momentum transfer cause strong nonlinear terms like $\nabla(u\nabla u)$ and $\nabla(\nabla^2 u\nabla u)$ in the governing equations (Riley, 1990). Nonlinearity may be affected by other fields, showing, for example, a strong dependence on the temperature (Breazeale, 1993). Even as high as the fourth-order expansion of the potential might be not sufficient. In ferroelectricity and shape-memory materials the potential is of the sixth order (Maugin et al., 1992) that leads to a response law with double hysteresis. The symmetry breaking is then to be taken into account.

Microstructure

The assumption on homogeneity of a continuum is certainly a simplified assumption justified in many cases, particularly in low-frequency processes. A usual solid has a microstructure that may affect wave motion strongly, especially in a high-frequency region. There are two possibilities to account for the microstructure (resp. inhomogeneities):

1. to model all the structural inhomogeneities in terms of observable variables;
2. to distinguish between two classes of variables: observable, possessing inertia and internal, dealing with relaxation and spatial localization (Maugin, 1990).

In the first case, the governing system of equation is presumably of the hyperbolic type discussed in the previous section. In the second case, however, a mixed system yields. The observable variables, as usual, are described by the hyperbolic-type equations, whereas the internal variables by the parabolic (kinetic) equations.

We start here from processes of the first class. Macroscopic approach used to model structural inhomogeneities is based on conventional conservation laws. The constitutive laws still are proposed for averaged quantities resulting in a generalized Hooke's law. The most striking phenomenon is the enormously high nonlinearity. Using the nonlinear parameter k (Engelbrecht, 1983),

$$k = 3(1 + m_0), \quad m_0 = \tfrac{1}{2}(\nu_1 + \upsilon_2 + u_3)(\lambda + 2\mu)^{-1}. \tag{16}$$

it is established (Nazarov et al., 1988) that besides the conventional values of k around 10 for metals, the values of k for soils and rocks may be several orders higher up to 10^4! That leads to certain limits in analysis that cannot be based on the assumption of weak nonlinearity. Experimentally it is shown (Beresnev and Nikolaev, 1988) that nonlinear seismic effects with high nonlinearity involved are measurable and besides energy transfer to higher harmonics, the demodulation may also occur. For a porous medium with certain regular porosity, a generalized Hooke's law may be derived by taking into account the deformation of pores (Nazarov et al., 1988). If pores are filled with a liquid, then using several assumptions, an evolution equation is derived (Sionoid, 1994) with a special type of nonlinearity ($u|u|$).

Another important phenomenon in a structured medium is dispersion. The macroscopic approach leads to the fourth-order terms in the equation of motion (Kunin, 1982) that finally gives the Korteweg–de Vries equation for longitudinal waves. However, the approach, based on lattice dynamics, has been very fruitful indeed, especially for wave motion in martensitic alloys and ferroelastic materials (Maugin and Cadet, 1991; Maugin, 1994). It is easy to show that starting from a discrete lattice and going to a continuum limit, the terms of the fourth, sixth, eight, . . . etc. order, are proved to be of certain importance in the equation of motion. Again, the solitons may exist in such systems.

Phenomenological modeling may also be used for describing the effect of the microstructure. For waves in layers with internal structure, a phenomenological body force (4) explains the possible amplification of waves due to energy release caused by waves themselves. In this case the governing equation is the Korteweg–de Vries equation with a special r.h.s. of the cubic character (Engelbrecht and Khamidullin, 1988; Engelbrecht and Peipman, 1992). An asymmetric solitary wave is possible in such a waveguide.

Finally, the stochastic approach is certainly useful for modeling structured medium. Ostoja–Starzewski (1991) has described random heterogeneous media using the Markov model and applied this model for waves in 1-D piecewise constant microstructure with randomness in constitutive moduli and grain lengths. The effect of spatial randomness of the material on the critical amplitude that ensures shock formation, is established.

In the second case, the notion of internal variables needs explanation. Based on a thermodynamical approach, internal variables possess the following properties (Lebon, Jou, and Casas-Vasquez, 1992):

1. they correspond to constraints inherent to the system that cannot be controlled by an observer external to the system;
2. they can be measured in both equilibrium and nonequilibrium situations; and
3. they are governed by first-order time differential (kinetic) equations, i.e., they are not inertial.

Maugin (1990) has shown how the formalism of internal variables can be applied to: nematic crystals, localization of damage, coupling to elasticity or plasticity, and localization of plastic strains in plasticity with hardening. Moreover, he has shown how the coupled hyperbolic-diffusion systems may exhibit either pure solitonic structures characteristic to hyperbolic systems or dissipative structures characteristic to evolution-diffusion systems. Under certain conditions, the evolution equations for observable variables can be derived even in this complicated case (Maugin and Engelbrecht, 1994).

COMPLICATED GEOMETRY

In practical applications, geometrical constraints applied to the continuum have a direct impact on wave motion. In the previous two sections, the basic theory was presented applicable for continua with dimensionality $K = 1, 2, 3$ without any constraint about boundaries. In this section, the attention is paid to processes in continua where the boundaries or constraints play a decisive role.

The most known and well-defined problem here is the wave motion in a half-space where in addition to longitudinal and transverse waves, the surface waves occur. The problems under discussion are schematically shown in Fig. 2. The surface waves are known for a long time but their nonlinear analysis has a much shorter history (see David and Parker, 1988). Recent results show that despite the complicated character of motion (the path of any particle in motion is an ellipse), there is a possibility to derive a nonlinear evolution equation for surface waves [Fig. 2(a)]. According to David and Parker (1988) and

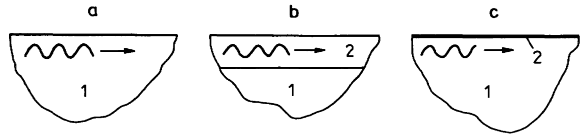

FIGURE 2 Typical problems of surface waves: (a) simple half-space 1; (b) half-space 1 with a layer 2; (c) half-space 1 with a thin film 2.

Parker (1994) one has

$$iJ\,\frac{\partial C(k, X)}{\partial X} + \int_{-\infty}^{\infty} \Lambda(k-\kappa, \kappa)(k-\kappa) \, C(k-\kappa, X)C(\kappa, X)\, d\kappa = 0, \tag{17}$$

where $C(k, X)$ is the Fourier transform of the surface elevation and k is the wavenumber. The kernel $\Lambda(k - \kappa, \kappa)$ determines the contribution to the (complex) rate of change in $C(k, X)$ from each pair of wave numbers with sum k and involves also material nonlinearities. The coefficient J measures the strain energy associated with the displacement field. Similar results have also been obtained by Zabolotskaya (1992). The distortion of wave profiles due to nonlinearity is essential in seismology as well as in acoustical devices based on surface waves.

Nonlinear waves in a layered half-space with one [Fig. 2(b)] or many layers are certainly more complicated to treat, but the general theory is known (Parker, 1994).

The case depicted in Fig. 2(c) is composed of a nonlinear isotropic elastic half-space and a superimposed linear elastic thin film (Maugin and Hadouaj, 1991; Maugin, 1994). The thin film as an interface of zero thickness introduces dispersion into the system that together with nonlinearity of the half-space governs the surface waves. A typical governing system of equations in terms of the complex amplitude a of the SH mode and the real x-gradients $n_1 = v_x$, $n_2 = w_x$ of the longitudinal and transverse vertical components of the Rayleigh mode is

$$ia_t + a_{xx} \pm \lambda\, |a|^2\, a + 2a(n_1 + n_2) = 0, \tag{18}$$

$$(n_1)_{tt} - c_L^2(n_1)_{xx} = -\mu_L(|a|^2)_{xx}, \tag{19}$$

$$(n_2)_{tt} - c_T^2(n_2)_{xx} = -\mu_T(|a|^2)_{xx}. \tag{20}$$

Here λ is the coefficient of self-interaction; μ_L, μ_T are the mutual interaction coefficients; c_L and c_T are the characteristic velocities. This system is

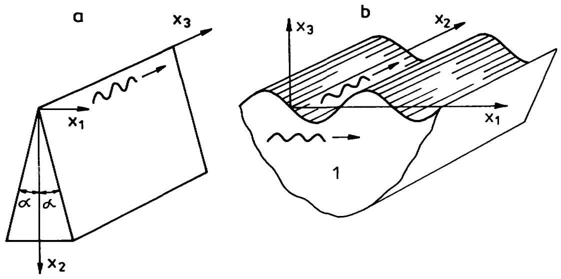

FIGURE 3 Guided waves: (a) a wedge; (b) a half-space with corrugated surface.

soliton-bearing, with a possible breather at the collision. For more details, the reader is referred to Maugin (1994). The results are applicable in nondestructive evaluation techniques and signal processing by using acoustic devices.

One special case of surface-guided waves is the propagation along a wedge [Fig. 3(a)]. The linear waves are nondispersive but nonlinear waves obey interactions between all the wave numbers (Parker, 1994). Despite the complicated geometry, even here one can derive the evolution equation. Antisymmetric modes, for example, are governed (Parker, 1994) by

$$iJ\frac{\partial C}{\partial Y}(k, Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Omega(k-\kappa-\nu, \kappa, \nu)(k-\kappa-\nu)\kappa\nu \times C(k-\kappa-\nu)C(\kappa)C(\nu)\, d\kappa d\nu = 0 \quad (21)$$

where $Y = \varepsilon^2 X_3$; $C(k, Y)$ as before is the Fourier transform of the wedge elevation; κ, ν are the wavenumbers and $\Omega(\sigma, \kappa, \nu)$ is the kernel. Parametric mixing and other effects are described by such an approach. This approach is also shown to be useful for analysis of waves in a half-space with the corrugated surface [Fig. 3(b)]. This is an intriguing problem where much is to be done, but a 2-D nonlinear Schrödinger equation is shown to serve as an evolution equation (see Parker, 1994)

$$i\frac{\partial a}{\partial y} + P\frac{\partial^2 a}{\partial z^2} + Q\frac{\partial^2 a}{\partial \tau^2} + R|a^2|a = 0, \quad (22)$$

where $a(x_1, x_2, t)$ is the amplitude, y is the evolution coordinate, z is the transverse distance, and τ is the retarded time while, P, Q, R are related to the geometry.

Two problems with special geometry should be mentioned in addition to surface waves. These are wave propagation in straight and helix elasticas. First, the transverse motion $w(x, t)$ in a straight half-finite ($x > 0$) elastica (string) subject to an excitation of the end is described by

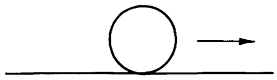

FIGURE 4 A loop soliton.

$$\frac{\partial^2 w}{\partial t \partial x} + \operatorname{sign}\left(\frac{dx}{ds}\right)\frac{\partial^2 w/\partial x^2}{[1 + (\partial w/\partial x)^2]^{3/2}} = 0 \quad (23)$$

where s is the arc length along the solution curve (Wadati, Konno, and Ichikawa, 1979). This equation has a solution in the form of the loop soliton propagating with an amplitude-dependent velocity. Such a soliton (shown in Fig. 4) has a remarkable property: the smaller the amplitude, the larger the velocity, i.e. the smaller loop overtakes the larger one (Konno and Jeffrey, 1994; Jeffrey and Engelbrecht, 1994).

Second, the excitation propagating in an elastic helix (Fig. 5) may lead to interesting phenomena. Complex static deformation patterns are known (Davies and Moon, 1993) involving quasi-periodicity and spatially stochastic deformation. The dynamics of such a structure needs exploration.

PROPAGATING INSTABILITIES

We return now to the definition of a wave as a state moving to another state. Defining instability as a state, it is easily concluded that a propagating instability is a wave. The question certainly is about the character of governing equations.

A good example to start is the domino effect (Stronge, 1987). Consider a regularly spaced sequence of slender rectangular blocks (dominoes) standing on end with a small space λ between the elements. The dominoes stand vertically in the gravitational field. Toppling one element initiates a sequence of collisions (Fig. 6). If the initial en-

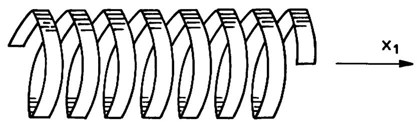

FIGURE 5 A helix as a wave-bearing structure.

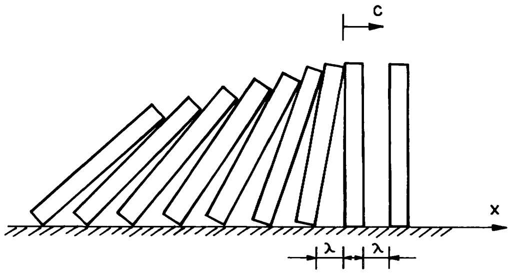

FIGURE 6 Toppling dominoes; c, wave velocity.

ergy is large enough, then a wave of collisions propagates over the entire array. This wave has a finite speed of propagation c determined from the condition of equality of the rates of diffusion and reaction (Stronge, 1989). The governing equations are of the reaction-diffusion type that means quite another physical phenomenon (c.f. Maugin, 1990). Nevertheless, the outcome is a wave, propagating in a mechanical system with a finite speed. Reaction-diffusion type systems are more known in chemistry and biology (Perelson et al., 1988), but this example shows a way to bridge such mechanical phenomena to other fields.

Noting that the toppling of one domino means generating an instability, we turn our attention to structures where instabilities may be buckles, bulges, discontinuities of deformation, or even of displacements, etc. An excellent review on propagating instabilities is presented by Kyriakides (1993). He treats the initiation and propagation of: bulges in inflated elastic tubes; buckles in long tubes and pipes under external pressure; buckles in long, confined cylindrical shells; and buckles in long shallow panels. The common characteristic of structures exhibiting such instabilities is the local unstable response characteristic shown in Fig. 1(c). In some sense the character of propagation is similar to that of toppling the dominoes.'' Once the geometric integrity of such structures is compromised, the instability has the potential of spreading over the whole structure'' (Kyriakides, 1993: 68). Such a situation is shown in Fig. 7. A pipeline is installed on the sea floor using a special vessel and if the technological characteristics of the process (movement of a vessel, loss of tension, etc.) are wrong, the pipe can buckle due to the combined effect of bending and pressure. Once initiated, the buckle can also propagate provided the external pressure is high enough. The governing equations for the beam-flexural mode A and the ring-flexural mode B (scaled over radius a) of the pipeline with a round cross section are (Sugimoto, 1989):

$$\rho_0 \frac{\partial^2 A}{\partial t^2} + \frac{Ea^2}{2}\frac{\partial^2}{\partial z^2}\left[(1 + \tfrac{3}{4}B + \tfrac{1}{16}B^2)\frac{\partial^2 A}{\partial z^2}\right] = T\frac{\partial^2 A}{\partial z^2}, \tag{24}$$

$$\rho_0 \frac{\partial^2 B}{\partial t^2} + \frac{3Eh^2}{5a^4}\left(1 - \frac{p}{p_c}\right)B + \frac{Ea^2}{20}\frac{\partial^4 B}{\partial z^4} = T_B\frac{\partial^2 B}{\partial z^2} - \frac{3E}{5}\left(1 + \frac{5B}{6}\right)\left(\frac{\partial^2 A^2}{\partial z^2}\right), \tag{25}$$

where z is the axial coordinate; ρ_0 and E are the density and Young's modulus, respectively; p and p_c denote the hydrostatic side pressure and the buckling pressure; h is the effective thickness of the pipe; and finally, T_A and T_B denote the effective tension for both modes, respectively. The propagation velocity of a buckle is determined from the expression

$$\left(\frac{v}{v_0}\right)^2 = \left(\frac{h}{2a}\right)^2\left[\tfrac{1}{2}(N + p/p_c) - \frac{2^{1/2}}{9S}(1 - p/p_c)^{1/2}\right] \tag{26}$$

where S and N denote the dimensionless maximum deflection and the scaled axial force, respectively, and $v_0^2 = E/\rho_0$. Note that the governing equations (24) and (25) are of the dispersive character and nonlinearly coupled, and describe actually the ovalization of the cross section. A simple experiment showing initiation and propagation of a buckle can easily be performed with a flexible steel tape measure.

In general terms, the propagation of instabilities can be explained in terms of the global load-deformation curve (Fig. 8). Following Kyriakides (1993), the explanation follows. Initially, the structure deforms uniformly (region $d_0 - d_1$),

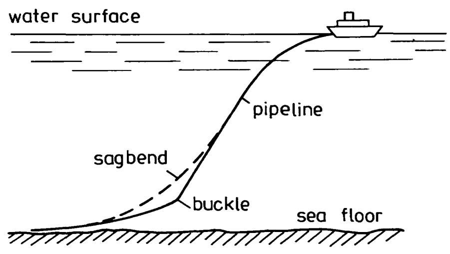

FIGURE 7 Initiating of a buckle during an installation of an offshore pipeline.

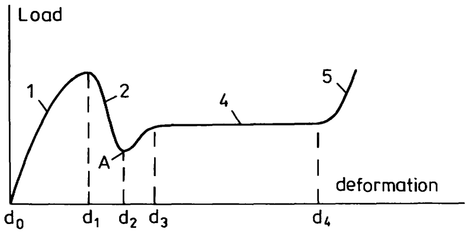

FIGURE 8 Global load-deformation response: 1, uniform deformation; 2, localization; A, arrest; 4, propagation, 5, uniform deformation.

that may mean also bifurcation at d_1. Then, in region $d_1 - d_2$, the instability is localized and at d_2 (a local minimum), it is arrested. Region $d_3 - d_4$ corresponds to the propagation. In this case, the load is lower than needed for the bifurcation at d_1.

With a certain flexibility we may also classify a phase boundary as an instability. Clifton (1993) reports on failure waves in glasses that are the propagating phase boundaries. There is experimental evidence that above a certain impact stress, there is a propagating boundary between low-spall-strength material close to the impact face and high-spall-strength material in the rest of the specimen. This phase change according to Clifton (1993) could be a transformation either to a crystalline phase or to an amorphous phase with a higher coordination number.

And last but not least, the propagation of the discontinuity of the displacement, i.e., crack. The propagation of a crack again corresponds to the definition of waves. Here, however, the plastic deformation at the crack tip should be taken into account. The finite velocity of the crack tip certainly depends on the stress state. For a straight expanding crack ($x < l(t)$), the velocity $\dot{l} < c_2$ where c_2 is the shear wave velocity, is determined by the following expression (Slepyan, 1990)

$$\dot{l}(t) = \left(c_2 - \frac{\pi K^2}{8\sigma^2 t}\right) H(t - t_*), \qquad (27)$$

$$t_* = \pi K^2 / 8\sigma^2 c_2, \qquad (28)$$

where K is the critical coefficient of intensity and σ is the uniform stress at the crack surface. Generally speaking, this velocity characterizes the propagation of the viscoplastic zone around the crack tip. Maugin (1994) calls this structure a "viscoplastic soliton." The resemblance to the soliton notion in a conservative system is striking, including the amplitude dependence on the propagation velocity.

CLOSING REMARKS

The importance of being nonlinear is widely accepted in physical sciences and wave motion is not an exception. In this review, an attempt is made to extract a leading thread in a nonlinear wave theory of solids. Summing up briefly, the philosophy of analyzing wave processes is the following.

There are a variety of mathematical models describing nonlinear wave motion in solids. Conventional hyperbolic models based on conservation laws serve as a backbone for explaining wave phenomena, where the finite velocity is the most important notion. Related to the eigenvalues of a governing hyperbolic system the finite velocities are easily understood theoretically, i.e. in the mathematical sense. In applications, the situation is far more complicated because of various changing rates, properties of continua made up by various components, constraints, etc., and the straightforward modeling may lead not only to mathematical complexities but also to problems in determining the physical parameters. This turns us to nonhyperbolic models like dispersive or reaction-diffusion.

However, this way or another, the mathematical models need simplification. "One of the principal objects of theoretical research in any department of knowledge is to find the point of view from which the subject appears in its greatest simplicity," said J. W. Gibbs (Wheeler, 1951: 155). In this sense, the derivation of evolution equations, the separation of solitonic and dissipative structures, and any other simplification approach are all of great importance. In other words, this is again extracting a backbone of the process, resulting in simpler models preserving all the important factors. In nonlinear dynamics, a good example is the Lorenz attractor derived to model unsteady convection in the atmosphere. Gleick (1987: 15) has said: "Lorenz had boiled weather down to the barest skeleton." Still, all the richness of chaotic motion was preserved. Such a simplification does not always work, as demonstrated above, but the main idea remains.

In nonlinear wave motion, there are many hot problems. Some of them were recently listed by

Engelbrecht (1993). Some of the new directions in this field are:

1. problems of continuum mechanics: the notion of pseudomomentum for describing material inhomogeneities, external stimuli; and dissipation; interaction of deformations, and stress with other fields (electric, magnetic, etc.), separation of observable and internal variables (see Maugin, 1994);
2. engineering analysis with attention to material properties and geometrical constraints: propagating instabilities (Kyriakides, 1993); waves in wave guides (Parker, 1994);
3. influence of microstructure on wave motion: material inhomogeneities (Maugin, 1993); smart materials (adaptive to the environment); damage (Barenblatt, 1993);
4. conservative systems: solitonics;
5. spatiotemporal chaos: predictability of wave motion.

This list is by no means exhaustive.

Nonlinearity is a natural property of the world around us including wave motion. Consequently, the richness of motion can be explored only by using nonlinear mathematical models. Based on the long experience in mechanics and flavored by contemporary needs of high technology, the future of theory and applications of nonlinear waves is promising.

APPENDIX: SELECTED ABSTRACTS FROM ARTICLES ON WAVES, SOLIDS, AND NONLINEARITIES

Boulanger, Ph., and Hayes, M., 1992. "Finite-Amplitude Waves in Deformed Mooney–Rivlin Materials," *Quarterly Journal of Mechanics and Applied Mathematics,* Vol. 45, pp. 575–593.
In a previous paper (1969; *Journal of the Institute of Mathematical Applications,* Vol. 5, pp. 140–161). P. Currie and Hayes showed that two linearly polarized finite-amplitude shear waves, polarized in directions orthogonal to each other and to the direction of propagation ν, may propagate along any direction in a Mooney–Rivlin material maintained in a state of arbitrary static finite homogeneous deformation. Here, we recover this result and obtain explicit expressions for the speeds of the two waves in terms of the angles that ν makes with special directions, called acoustic axes. These are the only directions such that the two wave speeds are equal. They are determined only by the basic static deformation of the material. There are two such directions if this deformation is triaxial, and one if it is biaxial. Then, we show that, although the theory is nonlinear, the superposition of the two waves propagating along any direction is also a solution. In particular, for propagation along an acoustic axis, elliptically and circularly polarized finite-amplitude waves are possible. Finally, the energy flux and energy density of the waves are considered.

Chen, C. F., 1990, "Long-Wave Morphologies in Directional Solidification," *Mechanics USA 1990: 11th U.S. National Congress of Applied Mechanics,* pp. S85–S88.
Long-wave instabilities in a directionally solidified binary mixture may occur in several limits. Sivashinsky identified a small-segregation-coefficient limit and obtained a weakly nonlinear evolution equation governing subcritical two-dimensional bifurcation. Brattkus and Davis identified a near-absolute-stability limit and obtained a strongly nonlinear evolution equation governing supercritical two-dimensional bifurcation. In this presentation these previous analyses are set into a logical framework, and a third distinguished (small-segregation-coefficient, large-surface-energy) limit identified. The corresponding strongly nonlinear, evolution equation links both of the previous and describes the change from sub- to supercritical bifurcations.

Chu, B.-T., 1964, "Finite Amplitude Waves in Incompressible Perfectly Elastic Materials," *Journal of the Mechanics and Physics of Solids,* Vol. 12, pp. 45–57.

Clifton, R. J., 1993, "Analysis of Failure Waves in Glasses," *Applied Mechanics Reviews,* Vol. 46, pp. 540–546.
Recent plate impact experiments have been interpreted as indicating the existence of "failure waves" during the compression of glass by impact at sufficiently high velocities. In experiments on soda-lime glass, Brar et al. (1991) reported the propagation of a wave across which the shearing strength dropped sharply from 2 GPa to 1 GPa and the spall strength dropped from 3 GPa to zero. Such a drop in spall strength has also been reported by Raiser et al. (1993) in an aluminosilicate glass. Kanel et al. (1993) interpreted a small jump in the rear surface particle velocity in experiments on K19 glass as the reflection of a recompression wave from a wave

front propagating at approximately the speed reported for "failure waves." In this article such "failure waves" are interpreted within the context of nonlinear wave theory. In this theory, the failure wave corresponds to a propagating phase boundary, called a transformation shock. The theory is analogous to the theory of liquefaction shocks in fluids.

Davies, M. A., and Moon, F. C., 1993, "3D Spatial Chaos in the Elastica and the Spinning Top: Kirchhoff Analogy," *Chaos,* Vol. 3, pp. 93–99.
The existence of spatially chaotic deformations in an elastica and the analogous motions of a free spinning rigid body, an extension of the problem originally examined by Kirchhoff (1959), are investigated. It is shown that a spatially periodic variation in cross-sectional area of the elastica results in spatially complex deformation patterns. The governing equations for the elastica were numerically integrated and Poincaré maps were created for a number of different initial conditions. In addition, three dimensional computer images of the twisted elastica were generated to illustrate periodic, quasiperiodic, and stochastic deformation patterns in space. These pictures clearly show the existence of spatially chaotic deformations with stunning complexity. This finding is relevant to a wide variety of fields in which coiled structures are important, from the modeling of DNA chains to video and audio tape dynamics, to the design of deployable space structures.

Engelbrecht, J., 1993, "Qualitative Aspects of Nonlinear Wave Motion: Complexity and Simplicity," *Applied Mechanical Review,* Vol. 46, pp. 509–518.
The nonlinear wave processes possess many qualitative properties that cannot be described by linear theories. In this presentation, an attempt is made to systematize the main aspects of this fascinating area. The sources of nonlinearities are analyzed in order to understand why and how the nonlinear mathematical models are formulated. The technique of evolution equations is discussed then as a main mathematical tool to separate multiwave processes into single waves. The evolution equations give concise, but in many cases, sufficient description of wave processes in solids permitting analysis of spectral changes, phase changes and velocities, coupling of waves, and interaction of nonlinearities with other physical effects of the same order. Several new problems are listed. Knowing the reasons, the seemingly complex problems can be effectively analyzed.

Engelbrecht, J. K., and Chivers, R. C., 1989, "Evolution Equations and Ultrasonic Wave Propagation in Biological Tissues," *Physics in Medicine and Biology,* Vol. 34, pp. 1571–1592.
The complexity of ultrasonic wave propagation in tissue arises from a combination of factors. First, there is the biochemical sophistication of the media concerned. Second, there is the variety of physical phenomena involved: the diffractive nature of the ultrasonic field, the presence of absorption, the presence of large-scale inhomogeneities and small-scale scatterers, and the possibility of finite amplitude propagation effects. The authors were concerned with finding a unified approach that permits each of the effects to be taken into account in relation to the others. This approach is based on the application of two-dimensional evolution equations modeling ultrasonic propagation in noncavitating soft tissues. The model incorporates all the propagation phenomenon known from experimental studies, indicating a need for knowledge of nine material parameters for a complete description. It thus provides a basis for numerical investigation of the relative significance of the parameters under different conditions.

Engelbrecht, J., and Peipman, T., 1992, "Nonlinear Waves in a Layer with Energy Influx," *Wave Motion,* Vol. 16, pp. 173–181.
A nonlinear evolution equation is derived to study the propagation of deformation waves in an elastic layer in which energy is not conserved due to possible energy release from the prestress field within the layer. The approach is phenomenological and the influence of nonlinearity, geometrical dispersion, and possible energy influx is all accounted for simultaneously. The corresponding KdV-type evolution equation with a r.h.s. is derived. An example is solved numerically for transient waves subject to pulse-type (soliton-type) and harmonic inputs. As a result it is demonstrated that stable solitary waves may form depending on the properties of the driving force.

Haddow, J. B., 1993, "Nonlinear Hyperbolic Waves in Hyperelastic Solids," *Applied Mechanical Review,* Vol. 46, pp. 527–539.
This article considers hyperbolic, one spatial dimension nonlinear wave propagation in a hyperelastic solid, and a discussion of the basic theory

is presented. Constitutive relations for compressible rubber-like materials, whose internal energies can be expressed as the sum of a function of specific volume only and a function of temperature only, are discussed. These relations are assumed for the analysis of a class of plane wave problems and similarity solutions are obtained. Thermal effects, including the effect of the jump in entropy across a shock for a problem of uncoupled longitudinal wave propagation, are taken into account. However heat conduction is neglected. Solutions for a piezotropic model, which is a model for which mechanical and thermal effects are uncoupled, are obtained for comparison purposes. An axisymmetric problem is also discussed.

Haddow, J. B., Wegner, J. L., and Jiang, L., 1992, "The Dynamic Response of a Stretched Circular Hyperelastic Membrane Subjected to Normal Impact," *Wave Motion,* Vol. 16, pp. 137–150.
Finite amplitude wave propagation in a circular isotropic hyperelastic membrane, initially subjected to an equibiaxial stretch followed by a suddenly applied pressure or normal impact of a projectile, is considered. Axially symmetric deformation is assumed, and the governing equations, in Lagrangian form, are expressed in terms of the principal components of Biot stress and the principal stretches. These equations are a system of five first-order quasilinear partial differential equations in conservation form with a source term. Solutions, obtained by the method of characteristics and a finite-difference scheme, are presented graphically for particular cases of the Mooney–Rivlin strain energy function.

Kestin, Joseph, 1992, "Local-Equilibrium Formalism Applied to Mechanics of Solids," *International Journal of Solids and Structures,* Vol. 29, pp. 1827–1836.
The lecture starts with an expression of good wishes to George Herrmann on the occasion of his seventieth birthday and continues with a lament that the majority of research workers in the field of solid mechanics have failed to appreciate the power and relevance of "conventional" thermodynamics which is based on the acceptance of the hypothesis of local equilibrium (principle of local state). The lecture then proceeds to motivate the essential concepts of conventional thermodynamics and emphasizes the differences between the description of nonequilibrium states in physical space and equilibrium states in the Gibbsian phase space. It is asserted that the subject acquires its simplest form by the recognition of the relevance of Bridgman's internal variables. With their aid it is possible to define the accompanying equilibrium state and the accompanying reversible process. An elimination of internal energy between the field equation of energy (First Law) and the Gibbs equation in rate form results in an explicit expression for the local rate of entropy production, theta. It is asserted that the preceding elements supplemented with appropriate rate equations result in a closed system of partial differential equations whose solution, subject to appropriate initial and boundary conditions, constitutes the process (history) under consideration.

Lebon, G., Jou, D., and Casas–Vazquez, J., 1992, "Questions and Answers About a Thermodynamic Theory of the Third Type," *Contemporary Physics,* Vol. 33, pp. 41–51.
Nonequilibrium thermodynamics has several faces: the most popular theory, referred to as thermodynamics of the first type, is the classical theory of irreversible processes developed essentially by Eckart, Meixner, Onsager, and Prigogine. Another more general but more formal approach, the so-called thermodynamic theory of the second type, was proposed by Coleman, Noll, and Truesdell and was named by its founders "rational thermodynamics." By thermodynamics of the third type is understood the so-called extended irreversible thermodynamics: the latter has fueled much interest during the last decade. In the work, the main ideas underlying this formalism are reviewed. Moreover, questions about the foundations, contents, and aims of this new theory are raised. Tentative answers are provided.

Maugin, G. A., 1990, "Internal Variables and Dissipative Structures," *Journal of Non-Equilibrium Thermodynamics,* Vol. 15, pp. 173–192.
The formalism of internal state variables is established when gradients of these variables are involved, thus allowing for a spatial localization of dissipative effects that give rise to dissipative structures. Therefore, evolution-diffusion equations, rather than usual evolution equations, are obtained. This leads to a comparison between dissipative structures and local structures exhibited by conservative systems. Three illustrative cases are briefly sketched out: nematic liquid

crystals, localization of damage coupled to elasticity or plasticity, and localization of plastic strains in plasticity with hardening. Thanks to the absorption of some terms in the extra entropy flux, the approach is valid to any order in the gradients and the accompanying heat equation has always the same form, providing in fact a possible basis for the observation of some of the dissipative structures of interest.

Maugin, G. A., 1992, "Applications of an Energy-Momentum Tensor in Nonlinear Elastodynamics: Pseudomomentum and Eshelby Stress in Solitonic Elastic Systems," *Journal of the Mechanics and Physics of Solids,* Vol. 40, pp. 1543–1558.
In continuum mechanics the pseudomomentum is the covariant material momentum whose associated flux in the balance law is Eshelby's "energy-momentum" tensor. The unbalance of pseudomomentum was previously shown to play a basic role in the formulation of configurational forces and path-independent integrals in the theory of elastic inhomogeneities and brittle fracture. It is further shown to provide a fundamental conservation law for dispersive nonlinear elastic systems that exhibit soliton solutions. This is illustrated by both classical and grade-two elasticity with applications to the Sine–Gordon, Boussinesq, Sine–Gordon–d'Alembert, and generalized Zakharov systems encountered in various bulk or surface wave-propagation problems. In these systems the nonconservation of global pseudomomentum may be used for a perturbational approach to nearly integrable systems to study the influence of dissipation and external sources (e.g. defects).

Maugin, G. A., and Cadet, S., 1991, "Existence of Solitary Waves in Martensitic Alloys," *International Journal of Engineering Science,* Vol. 29, pp. 243–258.
Starting from a discrete lattice model, the authors investigate the dynamics of nonlinear waves that are supposed to represent domain walls. They consider an atomic chain, of which each particle represents a crystal plane. Motion is allowed in the longitudinal direction as well as in the transverse one, because transverse deformations are large; this modeling can account for the change in volume. The nonlinear equations of motion yield solitary wave solutions of several types. Kink solutions represent domain walls either between austenite and martensite or between two martensite variants. They move only when an external force is applied and they obey a Rankine–Hugoniot equation. Pulse solutions correspond to a matrix of austenite or martensite containing a moving sheet of the other phase. The presence of longitudinal deformation and its coupling with transverse deformation strongly affects the stability of the various excitations.

Maugin, G. A., and Hadouaj, H., 1991, "Solitary Surface Transverse Waves on an Elastic Substrate Coated with a Thin Film," *Physical Review B (Condensed Matter)*, Vol. 44, pp. 1266–1280.
A proof is given of the existence of stable guided solitary surface acoustic waves propagating in the form of envelope solitons on a structure made of a nonlinear substrate and a superimposed linear elastic thermodynamical interface (a very thin film) of mathematically vanishing thickness. A thin gold film on top of a lithium niobate substrate is such a system. The mathematical analysis starting with the theory of material interfaces is carried out by using the Whitham–Newell technique of treatment of nonlinear, dispersive, small-amplitude, almost monochromatic waves. In the process, "wave-action" conservation equations and "dispersive" nonlinear dispersion relations are established for this type of surface waves that could also be approached by using Whitham's averaged-Lagrangian technique as modified by Hayes to account for the transverse-modal behavior. It is shown that the whole problem is reduced to studying a single nonlinear Schrodinger equation at the interface, thus providing solutions that are the mechanical analogs of optical solitons known to propagate in nonlinear optical fibers.

Ostoja–Starzewski, M., 1991, "Transient Waves in a Class of Random Heterogeneous Media," *Applied Mechanics Reviews,* Vol. 44, pp. S199–S209.
A stochastic method is developed for the analysis of transient waves propagating in one-dimensional random granular-type media. The method is suited to study transient dynamic responses of nonlinear microstructures with material randomness, of high signal-to-noise ratio, being present in constitutive moduli and grain lengths. It generalizes the classical solution techniques, based on the theory of characteristics, by taking advantage of the Markov property of the forward propagating disturbances. Pulses propagating in bilinear

elastic, nonlinear elastic, and linear-hysteretic media are studied. Additionally, a short review is given of an investigation of acceleration wave fronts making a transition into shocks in random nonlinear elastic/dissipative continua, where the Markov property can again be exploited. For the entire collection see MR 92f:73002.

Peipman, T., Valdek, U., and Engelbrecht, J., 1992, "Nonlinear Two-Dimensional Longitudinal and Shear Waves in Solids," *Acustica,* Vol. 76, pp. 84–94.
The two-dimensional (2D) problem of propagating nonlinear longitudinal and shear waves in solids by a bounded input is considered. Using appropriate scaling of dependent and independent variables, the 2D evolution equations are derived. The solutions to these nonlinear model equations are comparatively analyzed and the differences between the distortion of longitudinal and shear waves are explained. The novelty of the 2D evolution equation for shear waves is an integral-type term accounting for the nonlinear coupling effects. Both asymptotic and numerical solutions to the evolution equations under consideration are presented with an explanation of the simplified 1D models.

Pence, T. J., 1992, "On the Mechanical Dissipation of Solutions to the Riemann Problem for Impact Involving a Two-Phase Elastic Material," *Archive for Rational Mechanics and Analysis,* Vol. 117, pp. 1–52.
Continuum mechanical descriptions of displacive phase transitions in solids lead naturally to mathematical treatments involving displacement gradients that suffer jump discontinuities across phase boundaries. In a dynamical setting this gives rise to both conventional shock waves and traveling phase boundaries.

Zabolotskaya, E. A., 1992, "Nonlinear Propagation of Plane and Circular Rayleigh Waves in Isotropic Solids," *Journal of the Acoustical Society of America,* Vol. 91, pp. 2569–2575.
Nonlinear Rayleigh wave propagation in an isotropic solid is investigated theoretically. Hamiltonian formalism is used to derive a set of coupled equations for the harmonic amplitudes. Both plane and circular waves are considered. Numerical results are presented for an initially monochromatic wave that propagates in steel. It is shown that the horizontal component of the particle velocity wave forms a shock profile, while the vertical component forms a pulse. An evolution equation for the waveform is derived.

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