S. Mirza
Y. Alizadeh

Department of Mechanical
Engineering
University of Ottawa
Ottawa, ON K1N 6N5, Canada

Free Vibration of Partially Supported Cylindrical Shells

The effects of detached base length on the natural frequencies and modal shapes of cylindrical shell structures were investigated in this work. Some of the important applications for this type of problem can be found in the cracked fan and rotor blades that can be idealized as partially supported shells with varying unsupported lengths. A finite element model based on small deflection linear theory was developed to obtain numerical solutions for this class of problems. The numerical results were generated for shallow shells and some of the degenerate cases are compared with other results available in the literature. The computations presented here involve a wide range of variables: material properties, aspect ratios, support conditions, and radius to base ratio. © 1995 John Wiley & Sons, Inc.

INTRODUCTION

Many turbine and fan blade failures were attributed to the lack of information on the interaction between fracture, vibration, and modal displacements of blades due to dynamic forces. This necessitates further work in this area investigating the interaction of defects in blades and their dynamic behavior. One such defect, a through crack along the support, can be identified as a partial support. A literature survey shows that very little work has been done so far in this area of partially supported shell panels. A literature survey on vibration of turbine engine blades using shell analysis has been given by Leissa (1980). Other recent works on vibration of fan blades have been reported by Vinson and Chou (1975) and Khader and Abu-Farsakh (1990).

The finite element method is an ideally suited method for the dynamic analysis structures have irregular geometries or complex boundary conditions.

Several different finite element techniques have been used by various researchers to study shells with arbitrary geometry. The selection of a particular approach depends on the shell theory used. Olson and Lindberg (1971) reported on a doubly curved triangular element for shallow shells. The work presented by Ahmad et al. (1970) was more general in nature as they had included thick as well as deep shells. The assumed displacement field and element displacement method is a widely used procedure in finite element analysis.

In the present study a model was prepared for a cantilever cylindrical shell of varying geometrical shape defined by r/a and b/a . Furthermore, a crack of length c has been introduced along the support of the shell making it only partially supported. By idealizing the cracked edge of fan and rotor blades as a partially supported model, we are able to establish important information for design and analysis.

A considerable amount of data has been gen-

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erated for this important applied problem by varying the above-noted parameters. The results thus generated add much needed information to the data base on the topic of dynamics of isotropic shells with cracks.

SHELL ELEMENT

The degenerate shell element was employed to formulate the shell problem. This type of shell element was discussed by Ahmad et al. (1970) and Huang and Hinton (1986). In the analysis of shell structure by the degenerate curved shell elements, we need to define four coordinate systems as shown in Figure 1.

Global Coordinate System (x, y, z)

A global Cartesian coordinate system is used to define the geometry of the structure (Fig. 1). Nodal coordinates, displacements, global stiffness matrix, and applied forces are all defined in this coordinate system.

Nodal Coordinate System (V_{1i}, V_{2i}, V_{3i})

The nodal Cartesian coordinate system is attached at each nodal point of the shell element and its origin is located at the shell midsurface. This local coordinate is used to define the direction of the normal at node i . The nodal coordinate system is used not only to define the direction of the normal at each node, but also the two

rotations α_1, α_2 about the directions $\vec{V}_{2i}, \vec{V}_{1i}$, respectively.

Natural Coordinate System (ξ, η, ζ)

The natural coordinate system, located at the shell midsurface, is used to define shape function N_i . In this system the coordinates ξ, η , and ζ vary between -1 and $+1$ on the respective faces of the element.

Local Coordinate System (x', y', z')

This Cartesian coordinate system is defined at the sampling points to define local stresses and strains within the shell element. The direction z' is taken to be perpendicular to the surfaces $\xi = \text{constant}$. However, in general, the orthogonal axes $x'y'z'$ do not coincide with the natural coordinates ξ, η, ζ , although x', y' are in the ξ, η plane.

Geometric Definition of Shell Element

The geometry of the element is described by a set of pairs of points $i_{\text{top}}, i_{\text{bottom}}$, each with given global coordinates. The elements of the midvector are given by $i_{\text{mid}} = 1/2(i_{\text{top}} + i_{\text{bottom}})$. The global coordinates of any point of the shell, after some simplification, can be written as

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^{nn} N_i(\xi, \eta) \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{\text{mid}} + \sum_{i=1}^{nn} N_i(\xi, \eta) \frac{\zeta}{2} \vec{V}_{3i} \quad (1)$$

Here, the shape function $N_i(\xi, \eta)$ takes a value of unity at the node i and zero at all other nodes and nn = the number of nodes at the midsurface of each element. In the case of relatively thin shells, the vector $\vec{V}_{3i}$ may be replaced by a unit vector $\hat{v}_{3i}$ in the direction perpendicular to the midsurface. In this case the last term in eq. (1) will take the form

$$\sum_{i=1}^{nn} N_i(\xi, \eta) \frac{\zeta}{2} t_i \hat{v}_{3i} \quad (2)$$

where t_i is the shell thickness at the node i and $\hat{v}_{1i}, \hat{v}_{2i}, \hat{v}_{3i}$ are the unit vectors along

$$\vec{V}_{1i}, \vec{V}_{2i}, \vec{V}_{3i}$$

FIGURE 1 Coordinate system for shell element.

Displacement Field for Shell Element

Using the shape functions, the displacement field u, v, w , or simply $\{u\}$ is written in the matrix form,

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{i=1}^{nn} \begin{bmatrix} N_i & 0 & 0 & N_i \xi \frac{t_i}{2} v_{1i}^x & -N_i \xi \frac{t_i}{2} v_{2i}^x \\ 0 & N_i & 0 & N_i \xi \frac{t_i}{2} v_{1i}^y & -N_i \xi \frac{t_i}{2} v_{2i}^y \\ 0 & 0 & N_i & N_i \xi \frac{t_i}{2} v_{1i}^z & -N_i \xi \frac{t_i}{2} v_{2i}^z \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \\ a_{1i} \\ a_{2i} \end{Bmatrix} \quad (3)$$

where v_{1i}^x, v_{2i}^x , are x components of unit vectors v_{1i} and v_{2i} , respectively. In a more general form we write,

$$\{u\} = [N] \{\Delta\} \quad (4)$$

where Δ is the generalized nodal displacement vector.

Once the displacement field for the element is known, the local coordinate system is the most convenient system for expressing the stress and strain components in the shell structures. Thus, the strain matrix $\{\epsilon'\}$ is given by

$$\{\epsilon'\} = [L'] [N] \{\Delta'\} = [B] \{\Delta'\} \quad (5)$$

where $[L']$ is a 5×3 linear operator matrix consisting of the first differential with respect to x, y , and z coordinates.

EQUATIONS OF MOTION

The principle of virtual displacements may be used to formulate general equations of motion of shell structures. Using this principal we obtain the differential equation

$$[M] \{\ddot{\Delta}\} + [C_d] \{\dot{\Delta}\} + [K] \{\Delta\} = \{R_B\} \quad (6)$$

where the mass matrix $[M]$, the damping matrix $[C_d]$, the stiffness matrix $[K]$, and the force vector $\{R_B\}$ are defined as

$$[M] = \sum_m \int_{v^{(m)}} [N]^T \rho [N] dv \quad (7)$$

$$[C_d] = \sum_m \int_{v^{(m)}} [N]^T \mu_d [N] dv \quad (8)$$

$$[K] = \sum_m \int_{v^{(m)}} [B]^T [D] [B] dv \quad (9)$$

$$[R_B K] = \sum_m \int_{v^{(m)}} [N]^T \{f^b\} dv \quad (10)$$

The elemental stiffness matrix $[K]^m$ and the mass matrix $[M]^m$ are evaluated numerically by using Gaussian quadrature weighted coefficients.

$$[K]^m = \sum_{i=1}^{ng} \sum_{j=1}^{ng} \sum_{k=1}^{ng} H_i H_j H_k [B]^T [D] [B] \det [J] \quad (11)$$

$$[M]^m = \sum_{i=1}^{ng} \sum_{j=1}^{ng} \sum_{k=1}^{ng} H_i H_j H_k \rho [N]^T [N] \det [J]. \quad (12)$$

In the above equations, H_i is the weight coefficient of the Gaussian quadrature; ng is the order of integration, that is, the number of Gauss points along the three directions of integration; and J is the Jacobian matrix. It may be noted that order of integration in the thickness direction is less than the order of integration in plane ξ, η . As observed by Zienkiewicz et al. (1971) and Parisch (1979), the same order of integration in all three directions results in an over stiff element. Therefore, a selectively reduced integration scheme has been recommended in the thickness direction. However, the reduced integration and selective integration techniques are not always successful in overcoming the over stiff or the so-called locking phenomenon, especially in thin membranes or shells with highly constrained boundaries. It has been observed that they may produce rank deficiency of the stiffness matrix. In recent years, some attention has been focused on the membrane and shear locking phenomenon and an alternate "assumed strain technique" has been suggested (Huang and Hinton, 1986). It is seen that the membrane strains can only be separated from the bending strains in the local coordinates. The authors have preferred the "assumed strain technique" in which the membrane and shear strains are expressed in the local Cartesian and natural coordinates, respectively.

Having obtained elemental mass and stiffness matrices, the total stiffness and mass matrices can be easily assembled.

NUMERICAL RESULTS

In this study, an analysis has been presented for vibration of cylindrical shells that are partially supported along one boundary and free along the other three boundaries. The present study is based on the assumed strain formulation with the nine-node shell element of Huang and Hinton (1986). As mentioned earlier, this element is one of the degenerate shell elements, which has five degrees-of-freedom per node (5-DOF), that is, three translational displacements (u_i, v_i, w_i) in the direction of the global axes and two rotations (α_{1i}, α_{2i}) with respect to the axes in the plane of the middle surface.

All computations in this investigation were carried out by varying three essential parameters: the shallowness parameter r/a , aspect ratio b/a , and the unsupported length c/a , as shown in Figure 2. The results for several degenerate cases of cylindrical shell will full support ($c/a = 0$), have been generated and compared with previously published results. The more general results for partially supported shallow cylindrical cantilever shells follow this case.

Trial computations for several alternative grid forms, CPU time, and number of elements were conducted and it was concluded that good convergence is achieved with 25 nine-node elements. These computations have not been included here. The tables in this section contain frequencies for the first six modes.

For computations the curved length a along the support, is taken as 12.0 in. (304.8 mm) and the thickness $t = 0.12$ in. (3.048 mm). These are

treated as constant for all cases. The results have been presented in terms of a dimensionless frequency parameter Ω , defined as,

$$\Omega = 2\pi\omega a^2\phi^2 \left(\frac{12\rho}{Et^2} \right)^{0.5}.$$

The values of modulus of elasticity E and the mass density ρ have been assumed as unity. The Poisson's ratio is taken to be 0.3

Fully Supported Cylindrical Shells

Almost all results available in the literature are for the fully clamped case ($c/a = 0$). In the present calculations these cases have been generated to test the accuracy of the element and computations.

Table 1 shows the comparison of the lowest six frequency parameters Ω for uniform cylindrical shells, fully clamped along one side, and free on the other three sides (C-F-F-F). It may be seen that the present results are in very close agreement with other published results with the largest discrepancy of 1.4% occurring at the higher modes.

Partially Detached Cantilever Cylindrical Shells

The frequency parameters for the partially supported-free-free-free (P-F-F-F), cylindrical shells with the three values of r/a (1.0, 2.0, and 3.0) and for different aspect ratios b/a (1.0, 2.0, and 3.0) are presented in Tables 2–10 for several values of

FIGURE 2 Partially supported cylindrical shell: (a) crack length from one side and (b) crack length from the two sides.

Table 1. Frequency Parameter $\Omega = 2\pi\omega a^2\phi^2 (12\rho/Et^2)^{0.5}$ for C-F-F-F Case

Source of Results	$a = 12 \text{ in.}, r/a = 2.0, \phi = b/a = 1.0, a/t = 100$					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
Present (25 elements)	11.153	17.958	31.973	44.671	50.400	69.090
Noamis and Lau (1990)	11.080	17.909	31.824	44.064	49.004	67.642
Noamis and Lau (1990), exp.	11.080	17.264	33.370	45.223	50.892	68.414
Walker (1979)	11.158	17.986	32.082	45.223	51.343	68.865
Olson (1971)	11.158	17.934	32.377	44.914	50.686	68.723
Khader (1990)	11.003	17.702	31.670	45.340	52.078	67.976

Table 2. Frequency Parameter $\Omega = 2\pi\omega a^2\phi^2 (12\rho/Et^2)^{0.5}$ for P-F-F-F Isotropic Cylindrical Shell

(i) c/a	$a = 12 \text{ in.}, r/a = 1.0, \phi = b/a = 1.0, a/t = 100$					
	Detached Length from One Side					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	15.671	23.615	47.523	59.655	60.891	70.911
0.2	11.101	21.355	38.259	48.434	60.167	70.065
0.4	6.633	18.129	22.844	29.969	59.882	67.993
0.6	4.019	10.518	14.894	26.785	54.111	60.611
0.8	2.254	5.448	11.640	24.918	34.829	40.332
0.9	1.777	4.404	10.833	22.399	23.631	32.764
(ii) $2c'/a$	Detached Length from Both Sides					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	15.671	23.615	47.523	59.655	60.891	70.911
0.2	13.676	21.771	44.381	56.038	57.559	69.764
0.4	9.461	15.572	33.363	48.173	48.458	68.106
0.6	7.727	10.317	27.833	39.478	40.324	66.922
0.8	4.786	6.063	20.978	23.861	30.648	53.940

Table 3. Frequency Parameter $\Omega = 2\pi\omega a^2\phi^2 (12\rho/Et^2)^{0.5}$ for P-F-F-F Isotropic Cylindrical Shell

(i) c/a	$a = 12 \text{ in.}, r/a = 1.0, \phi = b/a = 2.0, a/t = 100$					
	Detached Length from One Side					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	21.878	41.795	91.433	98.040	116.295	187.387
0.2	18.652	33.430	84.024	97.152	111.407	171.906
0.4	12.543	24.031	65.650	93.743	103.105	126.164
0.6	7.490	17.548	38.397	88.219	93.205	119.253
0.8	3.996	11.937	26.128	55.503	91.891	94.717
0.9	3.128	10.442	23.860	36.105	79.445	93.651
(ii) $2c'/a$	Detached Length from Both Sides					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	21.878	41.795	91.433	98.040	116.295	187.387
0.2	20.686	38.969	86.385	97.939	112.347	183.023
0.4	17.111	25.628	76.169	96.344	105.579	170.632
0.6	15.217	15.967	70.777	93.754	101.882	155.318
0.8	7.388	13.210	55.144	75.006	87.658	96.346

Table 4. Frequency Parameter $\Omega = 2\pi\omega a^2\phi^2 (12\rho/Et^2)^{0.5}$ for P-F-F-F Isotropic Cylindrical Shell

$a = 12 \text{ in.}, r/a = 1.0, \phi = b/a = 2.0, a/t = 100$						
(i) c/a	Detached Length from One Side					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	28.512	43.930	109.341	186.882	213.310	245.843
0.2	25.697	37.219	102.641	174.008	210.438	232.576
0.4	17.234	29.339	95.305	148.914	190.135	204.451
0.6	9.948	24.374	68.336	107.942	138.158	197.386
0.8	5.308	18.874	44.086	66.839	114.422	188.652
0.9	4.154	17.082	39.026	46.761	105.106	178.578
(ii) $2c'/a$	Detached Length from Both Sides					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	28.512	43.930	109.341	186.882	213.310	245.843
0.2	27.597	42.068	105.154	181.687	212.518	236.479
0.4	24.489	30.407	94.296	167.843	210.187	210.353
0.6	19.923	22.482	87.907	161.498	185.715	209.283
0.8	9.725	20.401	75.632	114.117	142.046	171.364

the crack length ratio c/a and $2c'/a$. The latter ratio ($2c'/a$) has been used for the shell with detached boundaries from the two sides as shown in Figure 2. Each of these groups of tables contains two sets. The first one contains frequency parameters for crack length from one side and the lower half of the table, for equal crack length from both sides.

Typical modal shapes for the P-F-F-F shallow shell with $c/a = 0.8$ are shown in Figure 3.

CONCLUDING REMARKS

A nine-node degenerate shell element has been used to study the vibration characteristics of isotropic shells using the displacement based finite element technique. To overcome the shear locking problem, an enhanced interpolation of the transverse shear strains in the natural coordinate system is used. Also to avoid membrane locking behavior, an enhanced interpolation of the mem-

Table 5. Frequency Parameter $\Omega = 2\pi\omega a^2\phi^2 (12\rho/Et^2)^{0.5}$ for P-F-F-F Isotropic Cylindrical Shell

$a = 12 \text{ in.}, r/a = 1.0, \phi = b/a = 2.0, a/t = 100$						
(i) c/a	Detached Length from One Side					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	11.153	17.959	31.974	44.671	50.401	69.090
0.2	9.092	14.650	29.813	39.297	48.178	68.678
0.4	5.874	11.334	21.563	29.360	46.877	56.551
0.6	3.531	7.837	14.408	27.381	45.619	52.391
0.8	2.094	5.071	11.620	25.401	36.225	37.340
0.9	1.096	4.401	10.964	21.152	24.324	33.355
(ii) $2c'/a$	Detached Length from Both Sides					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	11.153	17.959	31.974	44.671	50.401	60.090
0.2	10.406	16.161	31.387	42.789	48.415	68.866
0.4	8.510	10.415	28.474	37.335	42.523	67.988
0.6	6.911	7.479	26.162	32.463	37.374	62.138
0.8	3.868	5.901	20.567	22.202	30.635	52.423

Table 6. Frequency Parameter $\Omega = 2\pi\omega a^2\phi^2 (12\rho/Et^2)^{0.5}$ for P-F-F-F Isotropic Cylindrical Shell

$a = 12 \text{ in.}, r/a = 1.0, \phi = b/a = 2.0, a/t = 100$						
(i) c/a	Detached Length from One Side					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	17.835	22.064	64.276	91.023	102.507	137.380
0.2	15.598	19.201	60.858	86.874	99.869	130.553
0.4	9.775	17.074	53.210	77.964	97.688	103.434
0.6	5.727	14.611	36.369	66.911	92.631	96.311
0.8	3.432	11.316	25.608	53.379	71.636	95.805
0.9	2.827	10.329	23.602	35.852	62.124	93.640
(ii) $2c'/a$	Detached Length from Both Sides					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	17.835	22.064	64.276	91.023	102.507	137.380
0.2	17.322	20.792	62.409	88.801	101.146	133.955
0.4	14.559	15.707	57.520	82.686	98.858	124.392
0.6	9.962	14.610	54.273	79.005	98.060	116.494
0.8	5.744	12.084	46.398	66.766	87.477	93.118

brane strains in the local Cartesian coordinate system is applied. A number of cases for the fully supported isotropic shells ($c/a = 0$) have been generated and compared with other published results to evaluate the accuracy of the finite element model. For this case, the largest discrepancy for the first mode is less than 1% and for higher modes it is less than 1.5%, showing an excellent accuracy.

To make the results general and to demonstrate the behavior of this class of problems rather than the specific shells, a nondimensional form of frequency was used in all tables and graphs. Further, the modulus of elasticity E and mass density ρ have been assumed as unity, which again makes the results general and more useful.

The first six natural frequencies and corre-

Table 7. Frequency Parameter $\Omega = 2\pi\omega a^2\phi^2 (12\rho/Et^2)^{0.5}$ for P-F-F-F Isotropic Cylindrical Shell

$a = 12 \text{ in.}, r/a = 1.0, \phi = b/a = 2.0, a/t = 100$						
(i) c/a	Detached Length from One Side					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	22.614	24.495	82.844	122.972	167.976	214.721
0.2	19.079	24.093	79.620	112.793	162.855	214.007
0.4	12.172	23.312	74.520	100.029	148.196	198.995
0.6	7.262	21.401	60.681	85.678	117.021	161.175
0.8	4.417	18.124	42.193	66.566	92.181	127.818
0.9	3.659	17.069	37.981	47.249	83.029	122.370
(ii) $2c'/a$	Detached Length from Both Sides					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	22.614	24.495	82.844	122.972	167.976	214.721
0.2	21.805	24.062	81.235	118.785	164.595	214.548
0.4	16.757	22.644	76.613	105.763	155.399	213.566
0.6	12.155	21.543	73.179	99.455	147.819	211.432
0.8	7.397	19.664	66.204	93.480	109.588	135.799

Table 8. Frequency Parameter $\Omega = 2\pi\omega a^2\phi^2 (12\rho/Et^2)^{0.5}$ for P-F-F-F Isotropic Cylindrical Shell

$a = 12 \text{ in.}, r/a = 1.0, \phi = b/a = 2.0, a/t = 100$						
(i) c/a	Detached Length from One Side					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	10.013	13.300	29.373	38.728	45.585	68.705
0.2	8.391	11.092	28.217	35.119	43.314	66.974
0.4	5.3254	9.256	20.819	29.063	41.248	50.812
0.6	3.253	7.020	14.240	27.492	39.630	48.930
0.8	2.037	4.878	11.582	25.442	33.966	38.119
0.9	1.670	4.310	10.970	20.993	24.344	32.368
(ii) $2c'/a$	Detached Length from Both Sides					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	10.013	13.300	29.373	38.738	45.585	68.705
0.2	9.592	12.017	29.127	37.576	44.017	68.571
0.4	7.956	8.269	27.353	33.238	39.409	66.614
0.6	5.634	7.398	25.483	29.432	35.544	57.660
0.8	3.534	5.824	20.072	21.325	30.404	47.283

sponding modal shapes, and nodal lines for variable detached based length have been generated in this work. These computations have been carried out for shallow shells for the two cases with three different aspect ratios as well as three different radius to base length ratios. These two cases are identified as: (i) detached based length from one side (c/a), (ii) detached base length from both sides ($2c'/a$). The results show that as

the ratio r/a increases or as the shell becomes shallower, the frequency ω decreases. As seen from Tables 2 to 10, the change in frequency parameters (Ω/ϕ^2) for case (i) is more pronounced than for case (ii). The tables also show that for constant radius of shell ($r/a = 3$) and for the higher aspect ratio, i.e., $b/a \geq 3$, the frequency parameter Ω/ϕ^2 stays relatively unaffected, even for longer detached base length. Thus, we con-

Table 9. Frequency Parameter $\Omega = 2\pi\omega a^2\phi^2 (12\rho/Et^2)^{0.5}$ for P-F-F-F Isotropic Cylindrical Shell

$a = 12 \text{ in.}, r/a = 1.0, \phi = b/a = 2.0, a/t = 100$						
(i) c/a	Detached Length from One Side					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	15.081	16.705	57.274	73.775	99.222	117.896
0.2	12.364	16.324	54.725	69.012	98.491	112.953
0.4	7.932	15.492	48.180	64.144	89.970	97.783
0.6	4.945	13.762	34.941	58.733	76.861	96.581
0.8	3.218	10.977	25.202	51.367	66.819	80.897
0.9	2.712	10.092	23.322	36.305	55.824	77.478
(ii) $2c'/a$	Detached Length from Both Sides					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	15.081	16.705	57.274	73.775	99.222	117.896
0.2	14.307	16.401	56.241	71.161	98.940	115.954
0.4	10.553	15.288	52.770	64.957	98.075	108.879
0.6	7.814	14.399	50.052	61.967	97.583	102.709
0.8	5.080	12.626	43.517	55.925	84.982	87.866

Table 10. Frequency Parameter $\Omega = 2\pi\omega a^2\phi^2 (12\rho/Et^2)^{0.5}$ for P-F-F-F Isotropic Cylindrical Shell

$a = 12 \text{ in.}, r/a = 1.0, \phi = b/a = 2.0, a/t = 100$						
(i) c/a	Detached Length from One Side					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	15.370	23.342	76.204	87.073	147.046	196.001
0.2	13.527	22.835	73.318	81.493	143.255	187.758
0.4	9.456	22.028	66.255	76.506	131.414	171.862
0.6	6.132	20.517	54.965	72.166	105.309	147.313
0.8	4.069	17.675	40.316	64.408	87.439	99.608
0.9	3.459	16.722	36.667	48.283	74.300	95.435

(ii) $2c'/a$	Detached Length from Both Sides					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.0	15.370	23.342	76.204	87.073	147.046	196.001
0.2	14.880	23.081	75.293	84.401	145.226	191.984
0.4	11.927	22.118	72.167	75.755	139.120	182.718
0.6	9.2321	21.259	69.400	71.307	133.659	177.072
0.8	6.392	19.462	63.372	67.513	108.974	123.826

clude that as the aspect ratio increases, shallower shells ($r/a \geq 3$) behave more like long plates and beams where the frequency is not sen-

sitive to the length of the base support. For these types of shells, the first fundamental twisting modes are replaced by symmetric modes. Figure 3 shows the first six mode shapes for case (i), that is, the detached base length from one side. The patterns have been plotted for $c/a = 0.8$ and for relatively shallower shells ($r/a = 2.00$). For the shorter shells (b/a), it can be seen that the shell first goes into a twisting mode followed by the bending mode.

$b/a = 1.0, a/t = 100.00, c/a = 0.8.$

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