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Vibration Analysis of Structures with Rotation and Reflection Symmetry

The article applies group representation theory to the vibration analysis of structures with C_{nv} symmetry, and presents a new structural vibration analysis method. The eigenvalue problem of the whole structure is divided into much smaller subproblems by forming the mass and stiffness matrices of one substructure and than modifying them to form mass and stiffness matrices in each irreducible subspace, resulting in the saving of computer time and memory. The modal characteristics of structures with C_{nv} symmetry are derived from theoretical analysis. Computation and modal testing are used to verify the validity of the theoretical deductions. © *1996 John Wiley & Sons, Inc.*

INTRODUCTION

Group theory is a mathematical tool in the study of symmetry. Symmetry is possessed by many engineering structures. We can apply a symmetry operation to such a structure that interchanges the positions of various points in it but results in the structure looking exactly the same as before the symmetry operation (the structure is in an equivalent position). When the operation is continuously repeated, the structure continues to be in an equivalent position. There is one type of symmetry operation for which one point in the body remains fixed. These are the point symmetry operations. Two such operations are C_n, which means rotation by $2\pi/n$ about an axis of the body in the sense of a right-hand screw, and σ_v, which means reflection in a plane containing the C_n axis. Repeating the symmetry operation C_n m times, we get an operation C_n^m, which means rotation by $2m\pi/n$ about the C_n axis [$m = 0, 1, \ldots, (n-1)$]. In particular C_n^0 is the identity operation E under which the structure is not rotated at all. The collection of E, $C_n^1, \ldots$, and C_n^{n-1} forms a group in the mathematical sense. This group is called the C_n point group.

Suppose there are n reflection planes all containing the C_n axis denoted by $\sigma^{(0)}, \sigma^{(1)}, \ldots, \sigma^{(n-1)}$, of which $\sigma^{(i)}$ has an angle π/n with its neighbor $\sigma^{(i+1)}$ ($i = 0, 1, \ldots, n-1$, and define $\sigma^{(n)} = \sigma^{(0)}$). By adding σ_v symmetry operations $\sigma_v^{(i)}$ across the reflection plane $\sigma^{(i)}$ ($i = 0, 1, \ldots, n-1$) to the C_n point group, a new point group C_{nv} can be obtained (Burns, 1977).

For convenience of analysis, the $2n$ symmetry operations of the C_{nv} group

$$E, C_n^1, \ldots, C_n^{n-1}, \sigma_v^{(0)}, \sigma_v^{(1)}, \ldots, \sigma_v^{(n-1)}$$

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are called sequentially the first, second, to the $2n$th symmetry operation.

A C_{nv} structure can be defined as the one that remains in an equivalent position when acted on by any one of the $2n$ symmetry operations of the C_{nv} group. According to the $2n$ symmetric operations of the C_{nv} group, a C_{nv} structure can be divided into $2n$ identical substructures denoted by $S^{(K)}$ ($K = 1, 2, \ldots, 2n$). Choose one of the substructures arbitrarily as the basic substructure and make it correspond to the first symmetry operation E of the C_{nv} group and define it as $S^{(1)}$; $S^{(k)}$ can be obtained by applying the kth symmetry operation on the basic substructures $S^{(1)}$. A C_{4v} structure (the square plate ABCD) is shown in Fig. 1, where the z axis coincides with the C_4 axis and the reflection plane $\sigma^{(0)}$ contains the x axis, $\sigma^{(1)}$ contains $\overline{AC}$, etc. As shown in the figure, $S^{(2)}$ is obtained by applying the rotation operation C_4^1 (the second element of the C_{4v} group) on $S^{(1)}$; $S^{(5)}$ is obtained by applying the reflection operation $\sigma_v^{(0)}$ (the fifth element of C_{4v} group) on $S^{(1)}$, etc. Thus the plate is divided into eight identical substructures according to the eight symmetry operations of the C_{4v} group.

Many engineering structures, such as the top cover of the supporting structure of a hydraulic turbine, are symmetric under group C_{nv}. Zhong and Qiu (1978a, b) applied group representation theory to static analysis of structures with C_n or C_{nv} symmetry; but by applying group theory to vibration analysis of these structures, we can divide the natural eigenvalue problems into a set of much smaller subproblems. This leads to a remarkable saving of computer time and memory and hence increases the efficiency of calculation. Also, modal characteristics of natural vibration, derived from theoretical analysis, can be expected to provide some theoretical basis for modal testing. Only C_{nv} structures with no center point with n an even integer are studied in this article.

COORDINATE SYSTEM FOR APPLYING GROUP THEORY

For the application of group theory to the study of C_{nv} structures, it is extremely inconvenient or even impossible to use the rectangular coordinate system as shown in Fig. 1. Thus, a multicoordinate system (MCS) is established according to the classification of nodal points in a C_{nv} structure. As shown in Fig. 2, an MCS denoted by $C_S^{(1)}$ is established in $S^{(1)}$, where the coordinate system $C_{S0}^{(1)}$ is used to describe the interior points of $C^{(1)}$, $C_{S1}^{(1)}$ is used to describe the first class symmetric points (i.e., points on the interface between $S^{(1)}$ and $S^{(n+1)}$), and $C_{S2}^{(1)}$ is used to describe the second class symmetric points (i.e., points on the interface between $S^{(1)}$ and $S^{(n+2)}$). The points classifi-

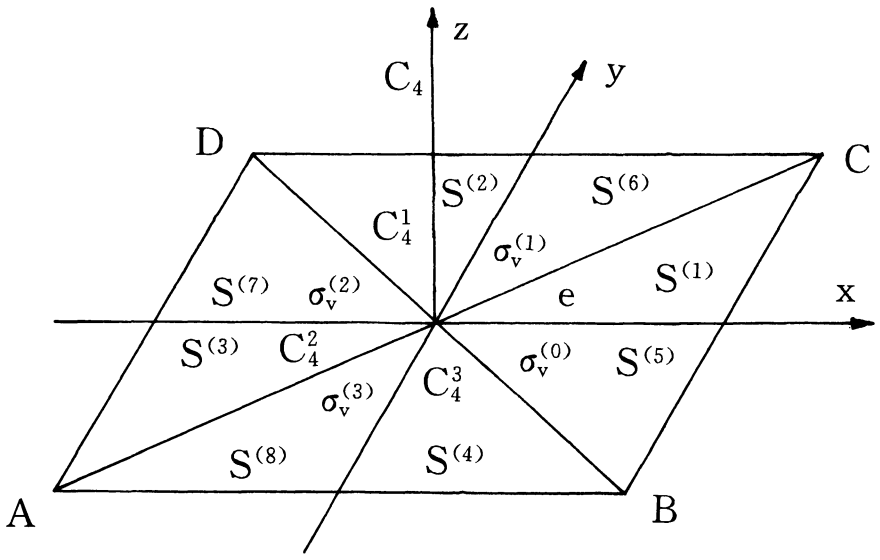

FIGURE 1 A C_{4v} structure.

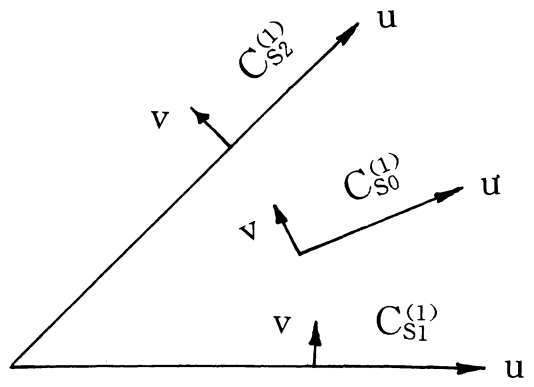

FIGURE 2 MCS $C_S^{(1)}$ for $S^{(1)}$.

cation and MCS $C_S^{(k)}$ for $S^{(k)}$ can be obtained by applying the kth symmetry operation of the C_{nv} group on $S^{(1)}$ and $C_S^{(1)}$. The assembly of $C_S^{(k)}$ (where $k = 1, 2, \ldots, 2n$) form the MCS denoted by C_S that describes the nodal displacement of the whole structure.

The advantage of the MCS C_S is that it ensures that every substructure has identical mass and stiffness matrices; however, in such an MCS, every symmetric point (including the first and second class symmetric points) is described by two coordinate systems, which is inconvenient. To solve this problem, an MCS C_T shown in Fig. 3 is used to describe the symmetric points.

Thus, in the MCS C_T, the coordinate systems for symmetric points of substructures $S^{(k)}$ ($k = 1, 2, \ldots, n$) are identical with that of the MCS C_S and the v axis of the coordinate system for symmetric points of substructures $S^{(k)}$ ($k = n + 1, n + 2, \ldots, 2n$) is of opposite direction to that of the C_S. Obviously the $C_S^{(k)}$ ($k = 1, 2, \ldots, n$) of C_S are right-hand systems while the $C_S^{(k)}$ ($k = n + 1, n + 2, \ldots, 2n$) are left-hand systems, which are due to the reflection operation.

Generally, six displacement components (i.e., the displacements u, v, w along axes x, y, z and the angular displacements θ_x, θ_y, θ_z around axes x, y, z, respectively) are needed to describe a nodal displacement of an engineering structure. Then the relationships between C_S and C_T are as follows for the first class symmetric points,

$$
\begin{aligned}
\{u^{(1)}, \ldots, u^{(n)}, u^{(n+1)}, \ldots, u^{(2n)}\}_{C_S} &= \{u^{(1)}, \ldots, u^{(n)}, u^{(1)}, \ldots, u^{(n)}\}_{C_T} \\
\{v^{(1)}, \ldots, v^{(n)}, v^{(n+1)}, \ldots, v^{(2n)}\}_{C_S} &= \{v^{(1)}, \ldots, v^{(n)}, -v^{(1)}, \ldots, -v^{(n)}\}_{C_T} \\
\{w^{(1)}, \ldots, w^{(n)}, w^{(n+1)}, \ldots, w^{(2n)}\}_{C_S} &= \{w^{(1)}, \ldots, w^{(n)}, w^{(1)}, \ldots, w^{(n)}\}_{C_T} \\
\{\theta_x^{(1)}, \ldots, \theta_x^{(n)}, \theta_x^{(n+1)}, \ldots, \theta_x^{(2n)}\}_{C_S} &= \{\theta_x^{(1)}, \ldots, \theta_x^{(n)}, -\theta_x^{(1)}, \ldots, -\theta_x^{(n)}\}_{C_T} \\
\{\theta_y^{(1)}, \ldots, \theta_y^{(n)}, \theta_y^{(n+1)}, \ldots, \theta_y^{(2n)}\}_{C_S} &= \{\theta_y^{(1)}, \ldots, \theta_y^{(n)}, \theta_y^{(1)}, \ldots, \theta_y^{(n)}\}_{C_T} \\
\{\theta_z^{(1)}, \ldots, \theta_z^{(n)}, \theta_z^{(n+1)}, \ldots, \theta_z^{(2n)}\}_{C_S} &= \{\theta_z^{(1)}, \ldots, \theta_z^{(n)}, -\theta_z^{(1)}, \ldots, -\theta_z^{(n)}\}_{C_T}.
\end{aligned} \tag{1}
$$

For the second class symmetric points,

$$
\begin{aligned}
\{u^{(1)}, \ldots, u^{(n)}, u^{(n+1)}, \ldots, u^{(2n)}\}_{C_S} &= \{u^{(1)}, \ldots, u^{(n)}, u^{(n)}, u^{(1)}, \ldots, u^{(n-1)}\}_{C_T} \\
\{v^{(1)}, \ldots, v^{(n)}, v^{(n+1)}, \ldots, v^{(2n)}\}_{C_S} &= \{v^{(1)}, \ldots, v^{(n)}, -v^{(n)}, -v^{(1)}, \ldots, -v^{(n-1)}\}_{C_T} \\
\{w^{(1)}, \ldots, w^{(n)}, w^{(n+1)}, \ldots, w^{(2n)}\}_{C_S} &= \{w^{(1)}, \ldots, w^{(n)}, w^{(n)}, w^{(1)}, \ldots, w^{(n-1)}\}_{C_T} \\
\{\theta_x^{(1)}, \ldots, \theta_x^{(n)}, \theta_x^{(n+1)}, \ldots, \theta_x^{(2n)}\}_{C_S} &= \{\theta_x^{(1)}, \ldots, \theta_x^{(n)}, -\theta_x^{(n)}, -\theta_x^{(1)}, \ldots, -\theta_x^{(n-1)}\}_{C_T} \\
\{\theta_y^{(1)}, \ldots, \theta_y^{(n)}, \theta_y^{(n+1)}, \ldots, \theta_y^{(2n)}\}_{C_S} &= \{\theta_y^{(1)}, \ldots, \theta_y^{(n)}, \theta_y^{(n)}, \theta_y^{(1)}, \ldots, -\theta_y^{(n-1)}\}_{C_T} \\
\{\theta_z^{(1)}, \ldots, \theta_z^{(n)}, \theta_z^{(n+1)}, \ldots, \theta_z^{(2n)}\}_{C_S} &= \{\theta_z^{(1)}, \ldots, \theta_z^{(n)}, -\theta_z^{(n)}, -\theta_z^{(1)}, \ldots, -\theta_z^{(n-1)}\}_{C_T},
\end{aligned} \tag{2}
$$

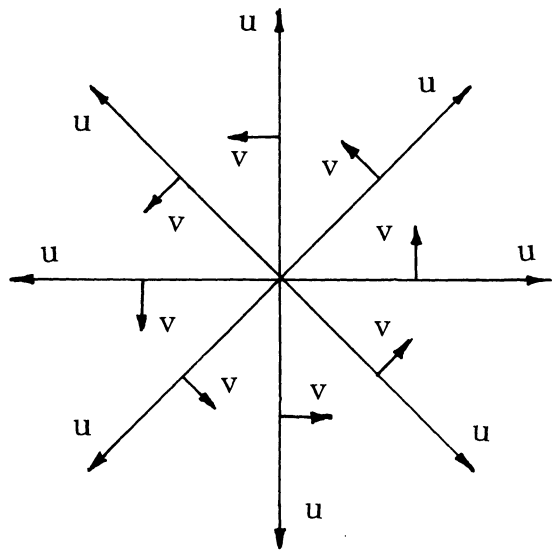

FIGURE 3 MCS C_T for symmetric points.

where the component $u^{(1)}$ of the displacement vector

$$\{u^{(1)}, \ldots, u^{(n)}, u^{(n+1)}, \ldots, u^{(2n)}\}_{C_S}$$

is the u displacement of a nodal point in substructure $S^{(1)}$ and $u^{(i)}$ ($i = 2, 3, \ldots, 2n$) is that of the corresponding nodal point of substructure $S^{(i)}$.

BASE VECTORS OF EACH CLASS OF POINTS IN MCS C_S

Selecting an interior nodal point in $S^{(1)}$ with corresponding nodal points in $S^{(i)}$ ($i = 2, 3, \ldots, 2n$) determined by the symmetry operations of C_{nv} group, we can get $2n$ corresponding nodal points. With the same displacement component of every nodal point we can form a $2n$ dimensional vector $\{\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(2n)}\}$. The vector belongs to the Euclidean space R^{2n} that can be determined as the representative space of the C_{nv} group for interior points. The representative space can be reduced into four 1-dimensional subspaces as V^{A_1}, V^{A_2}, V^{B_1}, V^{B_2} and $(n - 2)$ 2-dimensional irreducible subspaces as V^{EK_1}, V^{EK_2} ($k = 1, 2, \ldots, (n/2) - 1$). The base vectors of each irreducible subspace are

$$\begin{aligned}
\varphi^{A_1} &= \{1, 1, \ldots, 1; 1, 1, \ldots, 1\}^T \\
\varphi^{A_2} &= \{1, 1, \ldots, 1; -1, -1, \ldots, -1\}^T \\
\varphi^{B_1} &= \{1, -1, \ldots, -1; 1, -1, \ldots, -1\}^T \\
\varphi^{B_2} &= \{1, -1, \ldots, -1; -1, 1, \ldots, 1\}^T \\
\varphi_1^{EK_1} &= \{1, \cos k\theta, \ldots, \cos k(n-1)\theta; \\
&\quad \cos kn\theta, \ldots, \cos k(2n-1)\theta\}^T \\
\varphi_2^{EK_1} &= \{0, -\sin k\theta, \ldots, -\sin k(n-1)\theta; \\
&\quad \sin kn\theta, \ldots, \sin k(2n-1)\theta\}^T \\
\varphi_1^{EK_2} &= \{0, \sin k\theta, \ldots, \sin k(n-1)\theta; \\
&\quad \sin kn\theta, \ldots, \sin k(2n-1)\theta\}^T \\
\varphi_2^{EK_2} &= \{1, \cos k\theta, \ldots, \cos k(n-1)\theta; \\
&\quad -\cos kn\theta, \ldots, -\cos kn\theta, \ldots, \\
&\quad -\cos k(2n-1)\theta\}^T,
\end{aligned} \tag{3}$$

where $k = 1, 2, \ldots, (n/2) - 1$, $\theta = (2\pi/n)$, and in each base vector the two parts before and after the semicolons are all n dimensions. These are called the base vectors for the interior points.

Selecting a symmetric point in $S^{(1)}$ and applying the $2n$ symmetry operations of the C_{nv} group on it, we can get $2n$ corresponding points belonging to the $2n$ substructures. These are, however, physically n different points that should be described by the MCS C_T. Thus, the representative spaces of the C_{nv} group for symmetric points are n-dimensional. They can be reduced into irreducible subspaces, and the base vectors of these subspaces are n-dimensional under the MCS C_T. These base vectors must be expanded into $2n$-dimensional vectors according to the relationships between C_S and C_T, i.e., Eqs. (1) and (2). The expanded base vectors for symmetric points can be expressed as base vectors for interior points and their combinations.

The base vectors for u, w, θ_y in the first class points are φ^{A_1}, φ^{B_1}, $\varphi_1^{EK_1}$, and $\varphi_1^{EK_2}$; the base vectors for v, θ_x, θ_z in the first class points are φ^{A_2}, φ^{B_2}, $\varphi_2^{EK_1}$, and $\varphi_2^{EK_2}$; and the base vectors for u, w, θ_y in the second class points are

$$\varphi^{A_1}, \varphi^{B_2}, \cos\left(\frac{k\pi}{n}\right)\varphi_1^{EK_1} + \sin\left(\frac{k\pi}{n}\right)\varphi_2^{EK_1} \quad \text{and}$$

$$\cos\left(\frac{k\pi}{n}\right)\varphi_1^{EK_2} + \sin\left(\frac{k\pi}{n}\right)\varphi_2^{EK_2};$$

The base vectors for v, θ_x, θ_z in the second class points are

$$\varphi^{A_2},\ \varphi^{B_1},\ -\sin\left(\frac{k\pi}{n}\right)\varphi_1^{EK_1}+\cos\left(\frac{k\pi}{n}\right)\varphi_2^{EK_1}\quad\text{and}$$

$$-\sin\left(\frac{k\pi}{n}\right)\varphi_1^{EK_2}+\cos\left(\frac{k\pi}{n}\right)\varphi_2^{EK_2},$$

where $k = 1, 2, \ldots, (n/2) - 1$.

C_{nv} GROUP TRANSFORMATION

Because we obtained base vectors for every displacement component (i.e., u, v, w, θ_x, θ_y, and θ_z) of every kind of point (i.e., interior points, first class symmetric points, and second class symmetric points) under the MCS C_S, the $2n$-dimensional vectors $\{\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(2n)}\}^T$, in which $\delta^{(i)}$ is one of the six displacement components of a nodal point in $S^{(i)}$ ($i = 1, 2, \ldots, 2n$), can be expressed as the linear combination of base vectors in (3). The coefficients of any base vector are called the generalized displacements related to the corresponding irreducible subspace.

Denote the global physical displacement of $S^{(k)}$ as

$$\delta^{(k)} = \{\delta_1^{(k)}, \delta_2^{(k)}, \ldots, \delta_L^{(k)}\}^T,$$

where L is the total number of degrees of freedom of $S^{(k)}$ when the finite element discretization is made in $S^{(k)}$.

Denote I_1, I_2, I_3, I_4, and I_5 to be, respectively, the subscript sets corresponding to: the displacements of the interior points; u, w, and θ_y of the first class symmetric points; u, w, and θ_y of the second class symmetric points; v, θ_x, and θ_z of the first class symmetric points; and v, θ_x, and θ_z of the second class symmetric points. Let

$$\Phi = [\varphi^{A_1}, \varphi^{A_2}, \varphi^{B_1}, \varphi^{B_2}, \ldots, \varphi_1^{EK_1}, \varphi_2^{EK_1}, \varphi_1^{EK_2}, \varphi_2^{EK_2}, \ldots],$$

where Φ is the reducing matrix and is formed by arranging all the base vectors for the interior points sequentially.

Thus we have the matrix form of the group transformation from physical displacements to generalized displacements.

$$\begin{bmatrix} \cdots\ \delta_{i1}^{(1)}\ \cdots & \cdots\ \delta_{i2}^{(1)}\ \cdots & \cdots\ \delta_{i3}^{(1)}\ \cdots & \cdots\ \delta_{i4}^{(1)}\ \cdots & \cdots\ \delta_{i5}^{(1)}\ \cdots \\ \cdots\ \delta_{i1}^{(2)}\ \cdots & \cdots\ \delta_{i2}^{(2)}\ \cdots & \cdots\ \delta_{i3}^{(2)}\ \cdots & \cdots\ \delta_{i4}^{(2)}\ \cdots & \cdots\ \delta_{i5}^{(2)}\ \cdots \\ \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots \\ \cdots\ \delta_{i1}^{(2n)}\ \cdots & \cdots\ \delta_{i2}^{(2n)}\ \cdots & \cdots\ \delta_{i3}^{(2n)}\ \cdots & \cdots\ \delta_{i4}^{(2n)}\ \cdots & \cdots\ \delta_{i5}^{(2n)}\ \cdots \end{bmatrix}$$

$$= \Phi \begin{bmatrix} \cdots\ q_{i1}^{01}\ \cdots & \cdots\ q_{i2}^{01}\ \cdots & \cdots\ q_{i3}^{01}\ \cdots & \cdots\ 0\ \cdots & \cdots\ 0\ \cdots \\ \cdots\ q_{i1}^{02}\ \cdots & \cdots\ 0\ \cdots & \cdots\ 0\ \cdots & \cdots\ q_{i4}^{02}\ \cdots & \cdots\ q_{i5}^{02}\ \cdots \\ \cdots\ q_{i1}^{03}\ \cdots & \cdots\ q_{i2}^{03}\ \cdots & \cdots\ 0\ \cdots & \cdots\ 0\ \cdots & \cdots\ q_{i5}^{03}\ \cdots \\ \cdots\ q_{i1}^{04}\ \cdots & \cdots\ 0\ \cdots & \cdots\ q_{i3}^{04}\ \cdots & \cdots\ q_{i4}^{04}\ \cdots & \cdots\ 0\ \cdots \\ \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots \\ \cdots\ q_{i1}^{k1}\ \cdots & \cdots\ q_{i2}^{k1}\ \cdots & \cdots\ \cos\frac{k\pi}{n}\, q_{i3}^{k12}\ \cdots & \cdots\ 0\ \cdots & \cdots\ -\sin\frac{k\pi}{n}\, q_{i5}^{k12}\ \cdots \\ \cdots\ q_{i1}^{k2}\ \cdots & \cdots\ 0\ \cdots & \cdots\ \sin\frac{k\pi}{n}\, q_{i3}^{k12}\ \cdots & \cdots\ q_{i4}^{k2}\ \cdots & \cdots\ \cos\frac{k\pi}{n}\, q_{i5}^{k12}\ \cdots \\ \cdots\ q_{i1}^{k3}\ \cdots & \cdots\ q_{i2}^{k3}\ \cdots & \cdots\ \cos\frac{k\pi}{n}\, q_{i3}^{k34}\ \cdots & \cdots\ 0\ \cdots & \cdots\ -\sin\frac{k\pi}{n}\, q_{i5}^{k34}\ \cdots \\ \cdots\ q_{i1}^{k4}\ \cdots & \cdots\ 0\ \cdots & \cdots\ \sin\frac{k\pi}{n}\, q_{i3}^{k34}\ \cdots & \cdots\ q_{i4}^{k4}\ \cdots & \cdots\ \cos\frac{k\pi}{n}\, q_{i5}^{k34}\ \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \quad (4)$$

where $i_m \in I_m$ ($m = 1, 2, \ldots, 5$).

Denote the transpose of each row vector of the generalized displacement matrix in turn as follows

$$q^{01}, q^{02}, q^{03}, q^{04}, \ldots, q^{k_1}, q^{k_2}, q^{k_3}, q^{k_4}, \ldots,$$

where $k = 1, 2, n/2 - 1$, and that of the reducing matrix as Φ_j ($j = 1, 2, \ldots, 2n$), then

$$\Phi_j^T = \{\psi_{jo_1}, \psi_{jo_2}, \psi_{jo_3}, \psi_{jo_4}, \ldots, \psi_{jk_1}, \psi_{jk_2}, \psi_{jk_3}, \psi_{jk_4}, \ldots\}.$$

Here ψ_{jo_1} is the jth element of φ^{A_1}, and so on. From Eq. (4) we have

$$\delta^{(j)T} = \Phi_j^T[q^{01}, q^{02}, q^{03}, q^{04}, \ldots, q^{k_1}, q^{k_2}, q^{k_3}, q^{k_4}, \ldots]^T$$

that is,

$$\begin{aligned}\delta^{(j)} &= [q^{01}, q^{02}, q^{03}, q^{04}, \ldots, q^{k_1}, q^{k_2}, q^{k_3}, q^{k_4}, \ldots]\Phi_j \\ &= \sum_{k=0}^{(n/2)-1} (\psi_{jk_1} q^{k_1} + \psi_{jk_2} q^{k_2} + \psi_{jk_3} q^{k_3} + \psi_{jk_4} q^{k_4}).\end{aligned} \tag{5}$$

Therefore, the global displacement vector of $S^{(i)}$ is expressed as a linear combination of the generalized displacement vectors (*GDV*, i.e., q^{k_1}, q^{k_2}, q^{k_3}, and q^{k_4}, $k = 0, 1, 2, n/2 - 1$) of each irreducible subspace.

DECOMPOSITION OF NATURAL EIGENVALUE PROBLEM

Structural mass and stiffness matrices are formed first. For convenience, the mass and stiffness matrices of substructure $S^{(k)}$ are formed first under $C_{S0}^{(k)}$, which describes the interior points in MCS $C_S^{(k)}$, and then they are transformed into mass matrix M and stiffness matrix K, respectively, under MCS $C_S^{(k)}$. Because all the substructures and the MCS they adopt are identical, their mass matrices and stiffness matrices are identical too. Therefore, the potential energy of the entire structure is

$$V = \frac{1}{2} \sum_{j=1}^{2n} \delta^{(j)T} K \delta^{(j)}, \tag{6}$$

in which the terms in the summation are energy coupled in proper sequence, so that it is impossible to decompose the eigenvalue problem.

Using Eq. (5) in Eq. (6) and applying group theory, we can prove that

$$V = \frac{1}{2} \sum_{k=0}^{(n/2)-1} x_k^2 (q^{k_1 T} K q^{k_1} + q^{k_2 T} K q^{k_2} + q^{k_3 T} K q^{k_3} + q^{k_4 T} K q^{k_4}), \tag{7}$$

in which

$$x_k^2 = \begin{cases} 2n, & \text{when } k = 0, \\ n, & \text{when } k = 1, 2, \ldots, \dfrac{n}{2} - 1 \end{cases}$$

Compared with Eq. (6), the energy coupling problem in Eq. (7) has lessened greatly. But there are still two problems that must be solved: the first is that all the generalized displacement vectors are constrained, that is, they contain zero components that should be deleted; the second is that there is still coupling between generalized displacement vectors q^{k_1} and q^{k_2} in subspace V^{EK_1} as well as those between generalized displacement vectors q^{k_3} and q^{k_4} in subspace V^{EK_2} (where $k = 1, 2, \ldots, n/2 - 1$), which cannot be further decoupled. To solve these two problems, special steps are taken as follows.

Delete the zero components of *GDV* q^{01}, q^{02}, q^{03}, and q^{04}, which are in 1-dimensional subspaces V^{A_1}, V^{A_2}, V^{B_1}, and V^{B_2}, and change them into q^{A_1}, q^{A_2}, q^{B_1}, and q^{B_2}, respectively. The corresponding stiffness matrices are changed into K_{A_1}, K_{A_2}, K_{B_1}, and K_{B_2}, respectively.

Delete the zero components of *GDV* q^{k_1} and q^{k_2}, which are in the 2-dimensional subspace V^{EK_1}, and combine them into one vector denoted by q^{EK_1}. The two corresponding identical stiffness matrices are changed into one and denoted by K_{EK_1}.

Similarly, the two *GDV* q^{k_3} and q^{k_4} in 2-dimensional subspace V^{EK_2} are combined into q^{EK_2}, and the stiffness matrices into K_{EK_2}.

Notice that q^{k_1} and q^{k_2} as well as q^{k_3} and q^{k_4} are identical in form; therefore, $K_{EK_1} = K_{EK_2}$. And the potential energy can also be written as

$$\begin{aligned}V = n(&q^{A_1 T} K_{A_1} q^{A_1} + q^{A_2 T} K_{A_2} q^{A_2} \\ &+ q^{B_1 T} K_{B_1} q^{B_1} + q^{B_2 T} K_{B_2} q^{B_2}) \\ &+ \frac{n}{2} \sum_{k=1}^{(n/2)-1} (q^{EK_1 T} K_{EK_1} q^{EK_1} \\ &+ q^{EK_2 T} K_{EK_2} q^{EK_2}).\end{aligned} \tag{8}$$

Here any term in summation is completely independent of the others, and there is no energy coupling. In a similar way, the kinetic energy

expression is

$$\begin{aligned} T = n(&\dot{q}^{A_1 T} M_{A_1} \dot{q}^{A_1} + \dot{q}^{A_2 T} M_{A_2} \dot{q}^{A_2} \\ &+ \dot{q}^{B_1 T} M_{B_1} \dot{q}^{B_1} + \dot{q}^{B_1 T} M_{B_2} \dot{q}^{B_2}) \\ &+ \frac{n}{2} \sum_{k=1}^{(n/2)-1} (\dot{q}^{EK_1 T} M_{EK_1} \dot{q}^{EK_1} \\ &+ \dot{q}^{EK_2 T} M_{EK_2} \dot{q}^{EK_2}) . \end{aligned} \tag{9}$$

Substituting Eqs. (8) and (9) into the Lagrange equation, we get the generalized structural natural vibration equations completely independently as follows

$$M_{A_1} \ddot{q}^{A_1} + K_{A_1} q^{A_1} = 0 \tag{10}$$

$$M_{A_2} \ddot{q}^{A_2} + K_{A_2} q^{A_2} = 0 \tag{11}$$

$$M_{B_1} \ddot{q}^{B_1} + K_{B_1} q^{B_1} = 0 \tag{12}$$

$$M_{B_2} \ddot{q}^{B_2} + K_{B_2} q^{B_2} = 0 \tag{13}$$

$$M_{EK_1} \ddot{q}^{EK_1} + K_{EK_1} q^{EK_1} = 0 \tag{14}$$

$$M_{EK_2} \ddot{q}^{EK_2} + K_{EK_2} q^{EK_2} = 0 \tag{15}$$

where $k = 1, 2, \ldots, n/2 - 1$.

PRELIMINARY WORK ON MODAL CHARACTERISTICS OF C_{nv} STRUCTURES

From Eqs. (10) to (15) we know that because the mass matrices and stiffness matrices in the generalized eigenvalue problems of the four 1-dimensional subspaces V^{A_1}–V^{B_2} are different from each other, frequencies solved for them are all single frequencies (one single frequency has only one modal shape), but for the two 2-dimensional subspaces V^{EK_1} and V^{EK_2}, the mass and stiffness matrices in the generalized eigenvalue equations are identical; therefore, the frequencies solved for them are all duplicated frequencies (a duplicated frequency has two modal shapes).

In the subspace V^{A_1}, Eq. (10) can be solved to give q^{A_1} and hence q^{01}. Introducing q^{01} into Eq. (5), we obtain

$$\delta^{(j)} = \psi_{j0_1} q^{01} = q^{01}, \tag{16}$$

where $j = 1, 2, \ldots, 2n$. Here all substructures vibrate in phase with equal amplitudes and from the property of q^{01} we know that the values of v, θ_x, and θ_z are zero at the symmetric points.

In the subspace V^{A_2},

$$\delta^{(j)} = \psi_{j0_2} q^{02}, \tag{17}$$

where $j = 1, 2, \ldots, 2n$. From the property of φ^{A_2} and q^{02} it is known that all the corresponding points between the neighboring substructures vibrate in antiphase with equal amplitude and u, ω, and θ_y are equal to zero at the symmetric point.

In the subspace V^{B_1},

$$\delta^{(j)} = \psi_{j0_3} q^{03}, \tag{18}$$

where $j = 1, 2, \ldots, 2n$. Here, all the corresponding points among substructures as $S^{(2k-1)}$ ($k = 1, 2, \ldots, n$) vibrate in phase with equal amplitudes, and the corresponding points among $S^{(2k)}$ ($k = 1, 2, \ldots, n$) also vibrate in phase with equal-amplitude, but the corresponding points between $S^{(2k-1)}$ and $S^{(2k)}$ ($k = 1, 2, \ldots, n$) vibrate in antiphase with equal amplitude. In other words, a substructure must vibrate in phase with one of its two neighboring substructures and in antiphase with the other. Furthermore, for the symmetric points between the neighboring substructures that vibrate in phase, their v, θ_x, and θ_z are all zero; and for the symmetric points between the neighboring substructures that vibrate in antiphase, the values of u, w, and θ_y are zero.

In the subspace V^{B_2},

$$\delta^{(j)} = \psi_{j0_4} q^{04}, \tag{19}$$

where $j = 1, 2, \ldots, 2n$. Here the modal shapes are similar to those in V^{B_1}.

In the subspace V^{EK_1}, q^{k_1} and q^{k_2} can be solved from q^{EK_1}, hence

$$\delta^{(j)} = \psi_{jk_1} q^{k_1} + \psi_{jk_2} q^{k_2}, \tag{20}$$

where $j = 1, 2, \ldots, 2n$.

In the subspace V^{EK_2}, because $q^{k_3} = q^{k_1}$ and $q^{k_4} = q^{k_2}$, therefore,

$$\delta^{(j)} = \psi_{jk_3} q^{k_1} + \psi_{jk_4} q^{k_2}, \tag{21}$$

where $j = 1, 2, \ldots, 2n$.

The regularity of the modal shapes of duplicated frequencies is more complex and hence only its mathematical expressions are given as

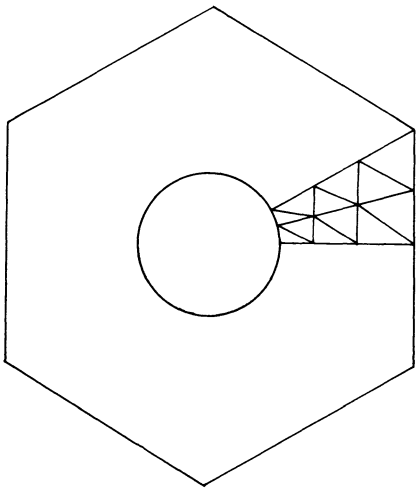

FIGURE 4 Finite element discretization of a C_{6v} plate.

Eqs. (20) and (21). It is easy to prove that the modal shapes described in (20) and (21) are orthogonal under C_s.

COMPUTATION AND EXPERIMENT

To verify the validity of the theoretical deductions above, programs were worked out and comparison made between the finite element and group theory methods. The example structure is a hexagonal plate with the inner boundary fixed and outer boundary free, as shown in Fig. 4.

The finite element discretization is made only in substructure $S^{(1)}$, and the others are obtained by applying the corresponding symmetry operations on them. For the sake of comparative calculation, special techniques were used for data input in the finite element method, that is, the nodal data input of $S^{(1)}$ is identical with that of the group

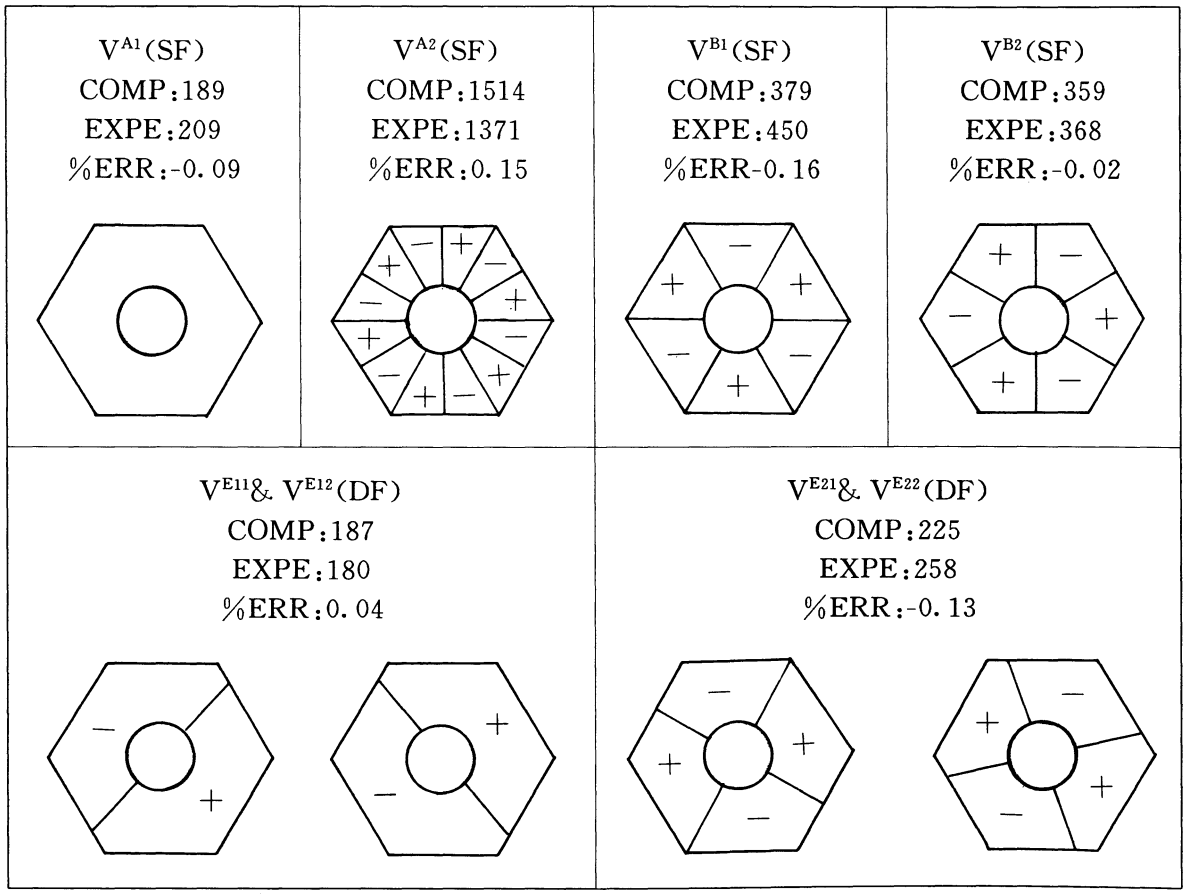

FIGURE 5 First-order frequency and modal shape in each subspace. SF, single frequency; DF, duplicated frequency; COMP, computational result; EXPE, experimental result; and ERR, error.

theory method. For the other substructures, the nodal data can be obtained automatically by that of $S^{(1)}$ through the program.

The calculation showed that the results of these two algorithms are identical but the finite element method used about 10 min while the group theory algorithm used less than 20 s. The computer storage needed by the group theory method is only one-tenth of that of the finite element method.

Having C_{6v} symmetry, the hexagonal plate has eight irreducible subspaces, i.e., four 1-dimensional subspaces V^{A_1}, V^{B_1}, V^{A_2}, and V^{B_2}; and four 2-dimensional subspaces $V^{E_{11}}$, $V^{E_{12}}$, $V^{E_{21}}$, and $V^{E_{22}}$. From the theoretical studies above, we can predict that vibrations in V^{A_1}, V^{B_1}, V^{A_2}, and V^{B_2}, must have single frequencies and modal shapes described by Eqs. (16)–(19) and vibrations in $V^{E_{11}}$ and $V^{E_{12}}$ as well as $V^{E_{21}}$ and $V^{E_{22}}$ must have duplicated frequencies and modal shapes described by Eqs. (20) and (21).

Computational and experimental results of the first-order frequency and modal shape in each subspace are shown in Fig. 5, in which the frequency unit is Hertz.

The results of frequency and modal shape testing also proved the validity of this method. The modal testing especially proves the theoretical prediction of modal characteristics quite well.

CONCLUSIONS

1. The eigenvalue problem of a structure with C_{nv} symmetry can be divided into much smaller subproblems by using group theory.
2. Calculation time as well as computer memory is reduced by using the method presented here.
3. The modal characteristics of C_{nv} structures derived from theoretical analysis are verified by modal testing.

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