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Grazing Orbits and Related Local Bifurcations of an Oscillator with Continuous and Piecewise-Linear Restoring Force

One critical case for the motion of a periodically excited oscillator with continuous and piecewise-linear restoring force is that the motion happens to graze a switching plane between two linear regions of the restoring force. This article presents a numerical scheme for locating the periodic grazing orbit first. Then, through a brief analysis, the article shows that the grazing phenomenon turns the stability trend of the periodic orbit so abruptly that it may be impossible to predict an incident local bifurcation with the variation of a control parameter from the concept of smooth dynamic systems. The numerical simulation in the article well supports the scheme and the analysis, and shows an abundance of grazing phenomena in an engineering range of the excitation frequency. © 1996 John Wiley & Sons, Inc.

INTRODUCTION

The dynamics of a periodically excited oscillator with piecewise-linear restoring force has drawn great attention in recent years, because the oscillator can model a large number of mechanical systems with clearances and elastic stops. Rattling gears, heat exchangers, and articulated mooring towers are but a few examples that fall into this category. Simple as the oscillator seems to be, it is not an easy task to gain insight into its complicated dynamics that many researchers, e.g., Natsiavas (1990) and Kleczka (1991), have studied from the viewpoint of smooth dynamic systems.

The vector field of an oscillator with piecewise-linear restoring force varies nonsmoothly from one linear region to another. If a periodic orbit of the oscillator starting from a certain initial state grazes a switching plane between two linear regions, the neighboring periodic orbit, due to a small variation of the initial state or a control parameter, either penetrates the switching plane or not. Thus, the oscillator may behave topologically different near a periodic grazing orbit. Such an orbit highlights the nature of an oscillator with piecewise-linear restoring force.

Nordmark (1991) and Foale and Bishop (1992) have intensively analyzed the new dynamics related to grazing orbits of an impact oscillator hav-

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ing very rigid stops with an instantaneous restitution law. The restoring force in their models is not continuous because the oscillator velocity jumps within an infinitely short time interval of collision. To the author's knowledge, the study on grazing phenomena of an oscillator with continuous and piecewise-linear restoring force remains rather untouched, although many elastic stops in engineering are not rigid enough to fit the instantaneous restitution law.

The purpose of this article is to study grazing phenomena of an oscillator with continuous and piecewise-linear restoring force, especially to determine the periodic grazing orbits, and to discuss their effect on local bifurcations from the viewpoint of numerical analysis based on the geometrical concepts of dynamic systems.

We will consider the motion of the oscillator subject to an excitation of period T_0 . The equation of motion described by state vector $\mathbf{v}$ yields

$$\dot{\mathbf{v}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ f(x, y, t, \lambda) \end{bmatrix} = \mathbf{f}(\mathbf{v}, t, \lambda) \quad (1)$$

where x and y are the displacement and the velocity of the oscillator, respectively; and $f(x, y, t, \lambda)$, which consists of the periodic excitation and a dissipative restoring force, is continuous and piecewise-linear with respect to state vector $\mathbf{v}$, sufficiently smooth with respect to time t and a control parameter λ , and of period T_0 as the excitation.

For the sake of simplicity, the term continuous and piecewise-linear oscillator (CPLO) will be used hereafter for the above oscillator. In addition, the symbol P - k will be used for the term period kT_0 of an orbit, where k is an integer.

Locating Periodic Grazing Orbits

To detect the existence of a periodic grazing orbit of the CPLO and to locate the orbit numerically, we concentrate on the case where all parameters of the oscillator are fixed. The path following of Foale and Thompson (1991) offers the method to trace the located phenomenon with the variation of control parameters.

If we define a globally transversal Poincaré section of the CPLO at a fixed time η , there follows the Poincaré mapping from the stroboscopic sampling

$$\begin{aligned} \mathbf{v}_{n+1}(\eta) &= \mathbf{P}(\mathbf{v}_n, \eta), \\ \mathbf{v}_n(\eta) &= \mathbf{v}(\eta + nT_0), \quad n = 0, 1, \dots \end{aligned} \quad (2)$$

Our task is to find a P - k fixed point $\mathbf{v}_o$ on a desired Poincaré section, say $\eta = 0$, such that the orbit starting from $\mathbf{v}_o$ grazes switching plane $x = x_s$ after a time delay η_s . An indirect but more efficient way to get $\mathbf{v}_o$ is to search for time η_s in time interval $[0, T_0]$ first, rather than the above $\mathbf{v}_o$ on the Poincaré section of two dimensions, so that grazing point $\mathbf{v}_s$ on the switching plane becomes the fixed point of the Poincaré mapping repeated k times

$$\mathbf{v}_k(\eta_s) = \mathbf{P}^k(\mathbf{v}_s, \eta_s) = \mathbf{v}_s = \begin{bmatrix} x_s \\ 0 \end{bmatrix}. \quad (3)$$

Then, $\mathbf{v}_o$ can be obtained by backward integrating Eq. (1) from $\mathbf{v}_s$ until the orbit arrives at the Poincaré section $\eta = 0$.

Before solving the nonlinear Eq. (3), consider the feature of the Poincaré mapping repeated k times

$$\mathbf{v}_k(\eta) = \mathbf{P}^k(\mathbf{v}_s, \eta) \quad \eta \in [0, T_0). \quad (4)$$

From the periodicity of the excitation and the solution continuity of Eq. (1) with respect to the time shift, the mapping determines a continuous and closed curve on the state plane when η varies in the interval $[0, T_0)$. It is easy to prove that the closed curve is an ellipse if the oscillator is linear and harmonically excited, and that the curve passes through point $\mathbf{v}_s$ at

$$\eta_s = \frac{1}{\omega} \left[\frac{\pi}{2} + \tan^{-1} \frac{2\zeta\omega}{1 - \omega^2} \right] \quad (5)$$

if and only if excitation amplitude e_0 yields

$$e_0 = x_s \sqrt{(1 - \omega^2)^2 + (2\zeta\omega)^2} \quad (6)$$

where ζ is the damping ratio and ω the dimensionless excitation frequency. When the oscillator is a CPLO, the closed curve, however, looks very complex and even intersects itself. The unknown existence of η_s and the complicated feature of the curve make it difficult to solve Eq. (3) by using traditional root solvers, such as the Newton–Raphson iteration scheme.

As an orbit of the CPLO near the grazing point is also of engineering interest, we turn to minimize the distance between grazing point $\mathbf{v}_s$ and the closed curve, rather than solve Eq. (3), from a more practical viewpoint

$$\min_{\eta \in [0, T_0)} \|\mathbf{v}_k(\eta) - \mathbf{v}_s\| < \delta, \quad k = 1, 2, \dots \quad (7)$$

where δ is the desired accuracy. This unconstrained optimization problem covers all possible periodic orbits that graze or almost graze the switching plane of concern.

After discretization of time interval $[0, T_0]$, the closed curve $\mathbf{v}_k(\eta)$, $\eta \in [0, T_0]$ becomes a polygon having vertices at $\mathbf{v}_k(\eta_j)$, $j = 1, 2, \dots, N$. The task is to judge whether the minimal distance between grazing point $\mathbf{v}_s$ and the polygon is less than tolerance δ .

Let x_j and y_j denote the displacement and the velocity corresponding to state vector $\mathbf{v}_k(\eta_j)$, respectively, and S_j denote the segment from $\mathbf{v}_k(\eta_j)$ to $\mathbf{v}_k(\eta_{j+1})$. The distance ρ_j between grazing point $\mathbf{v}_s$ and segment S_j depends on whether perpendicular line P_j from $\mathbf{v}_s$ to S_j intersects S_j or not

$$\rho_j = \begin{cases} \frac{|(x_s - x_j)(y_{j+1} - y_j) + (x_{j+1} - x_j)y_j|}{\sqrt{(x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2}}, & P_j \cap S_j \neq \emptyset \\ \min\{\sqrt{(x_j - x_s)^2 + y_j^2}, \sqrt{(x_{j+1} - x_s)^2 + y_{j+1}^2}\}, & P_j \cap S_j = \emptyset \end{cases} \quad (8)$$

while the intersection condition reads

$$\begin{aligned} & [(x_{j+1} - x_j)(x_s - x_j) - (y_{j+1} - y_j)y_j] \\ & [(x_{j+1} - x_j)(x_s - x_{j+1}) - (y_{j+1} - y_j)y_{j+1}] < 0. \end{aligned} \quad (9)$$

As soon as the minimal distance, say ρ_j , is in hand, there follows an approximate time η_s either from time η_j or η_{j+1} , or from an interpolation between them based on the intersection of P_j and S_j

$$\eta_s = \eta_j + \frac{|(x_{j+1} - x_j)(x_s - x_j) - (y_{j+1} - y_j)y_j|}{(x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2} (\eta_{j+1} - \eta_j). \quad (10)$$

When inequality (7) holds, we have a Poincaré section fixed at time $\eta = \eta_s$, where grazing point $\mathbf{v}_s$ is a P - k fixed point of the Poincaré mapping. The backward integration stated as above provides fixed point $\mathbf{v}_o$ on the Poincaré section $\eta = 0$ as usual.

LOCAL BIFURCATIONS INCIDENT TO GRAZING PHENOMENA

In this section we focus on the dynamics of a CPLO with the variation of control parameter λ in a neighborhood of grazing value λ_s , where a P -

k orbit of the oscillator grazes a switching plane at time t_s .

The path following (continuation) scheme is a popular technique of scanning the dynamics of a nonlinear system with the variation of a control parameter. To choose an appropriate forward step during path following, one can estimate the distance of the control parameter from the current value to an unknown bifurcation value by extrapolating a branching test function. As defined by Seydel (1988), the test function is real valued and it indicates possible bifurcations at its roots in the range of the control parameter. In tracing branches of a periodic orbit, the test function is usually based on the Jacobian of the Poincaré mapping.

More specifically, we set the Poincaré section at $\eta = 0$ as usual and rewrite the Poincaré mapping (2) to take control parameter λ into account

$$\begin{aligned} \mathbf{v}_{n+1}(\lambda) &= \mathbf{P}(\mathbf{v}_n, \lambda), \\ \mathbf{v}_n &= \mathbf{v}(nT_0), \quad n = 0, 1, \dots \end{aligned} \quad (11)$$

Let $\mathbf{V}(t)$ denote the Jacobian of state vector $\mathbf{v}(t)$ with respect to initial state vector $\mathbf{v}_0$ and $\mathbf{J}(\mathbf{v}, \lambda)$ denote the Jacobian of the Poincaré mapping with respect to $\mathbf{v}$. If $\mathbf{J}(\mathbf{v}, \lambda)$ exists, it will be $\mathbf{V}(T_0)$ through integrating the following variation equation

$$\begin{cases} \dot{\mathbf{V}}(t) = D_v \mathbf{f}(\mathbf{v}, t, \lambda) \mathbf{V}(t) \\ \mathbf{V}(0) = \mathbf{I} \end{cases} \quad (12)$$

where

$$\mathbf{V}(t) = \begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} \end{bmatrix} \quad D_v \mathbf{f}(\mathbf{v}, t, \lambda) = \begin{bmatrix} 0 & 1 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}. \quad (13)$$

The continuity and the piecewise linearity of $\mathbf{f}(\mathbf{v}, t, \lambda)$ with respect to $\mathbf{v}$ guarantee the existence of $\mathbf{V}(t)$ for $t \geq 0$, as well as that of $\mathbf{J}(\mathbf{v}, \lambda)$.

To obtain the derivative of Jacobian $\mathbf{J}(\mathbf{v}, \lambda)$ with respect to control parameter λ , one can differentiate Eq. (12) with respect to λ

$$\begin{cases} \frac{\partial \dot{\mathbf{V}}(t)}{\partial \lambda} = D_v \mathbf{f}(\mathbf{v}, t, \lambda) \frac{\partial \mathbf{V}(t)}{\partial \lambda} + [D_{vv}^2 \mathbf{f}(\mathbf{v}, t, \lambda) \frac{\partial \mathbf{v}}{\partial \lambda} + D_{v\lambda}^2 \mathbf{f}(\mathbf{v}, t, \lambda)] \mathbf{V}(t) \\ \frac{\partial \mathbf{V}(0)}{\partial \lambda} = 0. \end{cases} \quad (14)$$

Table 1. Periodic Grazing Orbits of CPLO in Frequency Range $\lambda \in [0.2, 5.0]$

λ	0.2298	0.7163	0.7176	0.7205	0.7230	0.7600	2.870	3.372	4.250
Orbits	S-1	U-3	U-3	U-3	U-3	U-3	S-3	S-2	S-3

If the derivative of $\mathbf{J}(\mathbf{v}, \lambda)$ with respect to λ exists, it will be the integral of Eq. (14) at time T_0 . Because a trajectory of the CPLO with control parameter λ_s grazes a switching plane at time t_s , the trajectory of the oscillator, when λ slightly differs from λ_s , either penetrates or does not touch the switching plane at any time near t_s . Furthermore, the second derivative $D_{vv}^2 \mathbf{f}(\mathbf{v}, t, \lambda)$ in Eq. (14) contains the Dirac functions, the integrals of Eq. (14) along two orbits determined by λ_s^+ and λ_s^- do not reach the same matrix when λ_s^+ and λ_s^- approach λ_s from different sides. It follows that the derivative of Jacobian $\mathbf{J}(\mathbf{v}, \lambda)$ with respect to control parameter λ does not exist at grazing value λ_s .

Now we go back to the test function. A simple but widely used test function for predicting saddle-node bifurcations reads

$$\tau(\mathbf{v}, \lambda) = \det(\mathbf{J}(\mathbf{v}, \lambda) - \mathbf{I}). \quad (15)$$

Obviously, the smoothness of test function $\tau(\mathbf{v}, \lambda)$ with respect to λ depends on that of Jacobian $\mathbf{J}(\mathbf{v}, \lambda)$. But from the above analysis, if a trajectory starting from $\mathbf{v}$ grazes a switching plane, the Jacobian $\mathbf{J}(\mathbf{v}, \lambda)$ is not smooth with respect to λ , neither is test function $\tau(\mathbf{v}, \lambda)$. Hence, the stability trend of the oscillator may abruptly change at the grazing value of λ , especially when the slope of the restoring force jumps dramatically on the switching plane. As a result, the CPLO may suddenly undergo a local bifurcation near the grazing phenomenon that the above test function cannot predict properly from extrapolation.

NUMERICAL SIMULATION

To support the analysis and the numerical scheme, a great number of numerical simulations were carried out. Presented here is a typical one for the sake of simplicity. The motion of the CPLO after scaling yields

$$\begin{cases} \dot{x} = y \\ \dot{y} = -g(x) - 0.1y + 10.0 \sin \lambda t \end{cases} \quad (16)$$

where trilinear function $g(x)$ describes the elastic restoring force caused by a pair of symmetrical elastic constraints

$$g(x) = \begin{cases} x, & |x| \leq 1, \\ 5.0x - 4.0 \operatorname{sgn}(x), & |x| > 1, \end{cases} \quad (17)$$

and excitation frequency $\lambda \in [0.2, 5.0]$ acts as the control parameter.

In the simulation the proposed numerical scheme was first used to search for the periodic grazing orbits of less than 1/5 subharmonics in the excitation frequency range. Then, the path following scheme was run to trace the branch of the located periodic orbit in the vicinity of the grazing frequency.

Table 1 lists the periodic grazing orbits found by the proposed scheme, where $S-k$ and $U-k$ denote the stable and unstable $P-k$ orbits, respectively. Figure 1 shows two typical periodic orbits that graze the switching plane(s) plotted in the dashed line(s). These observed periodic grazing

FIGURE 1 Two typical periodic grazing orbits of the CPLO: left, $\lambda = 0.2298$, $P-1$ orbit; right, $\lambda = 3.372$, $P-2$ orbit.

orbits indicate that if the excitation amplitude is large enough, the CPLO always possesses a number of periodic grazing orbits in a practical range of the excitation frequency.

An interesting observation in path following would be the stable branch of the P -1 orbit with slow decrease of the excitation frequency from $\lambda = 0.2300$ to $\lambda = 0.2297$. Figure 2 shows the test function of the CPLO in this interval by solid curve. Before λ reached 0.2298, the test function indicated that the CPLO was far from any bifurcations of the saddle-node type. At $\lambda \doteq 0.2298$, however, the P -1 orbit grazed two switching planes symmetrically as shown in Fig. 1 (left). Afterward the test function turned the stability trend of the oscillator abruptly and vanished at $\lambda \doteq 0.2297$, where a saddle-node bifurcation occurred. Only a small variation less than 0.05% of the excitation frequency totally denied the original estimation from the test function. This example fully shows the severity of grazing phenomena.

Very often in practice the corners of the restoring force of a CPLO with respect to the displacement can be locally smoothed, or the system model possesses the continuous and piecewise-analytic restoring force. In these cases, the vector field of the model is not piecewise-linear, but piecewise-nonlinear instead. Although the attention above is paid to the dynamics of a CPLO, the analysis to the local bifurcations connected with grazing orbits is obviously valid for any oscillators with a continuous and piecewise-analytic vector field, and so is the numerical scheme for locating periodic grazing orbits. In Figure 2, the dashed curve shows the test function of the locally smoothed model where a polynomial of order four was used to smooth every corner of the restoring force within

the range of ± 0.03 . This evidence of smooth test function encourages one to use locally smoothed models to avoid the difficulty caused by grazing phenomena in the bifurcation prediction.

CONCLUDING REMARKS

1. A periodically excited oscillator with continuous and piecewise-linear restoring force possesses abundant grazing phenomena where the periodic orbit of the oscillator happens to graze a switching plane between two linear regions of the restoring force. The numerical scheme proposed in the study can find all possible periodic grazing orbits with desired subharmonics.
2. The nonsmooth nature of the restoring force results in the nonsmooth Jacobian of the Poincaré mapping with respect to a control parameter at certain values where grazing phenomena occur. As a result, the oscillator may unexpectedly undergo a local bifurcation when the control parameter varies slightly near one of those values.
3. In the numerical simulation, the path following based on the concept of smooth dynamic systems failed to predict local bifurcations. A simple way to avoid the failure is to modify the model of the piecewise-linear restoring force by smoothing its corners locally.

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FIGURE 2 The test functions in tracing a P -1 orbit (—) CPLO model and (---) smoothed model.

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