Review

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Procedures to Compute Maximum Structural Responses from Predictions or Measurements at Selected Points

It is common to establish shock and vibration design and/or test criteria for equipment mounted on a structure by computing a conservative upper bound for the spectrum of the dynamic load induced response of the structure based upon predicted or measured spectra at various points. This task is usually accomplished by one of five procedures, namely, the computation from the available spectra of a simple envelope, a normal tolerance limit, a distribution-free tolerance limit, an empirical tolerance limit, or a normal prediction limit. These five procedures are reviewed and illustrated using the power spectra computed from vibration signals measured during the lift-off of a large launch vehicle at 12 locations in a structural region where equipment will be mounted. The results are compared and the merits and liabilities of the various procedures are discussed. © *1996 John Wiley & Sons, Inc.*

INTRODUCTION

The procedures commonly used to predict the response of structural systems to low frequency dynamic loads (below about the 10th normal mode of the structure) generally produce motions at the specific locations and/or forces in the specific structural members of interest in the formulation of design and test criteria. On the other hand, many procedures used to predict the structural responses to higher frequency dynamic loads, as well as the direct measurement of the responses on an existing structural system, yield only spectra for motions at point locations. In some cases, the response predictions or measurements might be made at the specific locations of interest in the formulation of design and/or test criteria, e.g., the attachment points for equipment mounted on the structure. In most cases, however, the locations for response predictions or measurements do not correspond exactly to the specific points of interest; even if they do, the equipment to be mounted at those points is often not modeled for the predictions or is not in place for the measurements. Hence, it is necessary to pool the predicted or measured responses at various locations to obtain a single spectrum that will conservatively bound the spectra for the responses at all points in a structural region, including the specific points of interest. This approach requires that the structural system be divided into a collection of structural regions, often referred to as "zones."

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A conservative bound on the spectra for the responses at all points in a single zone is commonly called the zone "limit."

BACKGROUND

Before detailing the various procedures for establishing limits on the dynamic load induced structural responses in a zone, several issues should be clarified, including the definition of zones, determination of zones, unbiased selection of response points, and definitions of response terms.

Definition of Zones

Assume the high frequency dynamic load induced responses at all points on a structural system (e.g., an aerospace or automotive vehicle) are of interest. The responses may be stationary or nonstationary random vibrations, periodic or other deterministic vibrations, or transients including pyroshocks. In any case, the responses at various points on the structure will typically vary widely from one location to another. The goal in zoning is to divide the structure into regions or zones such that the responses at all points within each zone are reasonably homogeneous, meaning the spectra for the responses at all points can be described by a single spectrum that will exceed most or all of the spectra at the individual points without severely exceeding the spectrum at any point. It is also required that the selected zones correspond to structural regions of interest in the formulation of design and test criteria. For example, if the ultimate goal is the definition of vibration inputs to equipment that will be mounted on the structure, as required to derive test specifications, a single zone should include all the attachment points for at least one equipment item, and preferably for several equipment items. On the other hand, it is not necessary for a zone to be a single contiguous structural region. For example, all frames of a given size in a structural system, no matter where they are located, might constitute a single zone if the responses of those frames are similar.

Determination of Zones

There are direct analytical techniques that can be used to arrive at structural zones based upon tests for the statistical equivalence of the spectra for the predicted or measured responses at various locations on the structure (e.g., Bendat and Piersol, 1986). However, the zoning operation is usually accomplished based upon engineering judgment, experience, and/or a cursory evaluation of predicted or measured spectra. For example, engineering judgment dictates that frame and panel structures should represent different zones, because the response of light panels will generally be higher than the response of heavier frames. For aerospace vehicles, experience suggests that the aft structural regions exhibit higher responses during flight than the forward structural regions, meaning even similar structures in the forward and aft regions of the vehicle usually must be represented by different zones. Responses normal to the surface of a panel structure are generally higher than those in the plane of the panel, so the responses along different orthogonal axes might be divided into separate zones. Beyond such engineering considerations, a visual inspection of the spectra for the predicted or measured responses over the structure can be used to group locations with responses that have similar spectral magnitudes.

Unbiased Selection of Response Points

All of the procedures to establish zone limits assume the available spectra for a given zone are predicted or measured at locations that are representative of all points of interest in that zone. Ideally, this would be achieved by a random selection from all possible response points within the zone. In practice, a random selection usually is not feasible because the predictions or measurements are commonly made before the zones are selected; in fact, the spectra for the predicted or measured responses are often used to establish the zones, as discussed above. In some cases, however, the predictions or measurements may be made at those points where equipment items are mounted. If the ultimate goal is to establish test criteria for those equipment items, this would constitute a good selection of response points, even though such mounting points might not be representative of all points within the zone. In any case, it is important to assess the locations represented by the available predicted or measured spectra to assure that they are typical of all points of interest in the zone.

Definitions of Response Terms

To simplify later developments and discussions, the following terms are defined. The spectrum

(spectral value versus frequency) for the dynamic load induced structural response at any point within the zone of interest is x. Assuming the structural response is predicted or measured in terms of acceleration, the spectrum may be a power spectrum (also called an autospectrum) in g^2/Hz for a random vibration response, an energy spectrum in g^2-sec/Hz for a transient response, a line spectrum in g for a periodic vibration response, or a shock response spectrum in g for a shock response, depending on the type of dynamic environment and the spectral analysis procedure employed. See Himelblau et al. (1994) for details on the spectral descriptions of structural responses. The spectrum for the dynamic load induced structural response that is predicted or measured at the ith point ($i = 1, 2, \ldots, n$) within the zone of interest is x_i. The spectral value in the jth frequency resolution bandwidth ($j = 1, 2, \ldots, m$) for the dynamic load induced structural response that is predicted or measured at the ith point ($i = 1, 2, \ldots, n$) within the zone of interest is x_{ij}. The largest spectral value in the jth frequency resolution bandwidth for the predicted or measured dynamic load induced structural responses at the n points where predictions or measurements are made within the zone of interest, x_{hj}, is the largest value of x_{ij} ($i = 1, 2, \ldots, n$).

ENVELOPE LIMITS

The most common way to arrive at a limit for the structural response spectra within a zone is to superimpose the spectral responses predicted or measured at n points within that zone, x_{ij} ($i = 1, 2, \ldots, n$), and then to select and plot the maximum spectral values in the various frequency resolution bandwidths, i.e., x_{hj} ($j = 1, 2, \ldots, m$). This yields an envelope that, for simplicity, is often smoothed using a series of straight lines (usually no more than seven lines with slopes of 0, ±3dB/octave, or ±6dB/octave). The computation of unsmoothed and smoothed envelopes is illustrated in Fig. 1 using the power spectra for 12 vibration measurements made at different locations in a selected structural zone of a large launch vehicle during lift-off.The individual spectra were all computed using a one-sixth octave band frequency resolution.

The envelope approach is simple to apply, but it poses two significant problems.

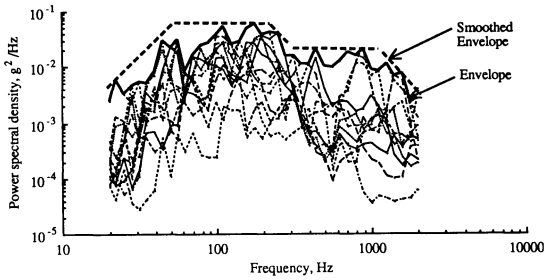

FIGURE 1 Envelope for 12 measured vibration response spectra within a zone.

1. It does not provide a specific probability that the envelope at a given frequency will exceed the spectrum for the response at some other location of interest. However, if the number of predicted or measured spectra is sufficiently large, this deficiency can be removed by a distribution-free tolerance limit computation, to be discussed later.
2. It can produce somewhat different results depending on the frequency resolution of the predicted or measured spectra; i.e., spectra computed in narrow bandwidths will display substantially higher peak magnitudes, and thus higher envelope values, than spectra computed in, say, one-third octave bandwidths.

There is a final problem that applies to all procedures. Specifically, for the usual case where the envelope is smoothed by a series of straight lines, the procedure becomes somewhat subjective in terms of the number of straight lines that are used, and whether all spectral peaks are enveloped or some are clipped. This problem is often addressed by using some systematic method for determining which spectral peaks will be either fully covered or partially clipped by the smoothed envelope. For example, a commonly used rule is that all narrowband spectral peaks should be clipped by 3 dB. Another rule is that all spectral peaks with a bandwidth less than 5% of center frequency should be clipped to the level where the bandwidth is 5% of center frequency. Perhaps the best approach is to compute all spectra with a resolution bandwidth that is proportional to frequency, say one-sixth octave bands, and then envelope all peaks without clipping, as is done in Fig. 1.

NORMAL TOLERANCE LIMITS

A more definitive way to arrive at a conservative limit for the structural response spectra within a zone is to compute a normal tolerance limit for the predicted or measured spectra in each frequency resolution bandwidth. Normal tolerance limits apply only to normally distributed random variables. The point-to-point (spatial) variation of structural responses to stationary, nonstationary, and transient dynamic loads is generally not normally distributed. However, there is considerable empirical evidence (e.g., Piersol, 1971) that the logarithm of the spectral values for any motion parameter describing the response or structures from one point to another does have an approximately normal distribution; i.e., the spatial distribution of structural response spectra in a specific frequency resolution bandwidth approximately fits a lognormal distribution. Hence, by simply making the logarithmic transformation

$$y = \log_{10} x, \tag{1}$$

a normal tolerance limit can be computed for the spectral values in the jth frequency resolution bandwidth from the transformed predictions or measurements, y_{ij} $(i = 1, 2, \ldots, n)$. Specifically, the normal tolerance limit for y_j, denoted by $\mathrm{NTL}_y(n, \beta, \gamma)$, is defined as the value that will exceed *at least* the β portion of all possible values of y_j with a confidence coefficient of γ, and is given by

$$\mathrm{NTL}_y(n, \beta, \gamma) = \bar{y}_j + k_{n,\beta,\gamma} \quad s_{y_j}, \qquad j = 1, 2, \ldots, m, \tag{2}$$

where $\bar{y}_j$ is the sample average and s_{y_j} is the sample standard deviation of y in the jth frequency resolution bandwidth, given by

$$\bar{y}_j = \frac{1}{n} \sum_{i=1}^{n} y_{ij}; \quad s_{y_j} = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (y_{ij} - \bar{y}_j)^2}. \tag{3}$$

The normal tolerance limit in the original engineering units of x can be retrieved by

$$\mathrm{NTL}_x(n, \beta, \gamma) = 10^{\mathrm{NTL}_y(n,\beta,\gamma)}. \tag{4}$$

In Eq. (2), the term $k_{n,\beta,\gamma}$ is called the normal tolerance factor, and is a tabulated value; a tabulation of $k_{n,\beta,\gamma}$ for selected values of n, β, and γ is presented in Table 1, which is extracted from Bowker and Lieberman (1975) and other sources. Note in Table 1 that for $n = \infty$,

$$\mathrm{NTL}_y(\beta) = \mu_{y_j} + z_\alpha \sigma_{y_j}, \quad \alpha = 1 - \beta, \tag{5}$$

where μ_{y_j} is the true mean and σ_{y_j} is the true standard deviation of y in the jth frequency resolution bandwidth, and z_α is the $\alpha = (1 - \beta)$ percentage point of the standardized normal distribution, which is tabulated in any statistics textbook (e.g., Guttman et al., 1982). The confidence coefficient associated with Eq. (5) is 100%, independent of the value of α. Also, Eq. (5) is sometimes routinely substituted for Eq. (2) with $\bar{y}_j$ and s_{y_j} replacing μ_{y_j} and σ_{y_j}, but this should never be done unless the number of available predictions or measurements within the zone is $n > 50$.

As an illustration, the normal tolerance limit versus frequency for the data in Fig. 1 computed

Table 1. Normal Tolerance Factors for Upper Tolerance Limit

	$\gamma = 0.50$			$\gamma = 0.75$			$\gamma = 0.90$		
n	$\beta = 0.90$	$\beta = 0.95$	$\beta = 0.99$	$\beta = 0.90$	$\beta = 0.95$	$\beta = 0.99$	$\beta = 0.90$	$\beta = 0.95$	$\beta = 0.99$
3	1.50	1.94	2.76	2.50	3.15	4.40	4.26	5.31	7.34
4	1.42	1.83	2.60	2.13	2.68	3.73	3.19	3.96	5.44
5	1.38	1.78	2.53	1.96	2.46	3.42	2.74	3.40	4.67
7	1.35	1.73	2.46	1.79	2.25	3.13	2.33	2.89	3.97
10	1.33	1.71	2.42	1.67	2.10	2.93	2.06	2.57	3.53
15	1.31	1.68	2.39	1.58	1.99	2.78	1.87	2.33	3.21
20	1.30	1.67	2.37	1.53	1.93	2.70	1.76	2.21	3.05
30	1.29	1.66	2.35	1.48	1.87	2.61	1.66	2.08	2.88
50	1.29	1.65	2.34	1.43	1.81	2.54	1.56	1.96	2.74
∞	1.28	1.64	2.33	1.28	1.64	2.33	1.28	1.64	2.33

with $\beta = 0.95$ and $\gamma = 0.50$ is shown in Fig. 2. This limit is commonly referred to as the 95% normal tolerance limit with 50% confidence, or simply the 95/50 limit, and is interpreted as the limit that will exceed the response spectral values for at least 95% of all points within the zone with 50% confidence. Note the 95/50 limit in Fig. 2 is similar to the envelope for the 12 measurements at most frequencies, but it could be higher or lower than the envelope depending on the values of β and γ. Also, computed normal tolerance limits are sometimes smoothed by a series of straight lines, as illustrated in Fig. 2. This step involves subjective judgments that pose the problems discussed earlier for the envelope procedure.

The normal tolerance limit provides two major advantages over the envelope approach.

1. It provides a limit that will not be exceeded at each frequency by the spectral values for a well-defined fractional portion β of all locations in the zone with a well-defined confidence coefficent γ.
2. Because the limit is computed using average values (the sample average and standard deviation), it is not as sensitive as the envelope to the frequency resolution of the predicted or measured spectra.

On the other hand, the procedure does pose one major problem, namely, its sensitivity to the assumption that the spatial distribution of the response spectral values in each frequency resolution bandwidth is lognormal within the zone. This assumption might come into question for the higher values of β and/or γ.

Distribution-Free Tolerance Limits

Distribution-free tolerance limits circumvent the primary problem associated with normal tolerance limits, namely, the lognormal assumption for the spatial distribution of the structural response spectral values x_{ij} within the zone. This assumption is eliminated by fixing the tolerance limit to be the maximum spectral value x_{hj} in each of the m frequency resolution bandwidths, i.e., the distribution-free tolerance limit, denoted by $\mathrm{DFL}_x(n, \beta, \gamma)$, is simply the unsmoothed envelope of the predicted or measured spectra. From Guttman et al. (1982), the fraction portion β of all response spectral values that will be less than the maximum spectral value x_{hj} with a confidence coefficient γ, can be determined using order statistics to be

$$\mathrm{DFL}_x(n, \beta, \gamma) = x_{hj}, \quad \gamma = 1 - \beta^n. \tag{6}$$

The interpretation of distribution-free tolerance limits is the same as for normal tolerance limits, i.e., x_{hj} is the value that will exceed *at least* the β portion of all possible values of x_{ij} with a confidence coefficient of γ. For example, the unsmoothed envelope for the data in Fig. 1 represents the tolerance limit that will exceed the response spectral values for *at least* 95% of all points in the zone with 46% confidence. As for the envelope procedure, the computed distribution-free tolerance limits are sometimes smoothed with a series of straight lines. This smoothing is illustrated in Fig. 1. Again, this step involves subjective judgments that pose the problems discussed earlier for the envelope procedure.

The primary advantage of the distribution-free tolerance limit is that it provides a statistical bound that is not sensitive to the spatial distribution of the response spectral values within the zone. However, it does pose two problems.

1. The procedure does not permit an independent selection of the values for β and γ. The usual approach is to select a desired value for the fractional portion β, and accept

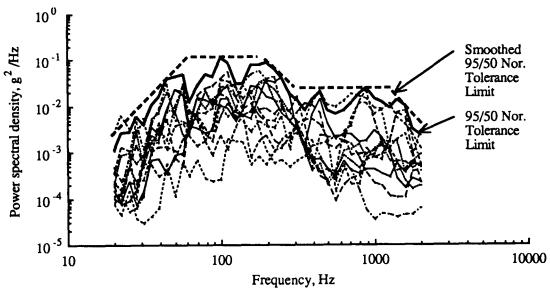

FIGURE 2 95/50 normal tolerance limit for vibration response spectra in Fig. 1.

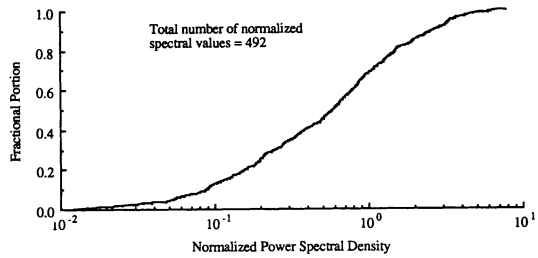

FIGURE 3 Empirical distribution for normalized vibration response spectra in Fig. 1.

whatever value of the confidence coefficient γ that results from Eq. (6). However, for a small number of predicted or measured spectral values, say $n \leq 13$, the confidence coefficient will be $\gamma < 0.5$ for values of $\beta \geq 0.95$.

2. As for the envelope approach, the procedure can produce somewhat different results depending on the frequency resolution of the predicted or measured spectra.

EMPIRICAL TOLERANCE LIMITS

A third method of selecting a conservative limit for the structural response spectra within a zone is to determine an empirical distribution function for the predicted or measured spectral values, and then select some large percentile x_β from the distribution (the value of x that exceeds β portion of the available values) to be a conservative limit, referred to as an empirical tolerance limit. Ideally, this would be done independently for the spectral values in each frequency resolution bandwidth. In practice, it is rare to have predictions or measurements at enough locations to allow an independent determination of a large percentile for the values in each frequency resolution bandwidth; i.e., for $\beta = 0.95$, at least 20 values are needed to determine a limit. Hence, it is common to normalize the spectral values in each frequency resolution bandwidth to a common mean value, and then pool all the spectral values to arrive at a single distribution function for the determination of a limit that covers β portion of the values. Of course, this pooling approach assumes the spatial distribution of the spectral values is the same in all the frequency resolution bandwidths.

Given a total of n predicted or measured spectra within the zone covering m frequency resolution bandwidths, $x_{ij}(i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m)$, the specific procedure is as follows:

1. Compute the average of the spectral values in each frequency resolution bandwidth, i.e.,

$$\bar{x}_j = \frac{1}{n} \sum_{i=1}^{n} x_{ij}. \tag{7}$$

2. Divide the spectral values in each frequency resolution bandwidth by the average in that bandwidth to obtain the normalized values u_{ij}, i.e.,

$$u_{ij} = \frac{x_{ij}}{\bar{x}_j}. \tag{8}$$

3. Pool the normalized values in all frequency resolution bandwidths to obtain a single set of values u_k $(k = 1, 2, \ldots, nm)$, and rank order the pooled normalized values from the smallest to the largest to obtain the set $u_{(k)}$, i.e.,

$(k) = 1$ is the smallest value of u_k,

$(k) = 2$ is the next smallest value of u_k,

. . .

$(k) = nm$ is the largest value of u_k.

4. Select that normalized value u_β that covers β portion of all the pooled normalized values, i.e.,

$$u_\beta = u_{(k)} \quad \text{where} \quad \frac{(k)}{nm} = \beta. \tag{10}$$

5. Determine the limit value $x_{\beta j}$ in each frequency resolution bandwidth by multiplying the average for each bandwidth by the normalized limit u_β to obtain the empirical tolerance limit, denoted by $\mathrm{ETL}_x(\beta)$, i.e.,

$$\mathrm{ETL}_x(\beta) = x_{\beta j} = u_\beta \bar{x}_j. \tag{11}$$

As an illustration, the empirical distribution for the normalized values of the spectral data in Fig. 1 is detailed in Fig. 3, and the 95% empirical tolerance limit is shown in Fig. 4. Note that the distribution in Fig. 3 is close

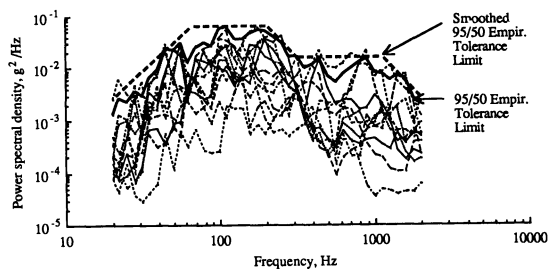

FIGURE 4 95/50 empirical tolerance limit for vibration response spectra in Fig. 1.

to lognormal, but the empirical tolerance limit approach is applicable to data with any spatial distribution. The multiplier determined from the rank-ordered, normalized spectral values in Fig. 3 to arrive at the 95% limit in Fig. 4 is $u_\beta = 3.37$. Of course, this value of u_β is an estimate that is computed from a statistical sample of 492 values and, hence, may be larger or smaller than the true value, v_β, for all points in the zone. The statistical variability of the estimate u_β is governed by the binomial probability function (see Guttman et al. 1982) and, hence, an upper bound, $U_\beta = Cu_\beta (C > 1)$, could be determined to produce a limit in Eq. (11) that exceeds the spectral values at β portion of the points in the zone with a high level of confidence. In practice, however, it is customary to use the value of u_β computed from Eq. (10) without modification, meaning the confidence coefficient associated with the limit in Eq. (11) is essentially $\gamma = 0.50$; i.e., the empirical tolerance limit derived in Eq. (11) is the value of x that will exceed the response spectral values at the 95/50 limit. As for the other procedures, the computed empirical tolerance limits are sometimes smoothed by enveloping with a series of straight lines. This smoothing is illustrated in Fig. 4. Again, this step involves subjective judgments that pose the problems discussed earlier for the envelope procedure.

Like the distribution-free tolerance limit, the empirical tolerance limit does not require a specific probability function to be assumed for the spatial distribution of the response spectral values within the zone. Also, like the normal tolerance limit, it is not as sensitive as the envelope to the frequency resolution of the predicted or measured spectra. However, the procedure does pose some other problems.

1. If the spectral values in all frequency resolution bandwidths are pooled to arrive at the empirical distribution, it is sensitive to the assumption that the spatial distribution is the same in all bandwiths.
2. It is most effective when response predictions or measurements are available at a reasonable number of points (generally, $n > 10$), and further requires extensive computations.
3. It provides a limit with a confidence coefficient of $\gamma = 0.50$ only, unless further extensive computations are performed.

NORMAL PREDICTION LIMITS

The final way to arrive at a conservative limit for the structural response spectra within a zone is to compute a normal prediction limit for the predicted or measured spectra in each frequency resolution bandwidth. As for normal tolerance limits, normal prediction limits apply only to normally distributed random variables. Hence, the logarithmic transformation in Eq. (1) is again required to obtain an approximately normal distribution for the transformed spectral values, $y_{ij} = 1, 2, \ldots, n$. The normal prediction limit, denoted by $\mathrm{NPL}_y(n, \gamma)$, is defined as the value that will exceed *the next* predicted or measured value of y_j with a confidence coefficient of γ, and is given by

$$\mathrm{NPL}_y(n, \gamma) = \bar{y}_j + \sqrt{1 + \frac{1}{n}}\, t_{(n-1);\alpha} s_{y_j}, \qquad (12)$$

$$\alpha = 1 - \gamma, \quad j = 1, 2, \ldots, m,$$

where $\bar{y}_j$ is the sample average and s_{y_j} is the sample standard deviation of y in the jth frequency resolution bandwidth, as defined in Eq. (3), and $t_{(n-1);\alpha}$ is the $\alpha = 1 - \gamma$ percentage point of Student's t variable with $n - 1$ degrees of freedom, which is tabulated in any statistics textbook. The normal prediction limit in the original engineering units of x can be retrieved by

$$\mathrm{NPL}_x(n, \gamma) = 10^{\mathrm{NPL}_y(n,\gamma)}. \qquad (13)$$

The normal prediction limit should not be confused with the normal tolerance limit given by Eq. (4), which defines an upper limit that will exceed *at least* β portion of *all* possible values of x with a confidence coefficient of γ.

Very few statistics textbooks cover normal prediction limits, so a brief derivation follows. Assume y is a normally distributed random variable that is sampled to obtain n predicted or measured values, y_i $(i = 1, 2, \ldots, n)$. Now let a new random variable v be defined as the difference between the next value of y that will be predicted or measured $(y_i, i = n + 1)$ and the average of the previous n values of y $(y_i = 1, 2, \ldots, n)$, i.e.,

$$v = y_{(n+1)} - \frac{1}{n}\sum_{i=1}^{n} y_i = y_{(n+1)} - \bar{y}. \qquad (14)$$

The mean value of v is zero, i.e.,

$$\mu_v = E[v] = E[y_{(n+1)}] - E[\bar{y}] = \mu_y - \mu_y = 0, \qquad (15)$$

where $E[\ \ ]$ denotes the expected value operation. The standard deviation of v is given by

$$\sigma_v = \sqrt{\sigma^2_{y_{(n+1)}} + \sigma^2_{\bar{y}}} = \sqrt{\sigma^2_y + \frac{\sigma^2_y}{n}} = \sigma_y \sqrt{1 + \frac{1}{n}}. \tag{16}$$

Assuming v is normally distributed, as it must be if y is normally distributed, the standardized value of v in terms of sample values, with $\mu_v = 0$, is

$$\frac{v}{s_v} = t_{(n-1)} = \frac{y_{(n+1)} - \bar{y}}{s_y \sqrt{1 + \frac{1}{n}}}, \tag{17}$$

where $t_{(n-1)}$ is Student's t variable with $n - 1$ degrees of freedom. Rearranging the terms in Eq. (17) yields

$$y_{(n+1)} = \bar{y} + \sqrt{1 + \frac{1}{n}}\, t_{(n-1)} s_y, \tag{18}$$

which gives the normal prediction limit in Eq. (12) for that α percentage point of Student's t variable corresponding to the desired confidence coefficient $\gamma = 1 - \alpha$. It should be mentioned that γ is sometimes referred to as a probability in this application, but it is more correctly referred to as a confidence coefficient because the magnitude of the limit is dependent upon the values of the predicted or measured data.

As an illustration, the normal prediction limit versus frequency for the data in Fig. 1 computed with $\gamma = 0.95$ is shown in Fig. 5. This limit is commonly referred to as the 95% prediction limit for the spectral values of the next predicted or measured response at a randomly selected point within the zone. Note that the 95% normal prediction limit in Fig. 5 falls slightly above the 95/50 normal tolerance limit for the same spectral data in Fig. 2 at all frequencies. As for tolerance limits, normal prediction limits are often smoothed by enveloping with a series of straight lines, as illustrated in Fig. 5. Again, this step involves subjective judgments that pose the problems discussed earlier for the envelope procedure.

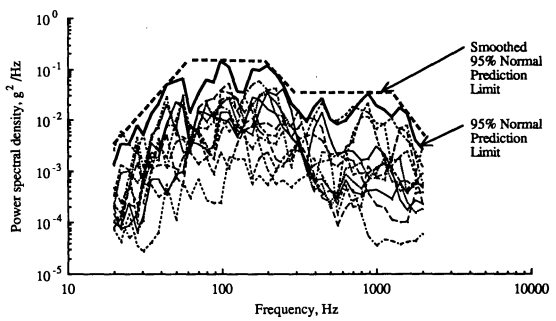

FIGURE 5 95% normal prediction limit for vibration response spectra in Fig. 1.

Equation (12) defines the prediction limit for the next value of y [i.e., $y_{(n+1)}$] based upon the sample average and standard deviation for the previous n values, y_i ($i = 1, 2, \ldots, n$). This equation can be used to establish the limit that will exceed the next two values of y [i.e., $y_{(n+1)}$ and $y_{(n+2)}$] by simply squaring γ. For example, if the $\mathrm{NPL}_y(n, \gamma)$ in Eq. (12) is determined for the next value of y with a confidence coefficient of $\gamma = 0.95$, that same limit will apply to the next two values of y with a confidence coefficient of $\gamma = 0.95^2 \approx 0.90$. Conversely, if a limit for the next two values of y with a confidence coefficient of $\gamma = 0.95$ is desired, Eq. (12) can be solved for $\mathrm{NPL}_y(n, \gamma)$ with a confidence coefficient of $\gamma = \sqrt{0.95} \approx 0.975$. The same procedure can be used to determine a normal prediction limit for any number of future values of y. It should be noted, however, that the normal prediction limit is unbounded as the number of values n becomes large.

The normal prediction limit is somewhat simpler to apply than any of the tolerance limits because it involves only one probability related parameter, namely, a confidence coefficient γ. Also, like the normal tolerance limit, it is not as sensitive as the envelope to the frequency resolution of the predicted or measured spectra. On the other hand, the normal prediction limit poses two important problems.

1. The procedure is sensitive to the assumption that the spatial distribution of the response spectral values in each frequency resolution bandwidth is lognormal within the zone. This assumption might come into question for the higher values of γ.
2. The procedure provides a conservative limit for only one or some other specific number of future response predictions or measurements at randomly selected points within the zone, rather than to the responses at all locations within the zone.

ASSESSMENTS

Using the $n = 12$ vibration measurements with the power spectra shown in Fig. 1, the five different methods for arriving at conservative limits for random dynamic load induced structural responses within a zone are compared in Fig. 6. The three tolerance limits in Fig. 6 cover the response spectra for at least $\beta = 0.95$ portion of all points within the zone, and are computed with similar confidence coefficients, namely, $\gamma = 0.46$–0.50. The normal prediction limit in Fig. 6 covers the response spectrum at the next randomly selected point within the zone with a confidence coefficient (probability) of $\gamma = 0.95$. The following interesting features in Fig. 6 should be noted.

1. The distribution-free and empirical tolerance limits are similar at most frequencies. This is consistent with the fact that these two tolerance limits do not involve a lognormal assumption and further have similar confidence coefficients, i.e., $\gamma = 0.46$ and 0.50, respectively.
2. The normal tolerance limit is somewhat higher at many frequencies than the distribution-free and and empirical tolerance limits, even though it has a similar confidence coefficient, i.e., $\gamma = 0.50$. This discrepancy probably reflects a slight inaccuracy in the lognormal assumption associated with the normal tolerance limit.
3. The normal prediction limit is higher than all three of the tolerance limits at most frequencies. However, if the tolerance limits were computed with a higher confidence coefficient, they would probably exceed the normal prediction limit.

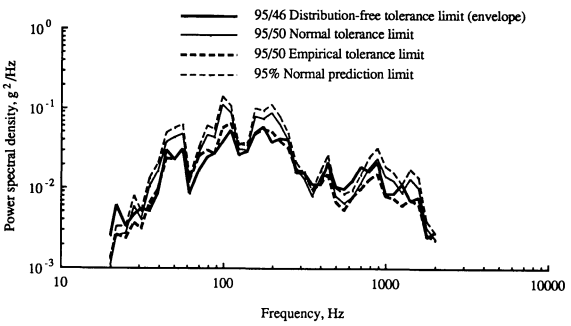

FIGURE 6 Comparison of various limits for vibration response spectra in Fig. 1.

It is also interesting to view how the various limits vary with an increasing number of predicted or measured spectra. Assuming the mean and standard deviation for the predicted or measured spectra do not vary with n, the following can be anticipated.

1. For the normal tolerance limit defined in Eq. (2), if the values of β and γ are fixed, the limit will decrease slightly as n increases due to the slightly smaller value of $k_{n,\beta,\gamma}$ given in Table 1. For example, if $\beta = 0.95$ and $\gamma = 0.50$, $k_{n,\beta,\gamma}$ decreases from 1.69 to 1.65 as n increases from 12 to 50.
2. For the distribution-free tolerance limit defined in Eq. (6), if the value of β is fixed (γ cannot be fixed independent of n), the limit will increase as n increases because the envelope of all predicted or measured spectra will generally increase with larger numbers of spectral values. However, the value of γ will also increase, meaning the higher limit corresponds to a higher confidence of exceeding the spectral values for at least β portion of all points in the zone. For example, if $\beta = 0.95$, the limit increases as the envelope increases, and γ increases from 0.46 to 0.92 as n increases from 12 to 50.
3. For the empirical tolerance limit defined in Eq. (11), if the value of β is fixed and $\gamma = 0.50$, there will be some statistical variations, but the expected value of the limit will not change as n increases, i.e., for 50% confidence with any value of β, the empirical tolerance limit is the same for $n = 12$ and 50.
4. For the normal prediction limit defined in Eq. (12), if the value of γ is fixed, the limit will decrease slightly as n increases due to small decreases in both the multiplier $\sqrt{1 + 1/n}$ and the value of $t_{(n-1);\alpha}$. For example, if $\gamma = 0.95$, the product $\sqrt{1 + 1/n}\, t_{(n-1);\alpha}$ decreases from 1.87 to 1.69 as n increases from 12 to 50.

As a final point of discussion, it should be emphasized that the various limits detailed here yield the maximum expected spectral response with any desired degree of conservatism at any point within a defined structural zone in terms of a motion parameter, commonly acceleration. Such limits are applicable to the general description of structural responses to dynamic loads, but care must be exercised when interpreting the limits as

Table 2. Comparative Merits of Procedures to Compute Maximum Structural Responses

Feature	Envelope	Tolerance Limit			Normal Prediction Limit
		Normal	Dist.-Free	Empirical	
Relatively easy to compute	Yes	Yes	Yes	No	Yes
Highly sensitive to frequency resolution band width of spectral data	Yes	No	Yes	No	No
Defines fractional portion β of responses exceeded with confidence coefficient γ	No	Yes	Yes	Yes	No
Fractional portion β and confidence coefficient γ can be independently selected	NA	Yes	No	With difficulty	NA
Provides statistically reliable results for a small sample size	NA	Yes	No	With data pooling	Yes
Applies to spectral data with any spatial distribution function	Yes	No	Yes	Yes	No

NA, not applicable.

a spectral input to equipment items that may be mounted on the structure. Specifically, when a spectral limit for a zone is used to formulate dynamic test criteria for equipment that will be mounted in that zone, it must be remembered that a relatively heavy equipment item will load the structure and, hence, see input motions that may be subtantially less than those given by the computed limit for the structural responses. This problem is particularly severe at the resonance frequencies of the equipment, where the apparent mass of the equipment becomes quite large (see Smallwood, 1976) and the equipment essentially behaves like a dynamic absorber (see Reed, 1988). There are several ways to address this problem, but the most desirable approach is to superimpose an input force limit on the input motion limit in the test criteria, as discussed by Scharton (1995).

CONCLUSIONS

The comparative merits of the various procedures for computing maximum structural responses from predictions or measurements at selected points are summarized in Table 2. There may be circumstances where any one of the procedures might be preferred over the others. In most cases, however, if it is reasonable to assume the structural responses follow a lognormal distribution, the normal tolerance limit is generally preferred because it can be easily computed with any desired combination of values for the fractional portion β of the structural responses covered by the limit and the confidence coefficient γ. On the other hand, if the number of available predictions or measurements is relatively large, say $n > 13$, and there is reason to question the lognormal assumption for the spatial distribution of the structural responses, either the distribution-free or empirical tolerance limit would be more reliable. Finally, if there is an interest only in the structural response at one location for one dynamic load event, the normal prediction limit is more appropriate than any of the tolerance limits.

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