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Numerical Study on Identification of Time Varying Parameters of Vibration Systems

An on-line least squares algorithm has previously been successfully applied to linear vibration systems in order to identify time varying parameters. In this article the limitations of the approach and the factors affecting the identification are further examined. The existence of the nonlinear term is determined by means of the time varying characteristics of the estimated linear parameters using the linear model and the data from a time invariant nonlinear system. The identification of the time varying linear parameters is also examined in accordance with the linear model by using the data with nonlinear elements. © *1997 John Wiley & Sons, Inc.*

INTRODUCTION

Parameter identification of vibration systems has become one of the most important research areas in inverse problems of structural dynamics. This increasing attention is a result of the need to reliably predict the response of complex structures and to validate and update finite element models. The vast majority of parametric identification techniques that are used make the assumption that the vibrationary characteristics are independent of time, i.e., the structural parameters remain constant throughout modal tests. A number of parameter identification procedures based on the time invariant assumption have been developed, which can be used successfully to estimate structural parameters of time invariant systems. For example, the accurate estimations of physical parameters of linear and nonlinear structures can be obtained by using the direct parameter estimation method, which was presented by Mahammad and colleagues (1990, 1992) and generalized by Liang and Cooper (1991, 1992, 1995). However, there are certain situations (e.g., a structure in a varying air flow or a structure undergoing failure) where the time invariant assumption does not hold. In order to analyze such situations, the so-called on-line identification techniques must be implemented.

The identification of time varying parameters of vibration systems has been followed with interest in recent years. A number of on-line identification techniques based on the least squares algorithm have previously been successfully applied to simulated and real data sets in order to identify time varying parameters. The ability to track time varying frequency and damping parameters using

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difference equation models and on-line versions of seven time domain system identification algorithms was examined by Cooper (1990). Three on-line system identification techniques were applied to real and simulated data sets by Cooper and Worden (1991a) in order to identify time varying physical parameters. Vibration tests on two mass varying systems were undertaken and the changes in the physical and modal parameters were tracked in Cooper and Worden (1991b). The sensitivity of the estimates to corruption of both input and measured sequences was examined by Liang and Wang (1993) using three different identification techniques. An optimum scheme for determining the forgetting factors was presented by Liang and Wang (1996) on the basis of a knowledge aftereffect, where the normalized error of the estimated parameters is taken as an objective function. It was demonstrated that it was possible to track the changes in the physical and modal parameters by applying the methods to the vibration systems, such as those containing an oscillatory stiffness or a damping element and those whose stiffness undergoes a step change or mass undergoes a gradual change. The limitations of the approaches and the factors affecting the identification are further examined in this article. The estimated frequencies obtained using the time domain and frequency domain are compared. The existence of the nonlinearities is determined by means of the time varying characteristics of the estimated linear parameters using the data from a time invariant nonlinear system. The estimates of the time varying linear parameters are also examined in accordance with the linear model using the data with nonlinear elements.

MATHEMATICAL MODEL

Consider a single degree of freedom system with time varying parameters. The equation of motion can be written as

$$m(t)\ddot{y}(t) + C(t)\dot{y}(t) + k(t)y(t) = x(t), \tag{1}$$

where $m(t)$, $c(t)$, and $k(t)$ are the unknown mass, damping, and stiffness coefficients, respectively; and $\ddot{y}(t)$, $\dot{y}(t)$, $y(t)$, and $x(t)$ are the measured acceleration, velocity, displacement, and input force, respectively. For the jth time instant, Eq. (1) can be rewritten as

$$[\ddot{y}_j \quad \dot{y}_j \quad y_j]\begin{Bmatrix} m \\ c \\ k \end{Bmatrix} = x_j. \tag{2}$$

Expanding the equations for N time instants gives the matrix equation

$$\begin{bmatrix} \ddot{y}_1 & \dot{y}_1 & y_1 \\ \ddot{y}_2 & \dot{y}_2 & y_2 \\ \vdots & \vdots & \vdots \\ \ddot{y}_N & \dot{y}_N & y_N \end{bmatrix}\begin{Bmatrix} m \\ c \\ k \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{Bmatrix}, \tag{3}$$

or

$$[\Phi]_N\{\boldsymbol{\theta}\}_N = \{x\}_N. \tag{4}$$

In reality there would be measurement errors on the data that would result in a residual vector on the right-hand side of Eq. (3). A least squares minimization of the squares of the residual terms results in the off-line least squares estimate described in Ljung and Söderström (1983),

$$\{\boldsymbol{\theta}\}_N = ([\Phi]_N^T[\Phi]_N)^{-1}[\Phi]_N^T\{x\}_N. \tag{5}$$

The on-line formulation of the least squares method is based on the relationship between the estimates found using N and $(N + 1)$ data points. It can be shown that

$$[\Phi]_{N+1}^T[\Phi]_{N+1} = [\Phi]_N^T[\Phi]_N + \{\boldsymbol{\alpha}\}_{N+1}\{\boldsymbol{\beta}\}_{N+1}^T \tag{6}$$

and

$$[\Phi]_{N+1}^T\{x\}_{N+1} = [\Phi]_N^T\{x\}_N + x_{N+1}\{\boldsymbol{\alpha}\}_{N+1}, \tag{7}$$

where

$$\{\boldsymbol{\alpha}\}_{N+1} = \{\boldsymbol{\beta}\}_{N+1} = \{\ddot{y}_{N+1} \quad \dot{y}_{N+1} \quad y_{N+1}\}^T. \tag{8}$$

Making use of the matrix inversion lemma (Ljung and Söderström, 1983),

$$\begin{aligned}([A] + [B][C][D])^{-1} &= [A]^{-1} \\ &- [A]^{-1}[B]([D][A]^{-1}[B] + [C]^{-1})^{-1}[D][A]^{-1},\end{aligned} \tag{9}$$

where $[A]$, $[B]$, $[C]$, and $[D]$ are matrices of compatible dimensions, the on-line least squares estimate is found by combining Eqs. (5)–(9) such that

$$\begin{aligned}\{\boldsymbol{\theta}\}_{N+1} = \{\boldsymbol{\theta}\}_N - [P]_N\{\boldsymbol{\alpha}\}_{N+1} \\ \times (\{\boldsymbol{\beta}\}^T_{N+1}[P]_N\{\alpha\}_{N+1} + 1)^{-1} \\ \times (\{\boldsymbol{\beta}\}^T_{N+1}\{\theta\}_N - x_{N+1}),\end{aligned} \tag{10}$$

where

$$\begin{aligned}[P]_{N+1} = [P]_N - [P]_N\{\alpha\}_{N+1} \\ \times (\{\beta\}^T_{N+1}[P]_N\{\boldsymbol{\alpha}\}_{N+1} + 1)^{-1}\{\beta\}^T_{N+1}[P]_N),\end{aligned} \tag{11}$$

where $[P]$, $\{\boldsymbol{\alpha}\}$, and $\{\boldsymbol{\beta}\}$ are the matrix and vectors formed of sampled acceleration, velocity, and displacement data, respectively; x is the measured force; and $\{\boldsymbol{\theta}\}$ is the unknown vector of physical parameters.

It is thus possible to obtain estimates of the mass, damping, and stiffness estimates for each time instant. In order to enable the algorithm to be adaptive, a forgetting factor is included so that Eqs. (10) and (11) become

$$\begin{aligned}\{\boldsymbol{\theta}\}_{N+1} = \{\boldsymbol{\theta}\}_N - [P]_N\{\boldsymbol{\alpha}\}_{N+1}(\{\boldsymbol{\beta}\}^T_{N+1}[P]_N\{\boldsymbol{\alpha}\}_{N+1} + \lambda)^{-1} \\ \times (\{\boldsymbol{\beta}\}^T_{N+1}\{\boldsymbol{\theta}\}_N - x_{N+1})\end{aligned} \tag{12}$$

and

$$\begin{aligned}[P]_{N+1} = ([P]_N - [P]_N\{\boldsymbol{\alpha}\}_{N+1}(\{\boldsymbol{\beta}\}^T_{N+1}[P]_N\{\boldsymbol{\alpha}\}_{N+1} + \lambda)^{-1} \\ \times \{\boldsymbol{\beta}\}^T_{N+1}[P]_N)/\lambda,\end{aligned} \tag{13}$$

with $0 < \lambda \leq 1$. The smaller the forgetting factor λ is, the more emphasis is placed upon the most recent data points. Some other on-line methods can be derived on the basis of the least squares algorithms, such as the double least squares method and the instrumental variables method. The basic on-line formulation is the same as that of the least squares except that Eq. (8) possesses different forms. Only the least squares algorithm is employed in this article. In all the simulated calculations the initial values of $\{\boldsymbol{\theta}\}$ and $[P]$ are taken as

$$\{\boldsymbol{\theta}\}_0 = \{0\}, \quad [P]_0 = \alpha[I], \tag{14}$$

where α is a large positive and $[I]$ is the unit matrix.

A normalized error function, defined as

$$\Delta E^{(i)} = \frac{1}{M - L + 1}\sum_{j=L}^{M} \frac{[T_j^{(i)} - E_j^{(i)}]^2}{[T_j^{(i)}]^2} \times 100, \tag{15}$$

will be used to compare the estimated results,

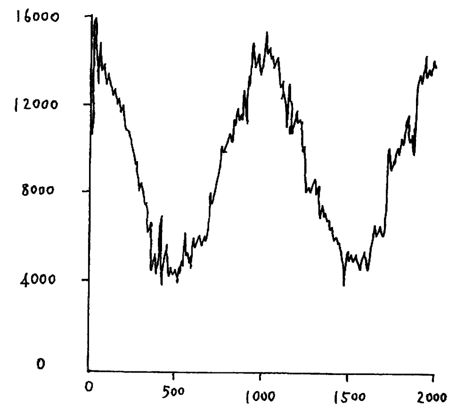

FIGURE 1 Stiffness values, $\zeta = 0.2$.

where $T_j^{(i)}$ and $E_j^{(i)}$ are the ith true and estimated parameter values at time instant $\Delta t \times j$, respectively; Δt is the sampling rate; and M is the total number of sampled data points. The value of L is set so that the algorithm has time to "warm up" before the error analysis is performed (Cooper, 1990).

NUMERICAL RESULTS AND DISCUSSION

The limitations of the least squares algorithm and the factors affecting the estimated results are examined and the estimates of nonlinearities are discussed by using a number of simulated systems. In all the examples the excitation is taken as white random noise. The parameters are as follows: forgetting factor $\lambda = 0.94$; sampling rate $\Delta t = 0.001$ s; mass $m = 1$ kg; linear stiffness $k = 10{,}000$ N/m; linear damping $c = 40$ Ns/m.

EXAMPLE 1. Consider a time varying system governed by the Mathieu equation

$$m\ddot{y} + c\dot{y} + k\left(1 + \frac{1}{2}\cos(2\pi t)\right)y = x(t). \tag{16}$$

A number of simulated tests using this model show that the estimated results are dependent on the damping ratio besides the selection of the forgetting factors. The estimates are bad for the very weakly damped case whereas the estimated accuracy related to the resonant frequencies for the systems possessed the same damping ratio. The estimates of the system with a large resonant frequency are better than those of the system with a small resonant frequency. Figures 1 and 2 show

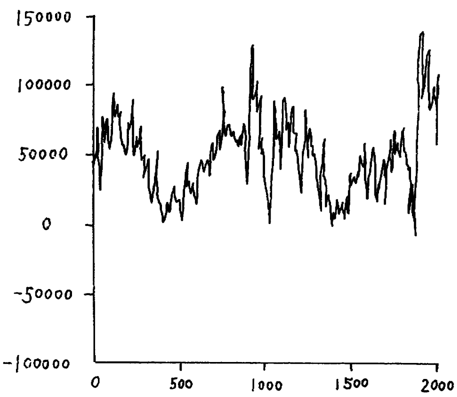

FIGURE 2 Stiffness values, $\zeta = 0.05$.

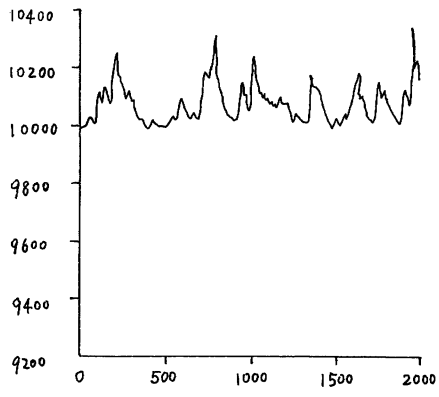

FIGURE 3 Stiffness values, $\gamma = 10{,}000$.

the estimated stiffness values using the data with a 10% signal to noise ratio for $\zeta = 0.2$ and 0.05, respectively. Table 1 shows the error function values for both the physical and modal parameter estimates to the fixed mass, damping, and the different stiffness. It can be seen from the table that the estimated accuracy of the resonant frequency is higher than those of the other parameters. The estimated frequencies obtained using the time domain and frequency domain methods are compared. The trend of the changes in the resonant frequency can be found employing the frequency domain method; however, the estimates obtained using the time domain method are superior to those obtained using the frequency domain method.

The discussion of Example 1 is about the limitations of the on-line least squares algorithm applied to linear time varying vibration systems and the factors affecting the estimates. Another object of this article is to determine the existence of the nonlinearities by means of the time varying characteristics of the estimated linear parameters using the linear model and the data from a time invariant nonlinear system. Examples 2–5 are discussed for this purpose.

Example 2. We consider a time invariant nonlinear system governed by the Duffing equation,

$$m\ddot{y} + c\dot{y} + ky + \gamma y^3 = x(t). \tag{17}$$

Figure 3 shows the estimated values of the time invariant linear stiffness using the linear model and nonlinear responses that are noise free. The amplitude of the excitation is taken as 2000 N and the nonlinear stiffness is taken as $\gamma = 10{,}000$ N/m^3. The linear stiffness should remain constant because of the invariant characteristics. However, from Fig. 3 it can be seen that the linear stiffness is time varying. Therefore, we can determine that the reason that the linear stiffness is time varying is the existence of the nonlinearity in the system. It can be seen that the factor of the nonlinearity with the same amount as the linear stiffness can be estimated when the amplitude of the excitation is large. The effect of the nonlinear term is not sensitive when the excitation level becomes small.

Table 1. Normalized Errors on Parameter Estimates (%)

Parameter Values		Normalized Errors (10% Noise)				
Stiffness	Damping Ratio	Mass	Stiffness	Damping	Frequency	Damping Ratio
2500	0.400	0.00360	0.47622	0.05804	0.17971	0.14608
6400	0.250	0.15498	1.15109	0.51019	0.30370	0.88373
10,000	0.200	0.39293	1.71078	1.12193	0.36117	1.96716
14,400	0.167	1.29076	4.36163	2.69510	1.35476	9.90119
25,600	0.125	3.45146	6.06879	5.65358	1.61590	22.52751
40,000	0.100	7.67902	11.09864	12.09864	1.85811	63.77441
160,000	0.050	52.15218	48.68893	56.76628	6.61184	2464.24000

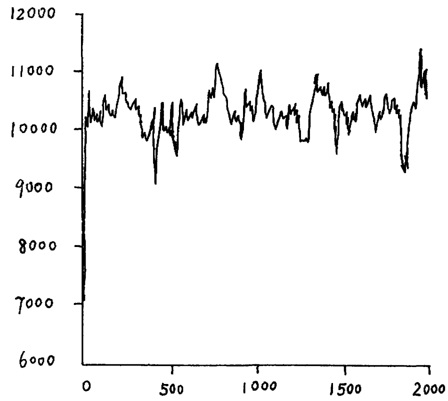

FIGURE 4 Stiffness values, $\gamma = 4 \times 10^8$.

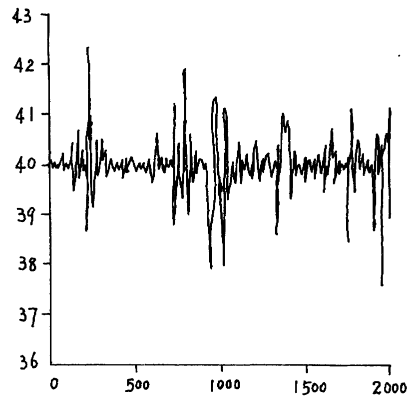

FIGURE 6 Damping values, $\gamma = 0.5$.

Figure 4 gives the estimated values of the time invariant linear stiffness. Here the amplitude of the excitation and the nonlinear term are taken as 20 N and $\gamma = 4 \times 10^8$ N/m^3, respectively. The time varying characteristics of the linear stiffness can obviously be seen; however, the order of the amount of the nonlinear term is much larger than that of the linear stiffness. The same conclusions can be obtained from the nonlinear system with a soft spring.

Example 3. Let us try a nonlinear system with tangent stiffness

$$m\ddot{y} + c\dot{y} + \frac{2k\gamma}{\pi} tg\left(\frac{\pi y}{2\gamma}\right) = x(t). \tag{18}$$

We still want to estimate the nonlinear term by using the linear model and the responses of the nonlinear system. The estimated results of the linear parameters are dependent on γ for this kind of system. The normalized errors of the estimated parameters become large as γ decreases. This is the reason that the effect of the nonlinear term becomes large as γ decreases. Figures 5 and 6 show the estimated damping values where the amplitudes of the excitations are taken as 20,000 and 2000 N and γ are 5 and 0.5, respectively. It can be seen that the damping coefficients are time varying. However, the estimated damping values of the linear system corresponding to Eq. (18) are time invariant. Therefore, the existence of the nonlinear term can be determined.

Example 4. A nonlinear system with a cross term is formed,

$$m\ddot{y} + \gamma\left(y^2 - 1 + \frac{c}{\gamma}\right)\dot{y} + ky = x(t). \tag{19}$$

This includes the van der Pol oscillator. The estimated results are dependent on c/γ for this kind of system. The normalized errors of the estimated linear parameters are very small when $c > \gamma$. Fig-

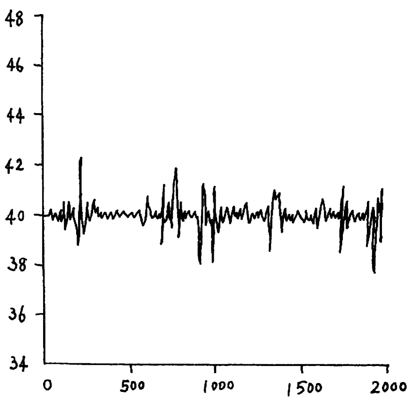

FIGURE 5 Damping values, $\gamma = 5$.

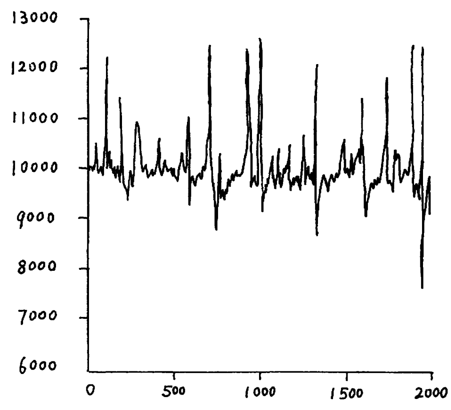

FIGURE 7 Stiffness values, $c/\gamma = 2$.

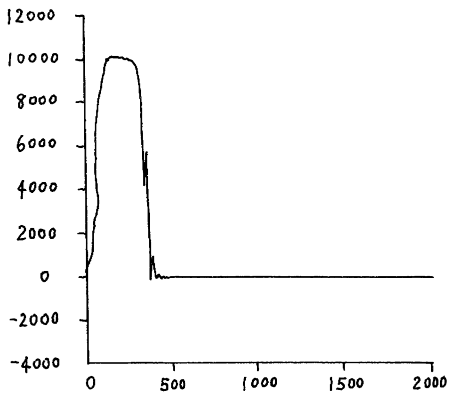

FIGURE 8 Stiffness values, $c/\gamma = 0.05$.

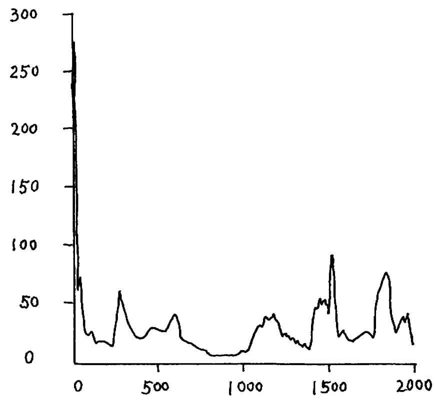

FIGURE 10 Stiffness k, $\gamma = 0.1$.

ure 7 shows the estimated values of the stiffness. In the simulation the amplitude of the excitation is taken as 20,000 N, $\gamma = 40$ Ns/m^3, and $c = 80$ Ns/m. Figure 8 gives another estimated stiffness, but the excitation and the parameters are taken as 20 N, $\gamma = 40$ Ns/m^3, and $c = 2$ Ns/m, respectively. The time varying characteristics of the stiffness can be seen from both figures. However, the estimated stiffness values of the linear system corresponding to Eq. (19) are time invariant. Therefore, the existence of the nonlinear term can be revealed. The estimated stiffness values from the second simulation are much worse than those from the first simulation. The estimated mass values from the first simulation are good but those from the second are bad. From lots of simulated experiments, we can conclude that the estimates using the linear model are not feasible when $c > \gamma$. Therefore, if $c < \gamma$ then the nonlinear term must be considered in the estimated model.

Example 5. Consider a nonlinear system with hyperbolic tangent stiffness,

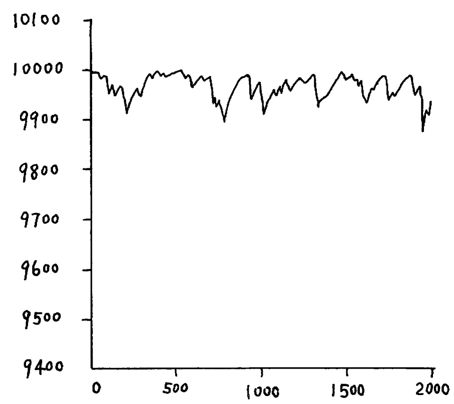

FIGURE 9 Stiffness k, $\gamma = 100$.

$$m\ddot{y} + c\dot{y} + \gamma th\left(\frac{ky}{\gamma}\right) = x(t). \tag{20}$$

The estimates in this example are different from those of Examples 2–4. In the previous examples the existence of the nonlinear terms is determined by means of the time varying characteristics of the linear parameters of the system, whereas the stiffness k is directly estimated using the linear model and the responses of the nonlinear system. The calculations show that the accuracy of the estimates decreases as γ becomes small. Figures 9 and 10 give the estimated stiffness k using the nonlinear system's responses with $\gamma = 100$ and 0.1 N, respectively. The time varying characteristics of the stiffness k can be seen from both figures. It reveals again the existence of the nonlinear term. However, the estimates from the latter are poor.

The above four examples are all time invariant nonlinear systems. A time varying nonlinear system will be considered in the following example. The estimates of the time varying linear parameters are examined in accordance with the linear model by using the data with nonlinear elements.

Example 6. Consider a time varying system with a cubic nonlinear stiffness element,

$$m\ddot{y} + c\dot{y} + k\left(1 + \frac{1}{2}\cos(2\pi t)\right)y + \gamma y^3 = x(t). \tag{21}$$

Table 2 shows the error function values for the physical and modal parameter estimates when the nonlinear stiffness increases. From the table it can be seen that the normalized errors are small when $\gamma \leq 10^9$ N/m^3. Therefore, the parameter estimates based on the linear model using the nonlinear sys-

Table 2. Normalized Errors on Parameter Estimates (%)

Nonlinear Stiffness	Normalized Errors (10% Noise)				
	Mass	Stiffness	Damping	Frequency	Damping Ratio
1×10^8	0.39778	1.24340	1.23758	0.29052	2.67705
1×10^9	0.54038	1.51908	1.58223	0.55246	2.81104
4×10^9	1.38180	8.63168	3.15504	3.39652	5.61757
5×10^9	1.73506	11.53996	3.70137	4.57352	6.82541
1×10^{10}	2.82847	26.13587	5.10254	10.36239	8.86525
3×10^{10}	3.08432	97.51439	5.26824	28.21210	7.80583
4×10^{10}	4.07068	122.12100	6.15813	35.58960	9.54249
5×10^{10}	7.97958	123.62850	8.77185	45.06074	15.78562
1×10^{11}	26.77550	153.46220	22.49960	108.44530	69.23538
1×10^{12}	58.42938	742.68570	43.13730	696.11890	208.51420

tem's responses are feasible even when the nonlinear term is strong. The normalized errors on physical and modal parameters versus the nonlinear stiffness are shown in Figure 11. Figure 12 shows the estimated linear stiffness coefficient varying in the sinusoidal form using the data corrupted with a noise of 10% signal to noise ratio from the nonlinear system ($\gamma = 5 \times 10^9$ N/m^3). From Table 2 it can be seen that the normalized error of the linear stiffness is 11.5% in this situation; however, it is still possible to track changes in the linear parameters. As the nonlinear term in the model increases, the estimated values of the linear parameters worsen. For example, when $\gamma = 10^{12}$ N/m^3, the estimated curves are chaotic and the estimated linear parameters cannot be used. Therefore, the nonlinear term must be considered in the estimated model.

CONCLUSIONS

In this article we studied the identification of linear time varying parameters of vibration systems, the estimates of nonlinearities in vibration systems, the limitations of the on-line least squares algorithm applied to time varying vibration systems, and the factors affecting the estimates. The simulated results based on certain typical time varying and time invariant nonlinear systems show that it is possible to track changes of the linear parameters in a time varying nonlinear system and to determine the existence of the nonlinearities by means of the time varying characteristics of the estimated linear parameters using the linear model and the data from a time invariant nonlinear system. The linear parameters can be identified within a certain range by employing a linear model

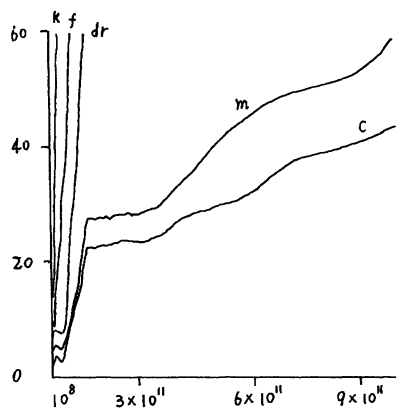

FIGURE 11 Normalized errors versus nonlinear stiffness.

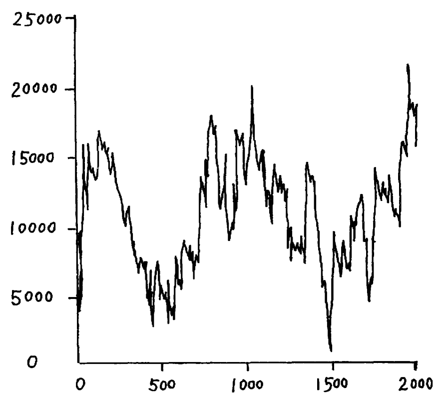

FIGURE 12 Stiffness values, $\gamma = 5 \times 10^9$.

and the responses of a time varying nonlinear system.

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