Vesta I. Bateman
Randall L. Mayes
Thomas G. Carne
Sandia National Laboratories
Albuquerque, NM 87185-0555, USA

Comparison of Force Reconstruction Methods for a Lumped Mass Beam *

Two extensions of the force reconstruction method, the sum of weighted accelerations technique (SWAT), are presented in this article. SWAT requires the use of the structure's elastic mode shapes for reconstruction of the applied force. Although based on the same theory, the two new techniques do not rely on mode shapes to reconstruct the applied force and may be applied to structures whose mode shapes are not available. One technique uses the measured force and acceleration responses with the rigid body mode shapes to calculate the scalar weighting vector, so the technique is called SWAT-CAL (SWAT using a calibrated force input). The second technique uses the free-decay time response of the structure with the rigid body mode shapes to calculate the scalar weighting vector and is called SWAT-TEEM (SWAT using time eliminated elastic modes). All three methods are used to reconstruct forces for a simple structure.

INTRODUCTION

For many structural tests, the structural response in the form of strain or acceleration is not a sufficient description of the dynamic event because the applied dynamic force is desired. Either accelerometers or strain gages can be used to measure the response of a structure to some transient excitation such as an impact. Force reconstruction is a process in which dynamic response signals are used to infer the externally applied force that produced those responses.

The sum of weighted accelerations technique (SWAT) (Bateman et al., 1992; Gregory et al., 1986; Priddy et al., 1988, 1989; Smallwood and Gregory, 1987) was developed at Sandia National Laboratories to reconstruct impact forces experienced by structures during full scale dynamic tests and has been extended elsewhere (Kreitinger et al., 1988; Wang et al., 1987). The SWAT technique was successfully applied at Sandia to data from field tests of a simulated bomb structure with an energy absorbing nose (Bateman et al., 1992) and a nuclear fuel cask colliding with an unyielding barrier (Bateman et al., 1991). Other techniques for force reconstruction (also called force identification) have been developed over a number of years that primarily use the frequency response function (FRF) matrix that relates the response measurements directly to the input forces. However, the FRF matrix has to be inverted to solve for the forces and this inversion causes many difficulties that limit the applicability of these techniques (Starkey and Merrill, 1989; Stevens, 1987). Although acceleration measurements are typically used for force reconstruction, Hillary and Ewins considered the use of strain gages (1984).

Received 4 February 1994; Revised 1 February 1997.

*This work was supported by the US Department of Energy under Contract DE-AC04-76-DP00789.

Shock and Vibration, Vol. 4, No. 4, pp. 231–239 (1997)
ISSN 1070-9622/97/$8.00

The SWAT approach to force reconstruction is to compute the force from a time domain summation of acceleration signals that have been multiplied by a set of scalar weights in the form of a weighting vector. Summation of data in the time domain has none of the numerical problems that make the FRF techniques so difficult. The only requirement in this technique is that the set of scalar weights must be calculated. To calculate the SWAT weights, the mode shape matrix of the structure with free boundary conditions must be obtained and inverted. The two extensions of SWAT presented in this article do not require the mode shape matrix to calculate the weights but need simpler force and acceleration response measurements from the structure.

With SWAT, the spatial distribution of the applied forces is not obtained, but the force is implicitly integrated over the structure. Consequently, only the resultant sum of all externally applied forces or the sum of all moments about the center of mass is calculated. For many applications this is adequate because the application point for the force is known. Even if the application point is unknown but the application area is small, the application point can be found by relating the resultant force to the resultant moment. In contrast, FRF techniques require exact knowledge of the location and application area of the force to produce acceptable results.

Two extensions of SWAT are presented in the next section. One technique uses the measured force and acceleration time responses in the laboratory with the rigid body mode shapes to calculate the weighting vector. The weights are calibrated with the laboratory input force, so the technique is called SWAT-CAL (SWAT using a calibrated force input). The second technique uses the rigid body mode shapes with the free-decay time response of the structure (after the removal of the response to the applied force) to calculate the weighting vector and is called SWAT-TEEM (SWAT using time eliminated elastic modes) (Mayes, 1994). In both extensions, the weights are extracted in the time domain and avoid the errors associated with the mode shape matrix. All three techniques are used to calculate the weighting vector for a lumped mass beam structure, and the results are compared.

SWAT THEORY

The result of the SWAT theory is that a force applied to a structure may be reconstructed from the measured acceleration responses with the equation,

$$f(t) = \sum w_i a_i(t), \tag{1}$$

where $f(t)$ is a reconstructed force, $a_i(t)$ are the measured response accelerations due to an applied force, and w_i are scalar weights that form the weighting vector, W. Equation (1) can be extended to solve for the applied moment (Bateman et al., 1992; Priddy et al., 1989).

The derivation of the theory of SWAT begins with the equations of motion for a multiple degree of freedom, freely supported structure that may be written as

$$M\ddot{u}(t) + C\dot{u}(t) + Ku(t) = f(t), \tag{2}$$

with the standard matrix notation. In (1), $f(t)$ represents all external forces, including applied forces and forces resulting from the boundary conditions, and M, C, and K represent the symmetric matrices for mass, damping, and stiffness, respectively. The displacement, u, may be written in terms of a summation of the free modes as

$$u(t) = \sum_i \phi_i q_i(t) = \Phi q(t), \tag{3}$$

where Φ is the matrix of free-body mode shapes and $q(t)$ is the vector of uncoupled generalized modal coordinates. Φ includes rigid body modes as well as elastic modes. Substitution of (3) into (2) and premultiplication of the equation by a single mode ϕ_r^T, which represents a rigid body mode of the free structure, results in

$$\begin{aligned} &\phi_r^T M\Phi\ddot{q}(t) + \phi_r^T C\Phi\dot{q}(t) + \phi_r^T K\Phi q(t) \\ &\quad = \phi_r^T f(t). \end{aligned} \tag{4}$$

For the rigid body modes (translation or rotation), there are no elastic deformations, and consequently, no spring or no damping forces. Thus,

$$K\phi_r = C\phi_r = 0, \quad \text{and} \quad \phi_r^T K = \phi_r^T C = 0, \tag{5}$$

using the symmetry of K and C. Substituting (5) into (4) and using the orthogonality of the modes with respect to the mass matrix produces our desired result,

$$m_r\ddot{q}_r(t) = \phi_r^T f(t), \tag{6}$$

where q_r is the modal coordinate for a rigid body mode and $m_r = \phi_r^T M\phi_r$ the modal mass. This equation shows that if the modal coordinate of a rigid body mode (translation or rotation) can be measured, then (6) can be solved for the sum of the external forces. For example, if the modal coordinate of the translation rigid body mode in any one direction is known, then Eq. (6) becomes Newton's second law for an elastic

structure as $m_r\ddot{q}_r = \sum f_i$ because $\phi_r^T = [1\ 1\ 1 \cdots]$, m_r is the total mass, and $\sum f_i$ is the sum of all the force components in the translation direction. Also, (6) relates the sum of all the moments about the center of mass to the pitch rigid body modes if ϕ_r^T is scaled such that m_r is the mass moment of inertia.

Equation (6) establishes the relation between the rigid body modal coordinates and the external forces. This equation may appear obvious as an expression of rigid body dynamics, but the ramification for force reconstruction is critically important: knowledge of the mass and the rigid body modal coordinates is completely sufficient to know the sum of all the externally applied forces.

The result in (6) may be used with (4) as

$$\phi_r^T M \Phi \ddot{q}(t) = \phi_r^T f(t). \tag{7}$$

A weighting vector may be defined as

$$W^T = \phi_r^T M. \tag{8}$$

If (8) is substituted back into (7) and the equation is rewritten in terms of the displacement, then

$$W^T \ddot{u}(t) = \phi_r^T f(t), \tag{9}$$

which reveals that the sum of all externally applied forces may be found by a time domain summation of the measured acceleration responses on a freely supported structure. Equation (9) is the basic SWAT equation. At this point, a practical limitation is imposed on all quantities. The assumption in Eq. (1) is that all degrees of freedom are included, or alternatively, that the bandwidth of all quantities is infinite. In practice, only a finite bandwidth can be achieved. From this point, all quantities are redefined as having a limited bandwidth and Φ are "reduced" free modes of the system with components only at the measured degrees of freedom. Equation (8) is postmultiplied by the mode shape matrix to obtain

$$W^T \Phi = \phi_r^T M \Phi. \tag{10}$$

When the right-hand side of (10) is multiplied out, it becomes simply a row vector of zeros except for the column corresponding to the rigid body mode for which a set of weights are being determined. This column is a one multiplied by the modal mass m_r. The usual SWAT formulation is given using the transpose of (10) or

$$\Phi^T W = \begin{bmatrix} m_r \\ \cdots \\ 0 \end{bmatrix}, \tag{11}$$

which has m_r in the first row for the case in which the weights for the first rigid body mode are being determined. The weights from (11) will determine the sum of all externally applied forces for the direction defined by the rigid body mode in (9). Equation (11) shows that the number of accelerometer locations, and consequently the number of weights, must be equal to or greater than the sum of the rigid body and elastic modes represented in Φ. Also, if the number of accelerometer locations (weights) exceeds the number of modes, an underdetermined equation results and a pseudoinverse technique must be used in this situation.

The derivation of the new alternative techniques, which do not require knowledge of the elastic modes or the mode shapes, continues by an examination of Eq. (9) from which the basic equation for SWAT-CAL results. Equation (9) simply states that the weighting vector may be determined in the laboratory by applying known forces to the structure and measuring the response accelerations. The weighting vector is computed from a least squares solution of (9) with the measured applied force and response acceleration time histories. This technique was originally proposed by Gregory et al. (1986) and Smallwood and Gregory (1987). However, in impact applications, the small amount of rigid body information is not sufficient because the rigid body information is only contained in the time histories during the very short application of the force. The problem is solved by partitioning Φ in Eq. (11) into rigid body and elastic parts as

$$\begin{bmatrix} \Phi_r^T \\ \cdots \\ \Phi_e^T \end{bmatrix} W = \begin{bmatrix} m_r \\ \vdots \\ 0 \\ \cdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{12}$$

where Φ_r is the matrix of rigid body mode shapes and Φ_e is the matrix of elastic mode shapes. Consequently, the rigid body partition of (12) is added to (9) to obtain the robust formulation of SWAT-CAL:

$$W^T = \begin{bmatrix} \Phi_r \\ \cdots \\ \ddot{u} \end{bmatrix} = \begin{bmatrix} m_r \\ \vdots \\ 0 \\ \cdots \\ \phi_r^T f \end{bmatrix}. \tag{13}$$

Each row in the partitioned matrix on the left-hand side of Eq. (13) represents either a rigid body mode shape or a sample of the acceleration measurements, and the acceleration portions, $\ddot{u}(t)$, include the entire

measured time histories at the designated locations. Equation (13) is solved for the weighting vector, W, in a least squares sense.

Alternatively, if there is no measurement of the applied force available, the acceleration time histories may be examined after the force is removed and while the structure is in free decay with a combination of Eqs. (9) and (11) as

$$\ddot{u}^T(t)W = f^T(t)\phi_r. \tag{14}$$

But because $f(t) = 0$, the right-hand side of (14) is zero. The result is an underconstrained set of equations with no rigid body acceleration information. To avoid the trivial solution ($W = [0]$) of (14), the rigid body portion of (12) is added to constraint (14) so that

$$\begin{bmatrix} \phi_r^T \\ \cdots \\ \ddot{u}^T \end{bmatrix} W = \begin{bmatrix} m_r \\ \vdots \\ 0 \\ \cdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{15}$$

Equation (15) is solved in the least squares sense for the weights. This is the SWAT-TEEM (SWAT using time eliminated elastic modes) formulation (Mayes, 1994). In some cases, a constrained least squares solution may be required.

During the early development of SWAT, the weighting vector was conceived as a representation of the mass matrix (Smallwood and Gregory, 1987). However, Eq. (11) shows that the weighting vector is the product of one rigid body mode with the mass matrix. The resultant vector is orthogonal to all the other mode shape vectors because of the mass orthogonality relationship. This is a common component of each formulation and allows the determination of the correct weighting vector by using the mode shapes in the original SWAT formulation, the measured force and response acceleration time histories in SWAT-CAL, or the free decay response of the structure after the force has been removed in SWAT-TEEM. The next section demonstrates this concept by computing the weight vector and the resulting reconstructed forces for a lumped mass beam structure.

MODAL TEST RRESULTS FOR LUMPED MASS BEAM STRUCTURE

A typical structure that might be used for a force reconstruction application is shown in Fig. 1. However, a simpler structure was used for this study. The

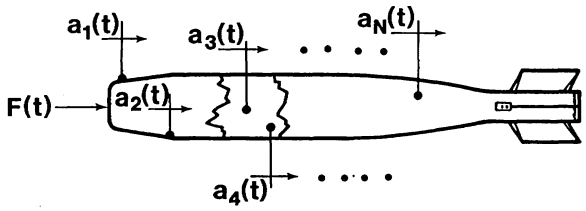

FIGURE 1 Typical structure for full-scale impact testing.

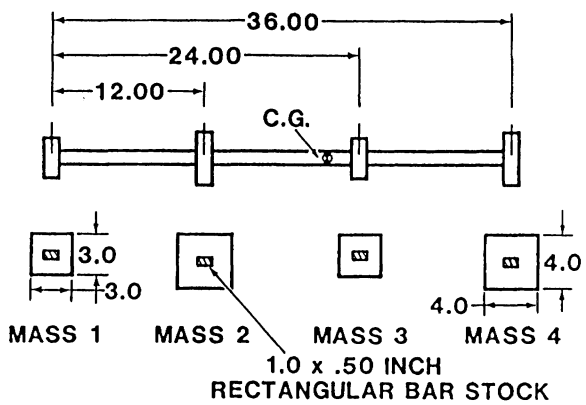

FIGURE 2 Lumped mass beam structure testing (all dimensions in inches); total weight = 18.68 lb.

lumped mass beam structure shown in Fig. 2 has been used successfully for other studies (Priddy et al., 1988; Smallwood and Gregory, 1987) and was used to demonstrate computation of the weight vector W^T and force reconstruction with the three techniques: SWAT, SWAT-CAL, and SWAT-TEEM.

A modal test was conducted of the lumped mass beam structure with the resulting modal frequencies and mode shapes shown in Table 1. The softwise bending (lateral) direction was used for force application and acceleration response measurement. Seven accelerometers were located on the masses and at the center of the beam sections. The pitch mode shows the actual instrumentation locations relative to the center of mass for the beam that is about 16.75 in. from the right side. As can be seen from Table 1, the mode shapes are not symmetrical. This is to be expected because the mass configuration is asymmetrical with respect to the center of mass. Table 1 contains all the information to form the Φ matrix for the SWAT technique. SWAT-CAL and SWAT-TEEM do not require a modal analysis but only that the number of modes in the desired bandwidth for the force reconstruction be known. With seven measurement locations, the SWAT theory allows seven modes (rigid body and elastic modes) to be included in the formulation, because there are only seven unknown scalar weights.

Table 1. Modal Frequencies and Mode Shapes for Lumped Mass Beam (Softwise Bending)

Mode	Measurement Location on Beam							Bending Frequency (Hz)
	Mass 1	Sec. 1	Mass 2	Sec. 2	Mass 3	Sec. 3	Mass 4	
Translation	1.	1.	1.	1.	1.	1.	1.	—
Pitch	19.25	13.25	7.25	0.25	–4.75	–10.75	–16.75	—
1	–0.58	–0.08	0.30	0.45	0.38	0.06	–0.34	31.6
2	0.34	–0.30	–0.44	0.22	0.62	0.48	–0.20	89.5
3	1.98	–9.13	–0.35	10.10	0.89	–11.04	0.93	259.6
4	–2.37	15.83	–2.65	–0.48	5.73	–9.93	0.25	350.3
5	1.11	–11.33	3.38	–15.63	2.83	2.95	–0.65	531.6

Table 2. Weighting Vectors for Lumped Mass Beam Calculated with Three Different SWAT Techniques

Accelerometer Location	SWAT Weighting Vector	SWAT-CAL Weighting Vector		SWAT-TEEM Weighting Vector	
		Sec. 2	Mass 4	Sec. 2	Mass 4
1	0.1458	0.1556	0.1585	0.1639	0.1485
2	0.0521	0.0531	0.0459	0.0340	0.0537
3	0.2787	0.2653	0.2682	0.2765	0.2728
4	0.0621	0.0530	0.0577	0.0694	0.0594
5	0.1508	0.1629	0.1592	0.1272	0.1564
6	0.0639	0.0538	0.0541	0.0814	0.0598
7	0.2466	0.2560	0.2565	0.2477	0.2489

WEIGHTING VECTOR CALCULATIONS

The next step in this force reconstruction investigation is to calculate the weighting vectors. The weighting vectors, W^T, were calculated from Eqs. (11), (13), and (15) for the three respective techniques, SWAT, SWAT-CAL, and SWAT-TEEM. The weighting vectors are shown in Table 2. The SWAT weighting vector was calculated using the mode shape information in Table 1. The weighting vectors for SWAT-CAL and SWAT-TEEM are averaged values that were calculated for five impacts in the softwise bending direction at each of two locations: section 2 (0.25 in. from the center of mass) and mass 4, which is a large mass (4 × 4 × 1 in. thick) on the right end of the beam. The data was digitized with a 4096-Hz sample rate and with analog filters whose cutoff frequency is 600 Hz. Each measured accelerometer response and force time history contained 1024 points. For the SWAT-CAL calculation, the entire record of 1024 points was used for the weighting vector calculation. For SWAT-TEEM the portion of the acceleration time history after the applied force was removed, starting at sample point 64, was used for the weighting vector calculation. Different portions of the remaining 961 points were used: 128, 256, 512, and 961 points. However, the 128-point calculation yielded the same results as the other acceleration time history lengths, so this was the length used for the averaged calculations. One additional change was made to the SWAT-TEEM algorithm; the first equation in (15), which models the rigid body translation mode, was given a weighting factor of 10. This emphasis of the translation mode constrains the weight summation to be m_r which is the desired value. As shown in Table 2, the different techniques and the different locations for multiple impacts give different weights but the weights do not differ by significant amounts.

FORCE RECONSTRUCTION RESULTS

Finally, the weighting vectors were used to reconstruct impact forces applied at different locations on the beam. Time history data were collected for the force reconstructions from impacts at all four mass and three beam section locations. Examples of the reconstructed forces in comparison to the measured impact forces applied to mass 1 are shown in Figs. 3–5. All weighting vectors yielded good reconstructions of the impact forces as demonstrated in the figures, but only the three examples are shown here because of limited

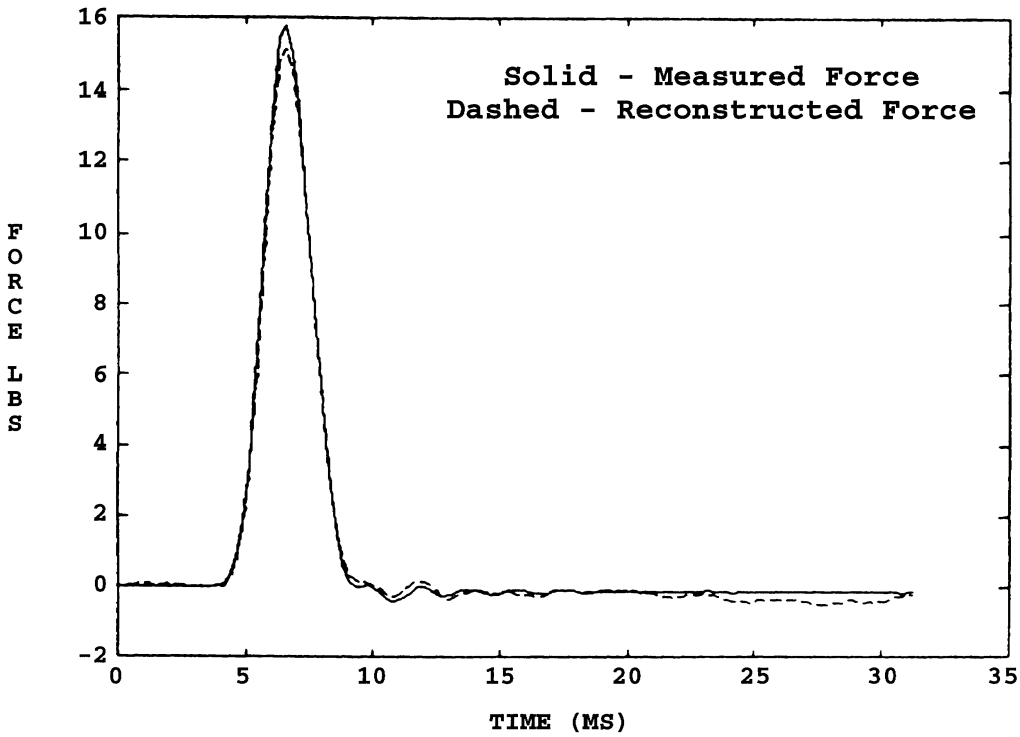

FIGURE 3 Force reconstructed from impact at mass 1 using the SWAT algorithm.

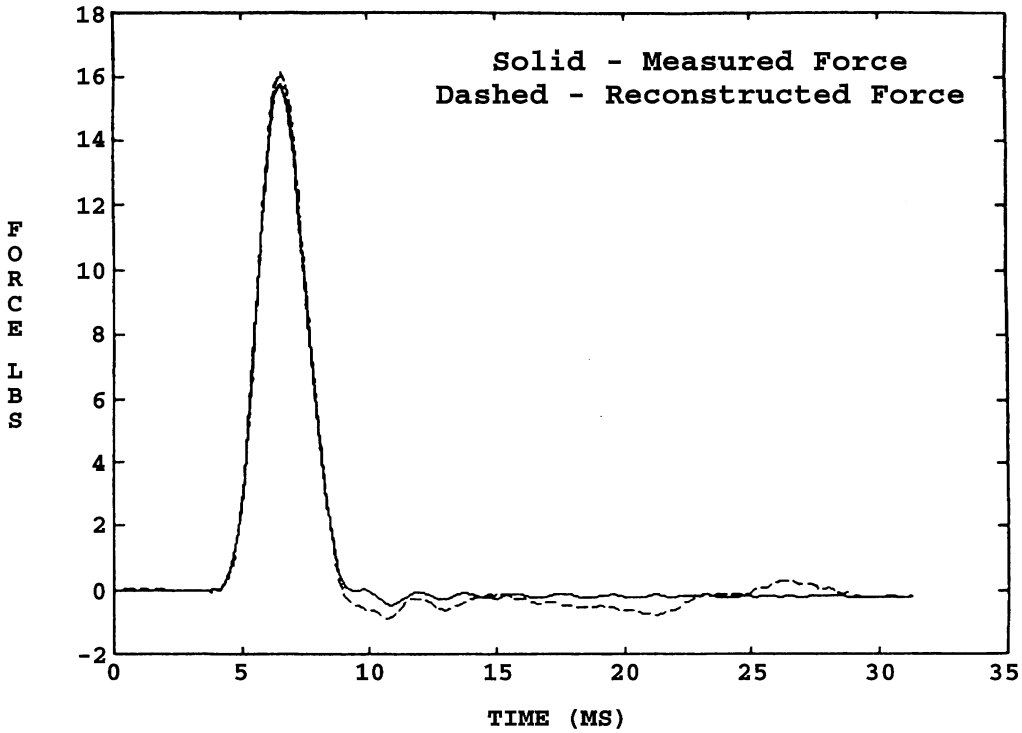

FIGURE 4 Force reconstructed from impact at mass 1 using the SWAT-TEEM algorithm (weights averaged from impacts at mass 4).

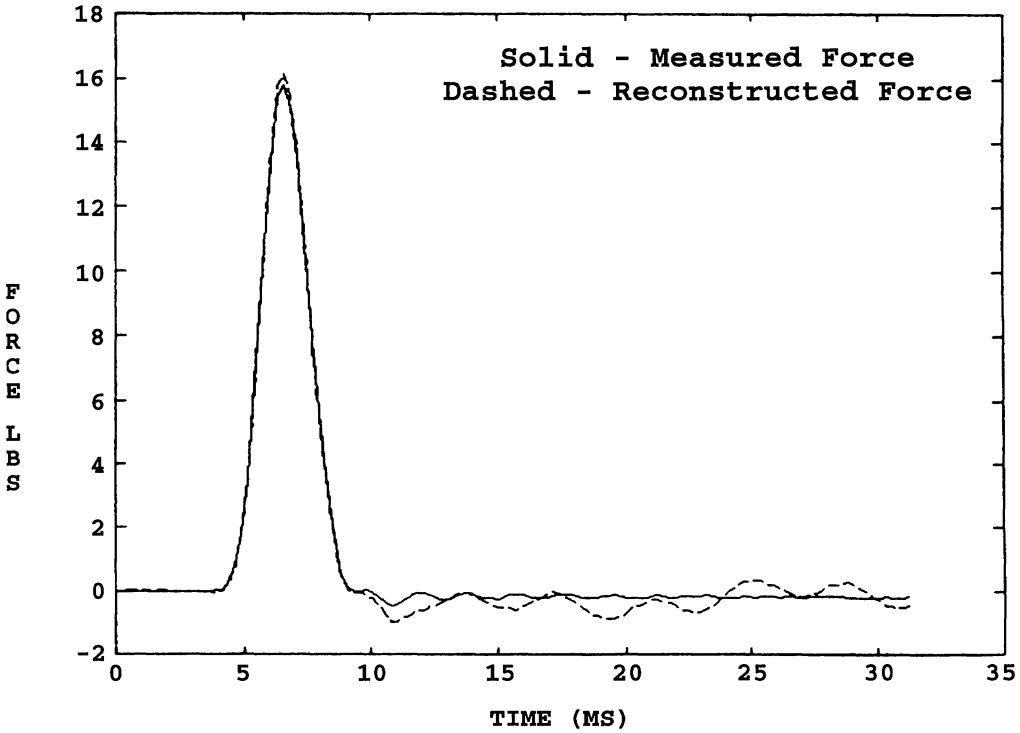

FIGURE 5 Force reconstructed from impact at mass 1 using the SWAT-CAL algorithm (weights averaged from impacts at section 2).

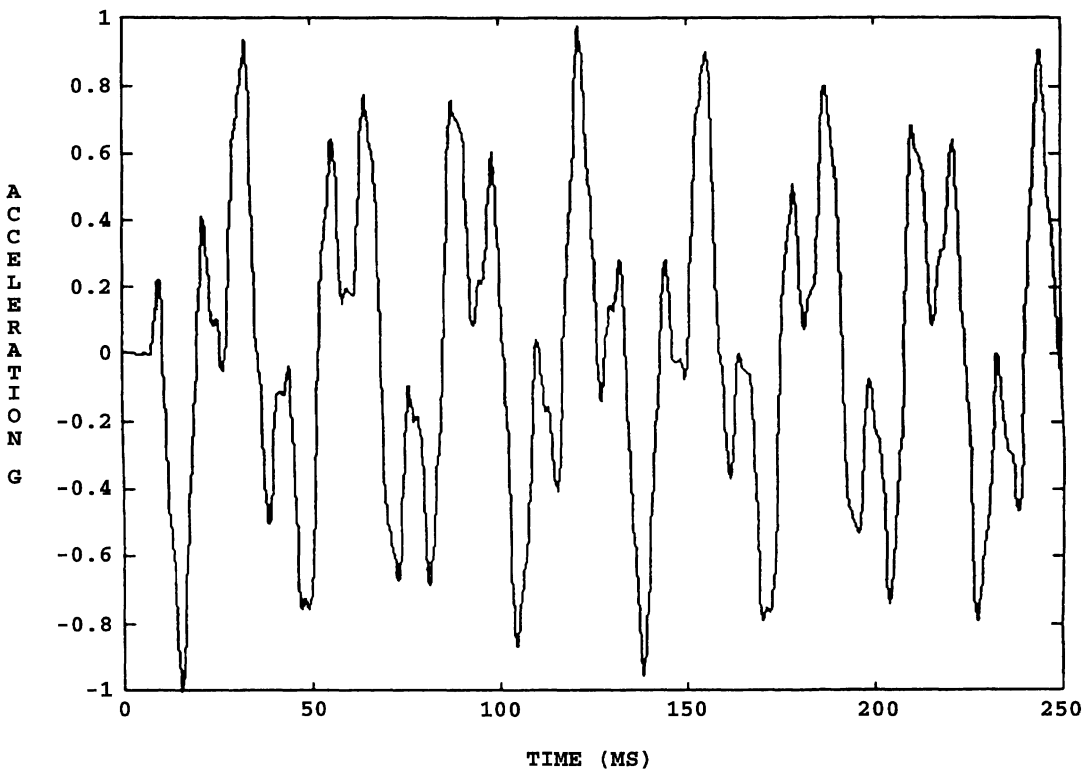

FIGURE 6 Individual acceleration response of mass 4 to impact applied at mass 1.

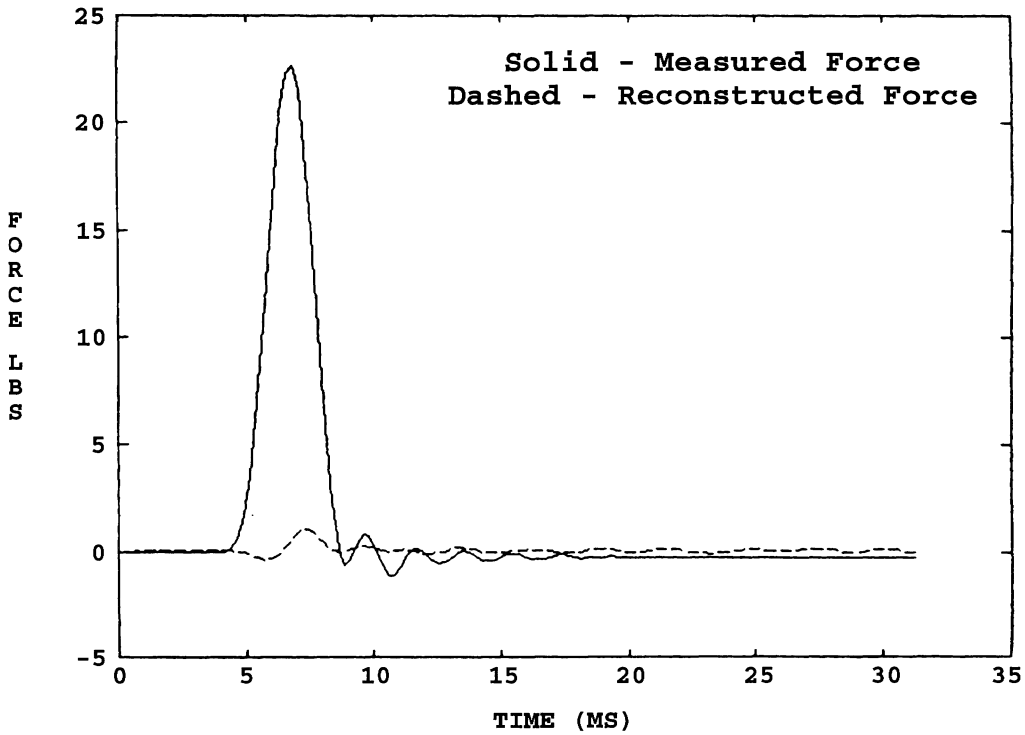

FIGURE 7 Lateral force reconstructed from axial impact at mass 1 using the SWAT-CAL algorithm (weights averaged from impacts at section 2).

space. The force reconstructions for all impact locations agreed with the measured forces to within 5%. A measured acceleration time history from mass 4 is shown in Fig. 6 for comparison with the measured and reconstructed forces. This individual acceleration response exhibits the resonant behavior of the lumped mass beam.

Additional impacts were made to the beam in the axial direction to test the robustness of these force reconstruction techniques. An example of the *lateral* force reconstruction using SWAT-CAL in response to an axial impact is shown in Fig. 7 where it can be seen that almost a zero force was reconstructed. This reconstruction shows that the SWAT algorithm is not sensitive to out of axis forces.

CONCLUSIONS

Three techniques for the reconstruction of dynamic forces from measured acceleration responses on a structure have been presented and applied to a lumped mass beam structure. All three techniques produce weighting vectors that reconstruct impact forces equally well for this structure. One technique, SWAT, requires that a modal analysis of the structure be conducted. The second technique, SWAT-CAL, requires that measurement of an applied force and the corresponding acceleration responses be made. The third technique, SWAT-TEEM, requires the measurement of free-decay acceleration responses but no applied force measurements, although the applied force is needed to check the laboratory force reconstruction results. SWAT-TEEM has the potential for being used to calculate a weighting vector during the free-decay response after a high level field impact test for comparison with the weighting vector calculated from low level laboratory data. The development of these techniques will continue in the future with applications to real structures at Sandia National Laboratories.

REFERENCES

Bateman, V. I., Carne, T. G., Gregory, D. L., Attaway, S. W., and Yoshimura, H. R., 1991, "Force Reconstruction for the Slapdown Test of a Nuclear Transportation Cask," *ASME Jounal of Vibration and Acoustics*, Vol. 113, No. 2.

Bateman, V. I., Carne, T. G., and McCall, D. M., 1992, "Force Reconstruction for Impact Tests of an Energy-Absorbing Nose," *The International Journal of Analytical and Experimental Modal Analysis*, Vol. 7, No. 1.

Gregory, D. L., Priddy, T. G., and Smallwood, D. O., 1986, "Experimental Determination of the Dynamic Forces Act-

ing on Non-Rigid Bodies," SAE Technical Paper 861791, *Aerospace Technology Conference and Exposition*, Long Beach, CA, October 1986.

Hillary, B., and Ewins, D. J., 1984, "The Use of Strain Gages in Force Determination and Frequency Response Function Measurements," *Proceedings of the 2nd International Modal Analysis Conference*, Orlando, FL, February 6–9, pp. 627–634.

Kreitinger, T. L., Wang, M. L., and Schreyer, H. L., 1988, "Analytical Approaches for Determining Effective Weights Used in the Method of Weighted Accelerations for Force Measurements," University of New Mexico.

Mayes, R. L., 1994, "Measurement of Lateral Launch Loads on Re-Entry Vehicles Using SWAT," *Proceedings of the 12th International Modal Analysis Conference*, Honolulu, HI, February, pp. 1063–1068.

Priddy, T. G., Gregory, D. L., and Coleman, R. G., 1988, "Strategic Placement of Accelerometers to Measure Forces by Sum of Weighted Accelerations," SAND87-2567.UC-38, National Technical Information Service, Springfield, VA, February.

Priddy, T. G., Gregory, D. L., and Coleman, R. G., 1989, "Measurement of Time-Dependent External Moments by the Sum of Weighted Accelerations," SAND88-3081.UC-38, National Technical Information Service, Springfield, VA, January.

Smallwood, D. O., and Gregory, D. L., 1987, "Experimental Determination of the Mass Matrix Using a Constrained Least Squares Solution," *Proceedings of the 28th Structures, Structural Dynamics and Materials Conference*, Monterey, CA, April.

Starkey, J. M., and Merrill, G. L., 1989, "On the Ill-Conditioned Nature of Indirect Force-Measurement Techniques," *The International Journal of Analytical and Experimental Modal Analysis*, Vol. 4, No. 3.

Stevens, K. K., 1987, "Force Identification Problems—An Overview," *Proceedings of the 1987 SEM Spring Conference on Experimental Mechanics*, Houston, TX, June 14–19, pp. 838–844.

Wang, T. J., Kreitinger, M. L., and Luo, H. L., 1987, "Force Identification from Structural Responses," *Proceedings of the 1987 SEM Spring Conference on Experimental Mechanics*, Houston, TX.