B. P. Wang
Department of Mechanical Engineering
University of Texas at Arlington
Arlington, TX 76019

F. H. Chu
Mechanical Systems
National Space Program Office
Taiwan, R.O.C.

Finite Element Model Optimization Using Diagonal Mass Matrix

By adjusting the analytic mass matrix or stiffness parameters, the correlation between measured and computed modal data can be improved. This article proposes a simple method for model optimization. Numerical examples will be included to illustrate the proposed approach. © 1997 John Wiley & Sons, Inc.

INTRODUCTION

Analytic model (finite element model) improvement has been studied for several decades. In general, the techniques can be classified into two main streams: one is based on the assumption that the analytic mass matrix is exact and then uses it to modify the measured mode shapes (Baruch and Bar-Itzhack, 1978; Gravitz, 1958; McGrew, 1969; Targoff, 1976). The other approach assumes the measured mode shapes are correct. The analytic mass matrix is then modified to get an improved mass matrix. The latter is more appealing in a number of published articles (Berman, 1979; Berman and Wei, 1981; Berman et al., 1980; Zhang and Zhao, 1986).

In this article the model optimization problem will be accomplished in a two-step process. In the first step we will identify an updated diagonal mass matrix by using measured mode shape data. The stiffness parameters of the system will then be adjusted to improve the correlation between measured and computed modal data. The eigenvalues and eigenvectors of the identified mass matrix may vary slightly from the measured data, and whether they are accepted or not can be checked by the modal assurance criterion (MAC) (Ewins, 1984).

The diagonal mass matrix required by this approach can be computed by solving a constrained optimization problem. The objective function of this problem is a weighted Euclidean norm of the mass distribution. To render meaningful results, the following constraints are imposed: the orthogonality condition of the identified mass matrix with respect to the measured modes and the conservation of total system mass as well as the position center of mass. Using the method of Lagrange multipliers, a closed form solution has been derived. However, in the above approach, if orthogonality conditions of the mass matrix with respect to the measured modes are strictly enforced, unreasonable entries may occur in the identified mass matrix (such as too big or negative). To circumvent the difficulty, the orthogonal conditions are to be relaxed as a set of inequality constraints. In such case, the diagonal mass matrix can be identified by solving a quadratic programming problem.

Once a satisfactory mass model is available, the

Received April 9, 1996; Accepted December 27, 1996.
Shock and Vibration, Vol. 4, No. 3, pp. 163–168 (1997)

CCC 1070-9622/97/030163-06

correlation between the computed modal data of the identified mass matrix and the measured data can be improved by adjusting the stiffness parameters. In the following sections, we will formulate the above ideas and present two simple numerical examples to demonstrate our proposed approaches.

IDENTIFICATION OF DIAGONAL MASS MATRIX FROM TEST DATA

Consider an analytic mass matrix $\mathbf{M}_a$ of rank n that is diagonal. Let the incomplete measured modal matrix be $\mathbf{\Phi}$ $(n \times m)$, where $n > m$, and let $\mathbf{M}$ be the unknown diagonal mass matrix of rank n to be identified. The objective is to find $\mathbf{M}$ subjected to the following constraints: orthogonality conditions,

$$\mathbf{\Phi}^T \mathbf{M} \mathbf{\Phi} = I, \tag{1}$$

where $\mathbf{\Phi}$ has been normalized to unit generalized mass with respect to $\mathbf{M}_a$; conservation of total mass,

$$\sum_{i=1}^{n} m_{ii} = W, \tag{2}$$

where m_{ii} is the diagonal term of $\mathbf{M}$ and W is the total mass of the system; conservation of center of mass,

$$\sum_{i=1}^{n} m_{ii} X_{ij} = C_j, \quad j \leq 3, \tag{3}$$

where X_{ij} is the coordinate of m_{ii} in the j direction and C_j is the center of mass in the j direction.

The left side of constraint (1) is a symmetrical matrix because $\mathbf{M}$ is assumed to be diagonal. Thus, by equating the upper triangular portion including the diagonal terms of both sides, the equation can be rearranged as

$$[\mathbf{A}] \begin{Bmatrix} m_{11} \\ m_{22} \\ \vdots \\ m_{nn} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{Bmatrix}, \tag{4}$$

or expressed as

$$[\mathbf{A}]\{m_{ii}\} = \{\mathbf{I}^{\mathrm{up}}\}, \tag{5}$$

where $[\mathbf{A}]$ is a constant matrix of order $m(m + 1)/2 \times n$ and $\mathbf{I}^{\mathrm{up}}$ is an $m(m + 1)/2 \times 1$ column formed by taking the upper triangular portion of the identity matrix $\mathbf{I}$ in the sequence from the diagonal term to the right end, from the top row to the bottom row.

The three constraint equations, Eqs. (2), (3), and (5) can be put together, for a 3-dimensional structure, as

$$\begin{bmatrix} [\mathbf{A}] \\ 1 \cdots 1 \cdots 1 \\ x_{11} \cdots x_{n1} \\ x_{12} \cdots x_{n2} \\ x_{13} \cdots x_{n3} \end{bmatrix} \begin{Bmatrix} m_{11} \\ \vdots \\ m_{nn} \end{Bmatrix} = \begin{Bmatrix} \mathbf{I}^{\mathrm{up}} \\ W \\ WC_1 \\ WC_2 \\ WC_3 \end{Bmatrix},$$

or expressed as

$$[\mathbf{H}]\{m_{ii}\} = \{D\}, \tag{6}$$

where $[\mathbf{H}]$ is an $m(m + 1)/2 + 4 \times n$ matrix.

If $m(m + 1)/2 + 4 < n$, Eq. (6) is undetermined. In this case, Eq. (6) can be partitioned as

$$[[\mathbf{B}][\mathbf{C}]] \begin{Bmatrix} m_b \\ \hline m_c \end{Bmatrix} = \{D\} \tag{7}$$

or

$$[\mathbf{B}]\{m_b\} + [\mathbf{C}]\{m_c\} = \{D\}, \tag{8}$$

where a proper set of m_b are chosen so that $[\mathbf{B}]$ is a square matrix of rank $m(m + 1)/2 + 4$.

From Eq. (8), $\{m_b\}$ can be expressed as:

$$\{m_b\} = [\mathbf{B}]^{-1}[\mathbf{D}] - [\mathbf{B}]^{-1}[\mathbf{C}]\{m_c\}. \tag{9}$$

We now formulate the optimization problem: minimize a weighted Euclidean norm,

$$\begin{aligned} \varepsilon &= \|\mathbf{M}_a^{-1/2}(\mathbf{M} - \mathbf{M}_a)\mathbf{M}_a^{-1/2}\| \\ &= \sum_{i=1}^{n} \left[\left(\frac{m_{ii}}{m_{aii}}\right)^2 - \left(\frac{2m_{ii}}{m_{aii}}\right) \right] + n, \end{aligned} \tag{10}$$

subject to

$$\{m_b\} = [\mathbf{B}]^{-1}[\mathbf{D}] - [\mathbf{B}]^{-1}[\mathbf{C}]\{m_c\}, \tag{11}$$

where m_{aii} is the diagonal term of $\mathbf{M}$.

Note that ε is a function of n variables that are not all independent. By substituting (11) into (10), the independent variables in ε are reduced to those of the submatrix $[\mathbf{m}_c]$ and ε automatically contains the three constraints (1), (2), and (3).

Differentiating of ε with respect to $[\mathbf{m}_c]$ and setting the results equal to zero will result in a set of linear equations for m_c,

$$\frac{\partial \varepsilon}{\partial [\mathbf{m}_c]} = 0. \tag{12}$$

Once $[\mathbf{m}_c]$ is found, it can be substituted into (11) to obtain $[\mathbf{m}_b]$ and the mass matrix $\mathbf{M}$ is then identified.

Using the above formulation, unreasonable results may occur such as too big, too small, or negative entries in the identified mass matrix. This may be due to the strong demand upon the orthogonality conditions. By relaxing the orthogonality condition constraints somewhat, the situation can usually be corrected. This corrective procedure leads to a quadratic programming problem. The formulation is: minimize a weighted Euclidean norm,

$$\varepsilon = \|\mathbf{M}_a^{-1}(\mathbf{M} - \mathbf{M}_a)\mathbf{M}_a^{-1/2}\|, \tag{13}$$

subject to relaxed orthogonality constraints,

$$\frac{|\Phi_i \mathbf{M} \Phi_j|}{\Phi_i \mathbf{M} \Phi_i} \le \alpha_i \quad \text{(such as 0.01)}; \tag{14}$$

$$\text{conservation of total mass [Eq. (2)]}, \tag{15}$$

$$\text{conversation of center of mass, [Eq. (3)]}, \tag{16}$$

and additional, we can impose bounds on m_{ii},

$$m_{ii}^L \le m_{ii} \le m_{ii}^u. \tag{17}$$

The above formulation can be solved by any quadratic programming package.

In Eq. (6), if $m(m + 1)/2 + 4 > n$, the equation becomes overdetermined. By using the least square estimation, $\{m_{ii}\}$ will be

$$\{m_{ii}\} = (H^T H)^{-1} H^T D. \tag{18}$$

Indicator MAC (9) can be used to compare the measured and computed mode shapes using updated mass matrix $\mathbf{M}$. The MAC is defined to be

$$\mathrm{MAC}(p, x) = \frac{\left| \sum_{j=1}^{n} (\Phi_x)_j (\Phi_p)_j \right|^2}{\left(\sum_{n=1}^{n} (\Phi_x)_j (\Phi_x)_j \right) \left(\sum_{j=1}^{n} (\Phi_p)_j (\Phi_p)_j \right)}, \tag{19}$$

where $(\Phi_x)_j$ is the measured mode shape of the jth degree of freedom and $(\Phi_p)_j$ is the computed mode shape of the jth degree of freedom when using the updated mass matrix $\mathbf{M}$.

The ideal case is $\Phi_x = \Phi_p$ from the same mode, then MAC will equal to one. In practice, MAC is close to one for correlated modes and close to zero for uncorrelated modes. Generally, if the value is greater than 0.9 for the former and less than 0.05 for the latter, then the modified mass matrix $\mathbf{M}$ may be considered acceptable.

IDENTIFICATION OF STIFFNESS TO CORRELATE PREDICTED AND MEASURED EIGENVALUES

Using the identified mass matrix $\mathbf{M}$ obtained in the previous section as the true mass matrix, model optimization will be achieved by adjusting the stiffness parameters to improve the correlation between measured and computed model data.

The formulation is to to find $\{K_c\}$ to minimize an eigenvalue norm with bounds on stiffness parameters and eigenvalues: minimize,

$$\sum_{r=1}^{m} (\lambda_r - \lambda_r^t)^2, \tag{20}$$

subject to

$$\begin{aligned} \frac{|Kc_i - Ka_i|}{Ka_i} &\le \beta_i \quad \text{(such as 0.1)}, \\ \frac{|\lambda_r - \lambda_r^t|}{\lambda_r^t} &\le \gamma_r \quad \text{(such as 0.01)}, \end{aligned} \tag{21}$$

where m is the number of modes, λ_r^t is the rth measured eigenvalue, and λ_r is the updated eigenvalue that is expected to minimize the norm. Ka_i is the ith stiffness parameter of the analytic modal and Kc_i is the correlated stiffness parameter to be identified.

The unknown eigenvalue λ_r can be linearized as

$$\lambda_r = \bar{\lambda}_r + \sum_{i=1}^{n} \frac{\partial \bar{\lambda}_r}{\partial K_i} (Kc_i - Ka_i), \tag{22}$$

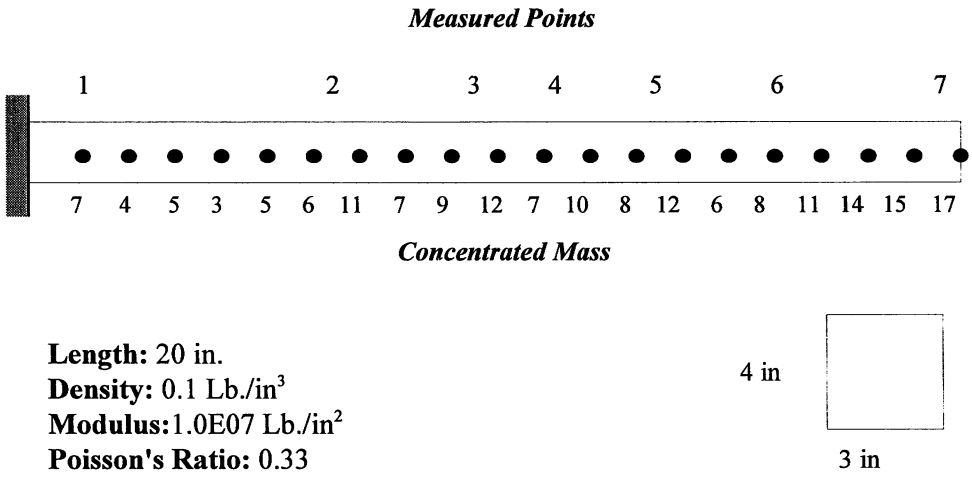


FIGURE 1 Cantiliever beam with concentrated mass.

where $\bar{\lambda}_r$ is the computed eigenvalue of the rth mode based on the analytic stiffness modal and the modified mass matrix **M**.

In Eq. (22), for a spring-mass model,

$$\frac{\partial \bar{\lambda}_r}{\partial K_i} = (\bar{\Phi}_r)^T \frac{\partial K}{\partial K_i} (\bar{\Phi}_r) = (K\bar{\Phi}_{i+1,r} - \bar{\Phi}_{i,r})^2, \quad (23)$$

where $\bar{\Phi}_{i,r}$ is the ith entry of the corresponding eigenvector of $\bar{\Phi}_r$.

The eigenvalue constraint (21) can be expanded as

$$-\lambda_r \lambda_t' \leq \lambda_r - \lambda_t' \leq \lambda_r \lambda_t'. \quad (24)$$

Substituting Eqs. (22) and (23) into (24), Eq. (24) becomes

$$(1 - \gamma_r)\lambda_r^t - \bar{\lambda}_r + \Theta_{i,r} Ka_i \leq \Theta_{i,r} Kc_i$$

and

$$\Theta_{i,r} Kc_i \leq (1 + \gamma_r)\lambda_r^t - \bar{\lambda}_r + \Theta_{i,r} Ka_i, \quad (25)$$

where $\Theta_{i,r}$ represents $\sum_{i=1}^{n} (\bar{\Phi}_{i+1,r} - \bar{\Phi}_{i,r})^2$.

The objective function (20) associated with linear constraints (25) will form a quadratic programming problem that can be solved by a quadratic programming package to obtain new stiffness parameters.

With the new stiffness parameters $[\mathbf{K}_c]$ obtained from the above formulations and combining with the modified mass matrix **M**, one can compute modal data such as eigenvalues and eigenvectors of this new model structure; and the values of MAC can be computed to see whether they are acceptable or not. Several iterations may be required to obtain a more acceptable improved stiffness model.

Table 1. Analytic Modal Data of Fig. 1

Analytic Mass by Lumped Mass Model (lb)	Assumed Measured Mode Shapes	
22.0333	0.01	0.06
36.2333	0.42	1.63
26.8333	0.75	1.96
19.9000	1.07	1.80
25.6666	1.36	1.27
36.2677	1.78	−0.15
33.4667	2.30	−1.80

Table 2. Improved Modal Data of Fig. 1

Improved Mass (lb)	Computed Mode Shapes	
24.1135	0.0094	0.0577
34.9885	0.4017	1.5686
23.7134	0.7598	1.9768
18.4510	1.0373	1.8174
25.7277	1.3361	1.2828
42.6286	1.8080	−0.0927
30.7773	2.2917	−1.8349

Comparison between Table 1 and Table 2: total weight change, 0.0001 lb; center of mass change, 3.5E-06 in; orthogonality condition check, Eq. (1) satisfied. MAC: for correlated modes, 1, 0.999; for uncorrelated modes, 0.008, 0.01.

Table 3. Assumed Measured Modal Data of Spring-Mass System

Mass[a]	Stiffness[a]	Mode Shapes[b]		
3.0	2200	0.0574	−0.1907	−0.2146
3.5	1800	0.1133	−0.3493	−0.3154
4.2	1400	0.1826	−0.4827	−0.2270
2.5	1600	0.2451	−0.4722	0.0752
1.8	1100	0.3238	−0.2979	0.4489
0.7	1000	0.3908	−0.0570	0.5054
1.2	2100	0.4207	−0.0666	0.4443
2.1	2500	0.4468	0.1866	0.3048
1.5	1100	0.4827	0.3847	−0.1896
2.5	1800	0.4962	0.4622	−0.4033

[a]Initial analytic model.
[b]Simulated test data.
Eigenvalues: 4.267, 10.574, 18.796.

Table 4. Modal Data after Relaxing Orthogonality Constraint

Mass	Mode Shapes		
2.94	0.0519	−0.1685	−0.2212
3.08	0.1137	−0.3425	−0.3531
2.92	0.1886	−0.4787	−0.2245
2.95	0.2456	−0.4617	0.0992
2.18	0.3163	−0.2931	0.4678
0.72	0.3813	−0.0334	0.4820
1.18	0.4098	0.0916	0.4257
1.88	0.4303	0.1916	0.3016
1.46	0.4631	0.3809	−0.1775
2.69	0.4763	0.4607	−0.4150

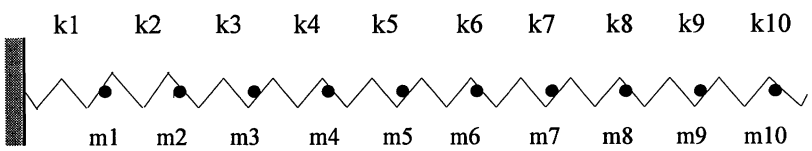
Eigenvalues: 4.301, 10.770, 19.581.
Stiffness parameters are the same as Table 3.

EXAMPLES

The first example is a cantilever beam with 20 concentrated mass evenly spanned along the beam (Fig. 1). Seven points have been arbitrarily picked up as the measurement points, and two simulated measured mode shapes are assumed. The simulated measured mode shapes are taken by solving the free vibration problem of Fig. 1 using finite element coefficients at these seven points. These data are rounded off to the second decimal place and then ±5% variation is added to them arbitrarily. The new mass **M** is solved by procedures defined in Eqs. (1)–(12). Table 1 shows the original structure conditions. Table 2 contains the results after optimization. The comparison shows that the results are quite satisfactory.

The second example is a 10 degrees of freedom (DOF) spring-mass system. Three lowest eigenvalues and their corresponding mode shapes (Table 3) derived from Fig. 2 are assumed to be in the measured modal data of the same spring-mass system but with different mass and stiffness parameters as shown in Table 3.

Following the steps defined in Eqs. (1)–(12), negative entries occurred in the identified mass matrix **M**, for the reason stated in the previous section. By relaxing the orthogonality constraint, the quadratic programming problem is solved. Table 4 shows the results of the new mass distribution and the corresponding eigenvalues and mode shapes after relaxing the orthogonality constraints. Comparing Tables 3 and 4, the difference between the modal data is expected due to the relaxed orthogonal conditions and the correlation can be improved by adjusting the stiffness parameters. After applying the stiffness updating procedures defined by Eqs. (20)–(25), the results of the new stiffness parameters associated with the corresponding modal data are shown in Table 5. The values of MAC for Tables 3 and 5 have also been calculated in the comparison data, showing that the final structure is acceptable.



Stiffness parameters:
k1 to k5: 2000 2000 1500 1500 1000
k6 to k10: 1000 2000 2000 1000 2000
Mass:
m1 to m5: 3 3 4 3 2 m6 to m10: 1 1 2 1 3

FIGURE 2 Assumed exact test model of 10-DOF spring-mass system.

Table 5. Modal Data after Adjusting Stiffness Parameters

Mass	Mode Shapes		
1980	0.0580	−0.1821	−0.2483
1694	0.1239	−0.3587	−0.3756
1540	0.1918	−0.4703	−0.2309
1456	0.2541	−0.4426	0.1576
1210	0.3178	−0.2851	0.4796
1100	0.3763	−0.0468	0.4738
1890	0.4077	0.0939	0.4024
2250	0.4301	0.2064	0.2629
1205	0.4598	0.3796	−0.1524
1620	0.4742	04691	−0.4093

Eigenvalues: 4.289, 10.723, 19.453.

Mass distributions are the same as Table 4. Comparison between Table 3 and Table 5: total weight change, 0; orthogonality condition, relaxed. MAC: for correlated mdoes, 0.984, 0.999, 0.998; for uncorrelated modes, 0.059, 0.012, 0.064, 0.011, 0.006, 0.028.

DISCUSSION

Application of a lumped mass model in finite element model optimization is proposed in this work. The advantage of using a diagonal mass matrix is that a modified structure model based on the new mass matrix **M** can be easily interpreted by engineers. The existence of solutions to the model optimization problem described here depends on the condition $m(m + 1)/2 + 4 < n$, where m is the number of measured modes and n is the number of DOF of the model to be identified. For example, to identify a model with 100 DOF, less than 13 measured mode shapes are required. On the other hand, using 15 measured modes, one can optimize a model with at least 125 DOFL. If $n < m(m + 1)/2 + 4$, one has to use the least square estimation (18) to identify the diagonal mass matrix.

REFERENCES

Baruch, M., and Bar-Itzhack, I. Y., 1978, "Optimal Weighted Orthogonalization of Measured Modes," *AIAA Journal,* Vol. 16, pp. 346–351.

Berman, A., 1979, "Mass Matrix Correction Using an Incomplete Set of Measured Modes," *AIAA Journal,* Vol. 17, pp. 1147–1148.

Berman, A., and Wei, F. S., 1981, "Automated Dynamic Analytical Model Improvement," NASA CR-3452.

Berman, A., Wei, F. S., and Rao, K. W., 1980, "Improvement of Analytical Models Using Modal Test Data," *Proceedings of AIAA/ASME/ASCE/AHS 21th SDM Meeting,* pp. 860–870.

Ewins, D. J., 1984, "Modal Testing: Theory and Practice," Wiley, New York, pp. 224–226.

Gravitz, S. I., 1958, "An Analytical Procedure for Orthogonalization of Experimentally Measured Modes," *Journal of Aerospace Science,* Vol. 25, pp. 1140–1150.

McGrew, J., 1969, "Orthogonalization of Measured Modes and Calculation of Influence Coefficients," *AIAA Journal,* Vol. 7, pp. 774–776.

Targoff, W. P., 1976, "Orthogonal Check and Correction of Measured Modes," *AIAA Journal,* Vol. 14, pp. 164–167.

Zhang, D. W., and Zhao, L. C., 1986, "The Minimum–Norm Perturbation Method For Modification of Structural Analytical Model," *Proceedings of the 4th International Modal Analysis Conference,* pp. 888–898.