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Bulging Modes of Circular Bottom Plates in Rigid Cylindrical Containers Filled with a Liquid

In this article the free vibrations of the bottom plate of an otherwise rigid circular cylindrical tank filled with liquid are studied, considering only the bulging modes (when the amplitude of the plate displacement is predominant with respect to that of the free surface). The tank axis is vertical, thus the free liquid surface is orthogonal to the tank axis. The liquid is assumed to be inviscid, and the contribution of the free surface waves to the dynamic pressure on the free liquid surface is neglected. Wet and dry mode shapes of the plate are assumed to be the same, so that the natural frequencies are obtained by using the nondimensionalized added virtual mass incremental (NAVMI) factors and the modal properties of dry plates. This simplifies computations compared to other existing theoretical approaches. NAVMI factors express the nondimensionalized ratio between the reference kinetic energy of the liquid and that of the plate and have the advantage that, due to their nondimensional form, they can be computed once and for all. Numerical results for simply supported and clamped bottom plates, as well as for supported plates with an elastic moment edge constraint are given. For more accurate results, and to exceed the limits of the assumed modes approach, the Rayleigh–Ritz method is applied and results are compared to those obtained by using the NAVMI factors and other existing methods in the literature. *© 1997 John Wiley & Sons, Inc.*

INTRODUCTION

The study of the free vibrations of circular cylindrical tanks has interested many researchers; this is obviously due to the wide application of tanks in mechanical, aeronautical, and civil engineering. Cylindrical tanks are often composed of a shell and a circular bottom plate. Many studies investigated vibrations of the shell and the bottom plate of these containers, and some are reported here. The liquid-filled tanks have two families of modes: the sloshing and the bulging ones. Sloshing modes are caused by the oscillation of the liquid free surface, due to the rigid body movement of the container; these modes are also affected by the flexibility of the container. The vibrations of the tank walls (bottom plate and shell) take the name of bulging modes when the amplitude of the wall displacement is predominant to that of the free surface; in this case, the tank walls and base oscillate with the liquid. The sloshing modes of circular cylindrical tanks having a flexible bottom and rigid shell wall were studied, for example, by Bleich (1956), Bhuta and Koval (1964a,b), Tong (1967), Siek-

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mann and Chang (1968), Bauer and Siekmann (1969), and Capodanno (1989); other references are given in Abramson (1966). Both sloshing and bulging axisymmetric modes of the bottom plates were experimentally and theoretically studied by Chiba, who also investigated the effect of the static deflection due to the fluid weight (Chiba, 1992, 1993) and the effect of a Winkler foundation (Chiba, 1994) on the bottom plate vibrations. Nagaia and Takeuchi (1984) studied plates of arbitrary shape in contact with a viscous fluid, and Nagaia and Nagai (1986) studied circular bottom plates on Winkler foundations in containers filled with viscous fluid. Nonlinear sloshing was studied by Bauer et al. (1971), and nonlinear sloshing and bulging modes of the bottom circular plate were experimentally investigated by Chiba (1992). A circular plate as the surface cover of a rigid circular cylindrical tank was studied by Bauer (1995). Regarding the shell vibrations, due to the great amount of literature, we only remember some works of Berry and Reissner (1958), Lindholm et al. (1962), Baron and Skalak (1962), and Kondo (1981), and that of Haroun and Housner (1981) on the earthquake response of storage tanks. Bauer and Siekmann (1971) and Bauer et al. (1972) studied the sloshing modes of circular cylindrical containers with both flexible bottom plates and flexible shells.

It is worth mentioning that the first modern day studies on the vibrations of circular plates in contact with fluids can be attributed to Rayleigh (1877) and Lamb (1921). However, they were interested in plates vibrating in a circular aperture of an infinite rigid wall, so that the fluid was unlimited; this is a different problem from that given by a cylindrical tank. This is also the case in the works of Kwak (1991), Kwak and Kim (1991), Ginsberg and Chu (1992), and Amabili et al. (1995a,b).

In this article attention is focused on the bulging modes of the flexible bottom plate of an otherwise rigid circular cylindrical container with a vertical axis and filled with liquid, so that the free surface of the liquid is orthogonal to the tank axis. The volume occupied by the liquid is cylindrical and the liquid velocity potential can be obtained by using the variable separation. This technique was used in the quoted studies to find the velocity potential of the inviscid liquid for sloshing and bulging modes. All these studies which include the effect of the free surface waves of the fluid and, in some cases, also the effect of the superficial tension of the liquid (Bauer and Siekmann, 1971) or the in-plane stress of the plate (Chiba, 1993), give quite complex solutions that must be numerically solved for each specific case; therefore, few numerical results are available, especially for bulging and asymmetric modes. On the contrary, in this work the effects of the free surface waves on the dynamic pressure at the free surface, the superficial tension, and the hydrostatic pressure are neglected (Morand and Ohayon, 1992), so that the plate vibrations, only considering bulging modes, are studied by using a simplified theory. As a consequence, the nondimensionalized added virtual mass incremental (NAVMI) factor approach, already successfully used for circular plates by Kwak and Kim (1991), Kwak (1991), Amabili et al. (1995a,b), and Amabili and Dalpiaz (1995), is applied so that all numerical computations can be made nondimensional and the natural frequencies of the plate in contact with the liquid can be obtained directly from those in a vacuum, considering the same plate boundary condition. This is a computational simplification with respect to other existing theoretical approaches; moreover, due to their nondimensional form, NAVMI factors can be computed once and for all. The proposed approach is based on the Rayleigh quotient for coupled vibrations (Zhu, 1994) and on the hypothesis that the dry (in vacuum) and wet (in liquid) mode shapes of the plate are unchanged (assumed modes approach); the accuracy of this approach is checked by using the Rayleigh–Ritz method (Zhu, 1995) that removes the restrictive hypothesis on the wet mode shapes. In particular, the wet mode shapes are developed in a series by using the dry mode shapes as admissible functions.

The Rayleigh–Ritz approach allows us to exceed the limits of the assumed modes approach, but it retains a remarkable simplicity of computation, with respect to other analytical techniques already applied to this problem. Moreover, nondimensional results are very useful for engineering applications. It is also shown that the results of the Rayleigh–Ritz method match very well with numerical data available in the literature (Chiba, 1993) for bulging modes, obtained by using more complex theories, and thus proving that the very simple NAVMI factor approach is accurate enough for many engineering applications.

THEORETICAL BACKGROUND

A thin circular plate having the thickness, h, and the mass density, ρ_P, vibrating in a vacuum, is considered. The plate material is assumed to be linearly elastic, homogeneous, and isotropic; the effects of shear deformation, rotary inertia, and damping are neglected. The equation of motion for the transverse displacement, w, of the plate is governed by (Leissa, 1969)

$$D\,\nabla^4 w + \rho_P h \frac{\partial^2 w}{\partial t^2} = 0, \tag{1}$$

where $D = Eh^3/[12(1-\nu^2)]$ is the flexural rigidity of the plate and ν and E are Poisson's ratio and Young's modulus, respectively. In addition,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{2}$$

is the Laplace operator in the polar coordinates r and θ. The solution of Eq. (1) is obtained by using the variable separation. In the case where boundary conditions at the edge are uniform, the solution takes the following form (Leissa, 1969):

$$w(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} W_{mn}(r)\cos(m\theta) f(t), \tag{3}$$

where

$$W_{mn}(r) = \left[A_{mn} J_m\left(\frac{\lambda_{mn} r}{a}\right) + C_{mn} I_m\left(\frac{\lambda_{mn} r}{a}\right)\right], \tag{4}$$

$$f(t) = e^{i\omega t}, \tag{5}$$

in which m and n represent the number of nodal diameters and circles; A_{mn} and C_{mn} are the mode shape constants, whose ratio is determined by the boundary conditions; J_m and I_m are the Bessel function and the modified Bessel function of the first kind; λ_{mn} is the frequency parameter, which is also determined by the boundary conditions; a is the plate's radius; and i is the imaginary unit. The frequency parameter λ_{mn} is related to the circular frequency ω of the plate by

$$\omega = \frac{\lambda_{mn}^2}{a^2}\sqrt{\frac{D}{\rho_P h}}. \tag{6}$$

The values of the modal parameters λ_{mn}, A_{mn}, and C_{mn} are given by Leissa (1969) for clamped circular plates, by Leissa and Narita (1980) for simply supported circular plates, and by Amabili and colleagues (1995b) for free-edge circular plates; λ_{mn} for circular plates with elastic edge supports are computed in Azimi (1988); and the mode shape constants can be obtained by using the boundary conditions.

The Rayleigh quotient for coupled vibrations (Zhu, 1994) is used to evaluate natural frequencies of the plate in contact with the liquid. The original idea is attributed to Rayleigh (1877) and Lamb (1921). Hence, we may write the following for each case:

$$f_V^2 \propto (V_P/T_P^*)_{\text{vacuum}} \quad f_L^2 \propto (V_P/(T_P^* + T_L^*))_{\text{liquid}}, \tag{7}$$

where V_P is the maximum potential energy of the plate and T_P^* and T_L^* are the reference kinetic energies of the plate and the liquid, respectively. Based on the hypothesis that the wet mode shapes are equal to the dry mode shapes, natural frequencies of free vibration in a liquid, f_L, can be related to natural frequencies in a vacuum, f_V. Thus, the following formula is obtained:

$$f_L = \frac{f_V}{\sqrt{1+\xi_{mn}}}, \tag{8}$$

where ξ_{mn} is the added virtual mass incremental (AVMI) factor. This factor is given by the ratio between the reference kinetic energy of the liquid, whose movement is induced by the vibration of the structure, and that of the plate

$$\xi_{mn} = \frac{T_L^*}{T_P^*} = \Gamma_{mn}\frac{\rho_L}{\rho_P}\frac{a}{h}, \tag{9}$$

where Γ_{mn} is the NAVMI factor and ρ_L is the mass density of the liquid.

NAVMI FACTORS

A circular plate is considered to be the flexible bottom of a rigid circular cylindrical tank filled with an incompressible and inviscid liquid (see Fig. 1); the liquid movement, considered as only caused by the plate vibration, is assumed to be irrotational. Then, the liquid movement associated with each mode shape can be described by the spatial velocity potential, Φ_{mn}, that satisfies the Laplace equation

$$\nabla^2\,\Phi_{mn} = 0. \tag{10}$$

The spatial velocity potential can be written as

$$\Phi_{mn} = \phi_{mn}(r,z)\cos(m\theta), \tag{11}$$

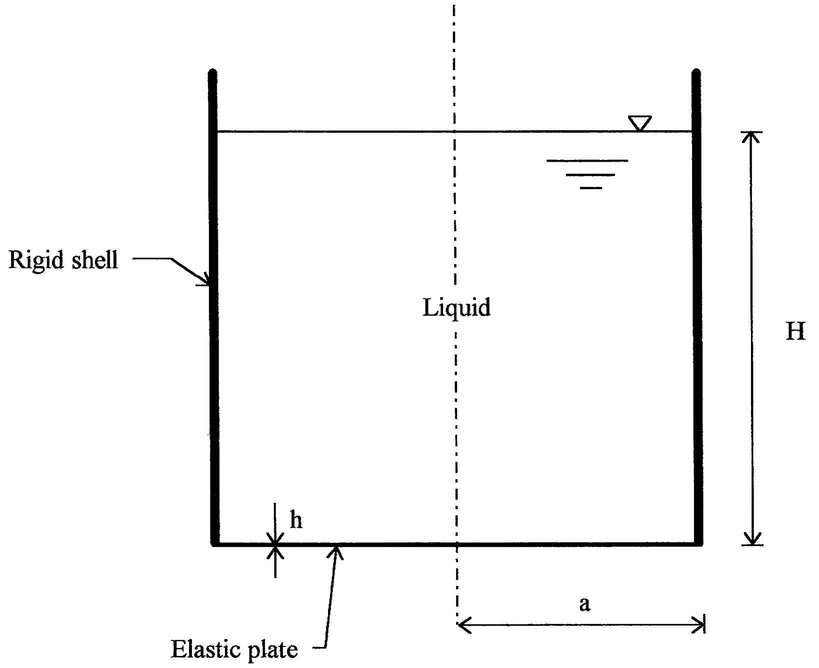

FIGURE 1 Liquid-filled tank with the flexible base.

where z is the axial coordinate. Substituting Eq. (11) in Eq. (10), results in

$$\frac{\partial^2 \phi_{mn}}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_{mn}}{\partial r} + \frac{\partial^2 \phi_{mn}}{\partial z^2} - \frac{m^2}{r^2}\phi_{mn} = 0. \quad (12)$$

The free liquid surface condition is described by the zero dynamic pressure condition at $z = H$ (Morand and Ohayon, 1992; Zhu, 1994, 1995),

$$\phi_{mn}(r, H) = 0, \quad (13)$$

where H is the level of the liquid in the container. This boundary condition is obtained neglecting the contribution of the free surface waves and superficial tension to the dynamic pressure of the liquid at $z = H$. This simplification does not give significant errors for tanks when only bulging modes are studied, as is shown in the discussion and comparison of numerical results. Studying the shell vibrations, Kondo (1981) discussed this phenomenon observing that wave heights of the free surface for bulging modes of circular cylindrical tanks are so small that they almost coincide with the undisturbed liquid level. The introduced simplification is the same obtained considering zero gravity (without superficial tension) and does not constrain the vertical velocity of the liquid. As a consequence of the hypotheses, the free surface does not exhibit an intrinsic capability to oscillate; thus, the liquid free surface is not subjected to a restoring force once it has moved, and sloshing modes cannot be studied.

The condition of impermeable walls at the liquid–rigid tank interface for a noncaviting liquid is

$$\left(\frac{\partial \phi_{mn}}{\partial r}\right)_{r=a} = 0, \quad (14)$$

and the liquid–flexible plate interface is

$$\left(\frac{\partial \phi_{mn}}{\partial z}\right)_{z=0} = -W_{mn}(r). \quad (15)$$

The solution of Eq. (12) with the condition given by Eq. (13) is, for axisymmetric modes ($m = 0$),

$$\phi_{0n}(r, z) = K_{0n0}(z - H) + \sum_{k=1}^{\infty} K_{0nk} J_0\left(\varepsilon_{0k}\frac{r}{a}\right) \left[\cosh\left(\varepsilon_{0k}\frac{z}{a}\right) - \frac{\sinh\left(\varepsilon_{0k}\frac{z}{a}\right)}{\tanh\left(\varepsilon_{0k}\frac{H}{a}\right)}\right], \quad (16)$$

and for asymmetric ($m \neq 0$) modes is

$$\phi_{mn}(r, z) = \sum_{k=0}^{\infty} K_{mnk} J_m\left(\varepsilon_{mk}\frac{r}{a}\right) \cdot \left[\cosh\left(\varepsilon_{mk}\frac{z}{a}\right) - \frac{\sinh\left(\varepsilon_{mk}\frac{z}{a}\right)}{\tanh\left(\varepsilon_{mk}\frac{H}{a}\right)}\right], \quad (17)$$

where K_{mnk} are constants. Equations (16) and (17) can be obtained by those of Bauer and Siekmann (1971) considering $g = \sigma = 0$; these equations must satisfy all the boundary conditions. Equations (16) and (17) satisfy the boundary condition (14) if ε_{mk} are solutions of the following equation

$$J'_m(\varepsilon_{mk}) = 0, \quad (18)$$

where J'_m is the derivative of the Bessel function in respect to its argument. The constants K_{mnk} are calculated in order to satisfy Eq. (15). For asymmetric modes we have

$$\sum_{k=0}^{\infty} K_{mnk} J_m\left(\varepsilon_{mk}\frac{r}{a}\right) \frac{\varepsilon_{mk}}{a \tanh\left(\varepsilon_{mk}\frac{H}{a}\right)} = \left[A_{mn} J_m\left(\frac{\lambda_{mn} r}{a}\right) + C_{mn} I_m\left(\frac{\lambda_{mn} r}{a}\right)\right]. \quad (19)$$

Recalling the orthogonality condition of the Bessel function (Spiegel, 1974), we have

$$\frac{1}{a^2}\int_0^a J_m\left(\varepsilon_{mk}\frac{r}{a}\right) J_m\left(\varepsilon_{mn}\frac{r}{a}\right) r\, dr = \alpha_{mk}\, \delta_{kn}, \quad (20)$$

where δ_{kn} is the Kronecker delta and α_{mk} is given by

$$\alpha_{mk} = \frac{1}{2}[1 - (m/\varepsilon_{mk})^2][J_m(\varepsilon_{mk})]^2. \quad (21)$$

Moreover, we have (Wheelon, 1968)

$$\frac{1}{a^2}\int_0^a J_m\left(\varepsilon_{mk}\frac{r}{a}\right) J_m\left(\lambda_{mn}\frac{r}{a}\right) r\, dr = \beta_{mnk}, \quad (22)$$

where

$$\beta_{mnk} = \frac{\lambda_{mn}}{\varepsilon_{mk}^2 - \lambda_{mn}^2} J'_m(\lambda_{mn}) J_m(\varepsilon_{mk}). \quad (23)$$

We also introduce the following integral (Wheelon, 1968)

$$\frac{1}{a^2}\int_0^a J_m\left(\varepsilon_{mk}\frac{r}{a}\right) I_m\left(\lambda_{mn}\frac{r}{a}\right) r\, dr = \gamma_{mnk}, \quad (24)$$

where

$$\gamma_{mnk} = \frac{\lambda_{mn}}{\varepsilon_{mk}^2 + \lambda_{mn}^2} I'_m(\lambda_{mn}) J_m(\varepsilon_{mk}). \quad (25)$$

If we multiply Eq. (19) by $(1/a^2) J_m[\varepsilon_{mk}(r/a)]r$ and then we integrate, this yields

$$K_{mnk}\alpha_{mk} \frac{\varepsilon_{mk}}{a \cdot \tanh\left(\varepsilon_{mk}\frac{H}{a}\right)} = A_{mn}\beta_{mnk} + C_{mn}\gamma_{mnk}. \quad (26)$$

Therefore, the constants K_{mnk} are given by

$$K_{mnk} = \frac{(A_{mn}\beta_{mnk} + C_{mn}\gamma_{mnk})}{\alpha_{mk}\varepsilon_{mk}} a \cdot \tanh\left(\varepsilon_{mk}\frac{H}{a}\right). \quad (27)$$

If one applies the Green's theorem to the harmonic function Φ_{mn}, the reference kinetic energy of the liquid can be computed as a boundary integral (Lamb, 1945; Amabili, 1995):

$$T_L^* = \frac{1}{2}\rho_L \iiint_V \nabla\Phi_{mn} \cdot \nabla\Phi_{mn}\, dV = -\frac{1}{2}\rho_L \iint_{\partial V} \Phi_{mn}(\partial\Phi_{mn}/\partial s)\, dS, \quad (28)$$

where V is the liquid volume, ∂V is the boundary of the volume V, and s is the direction normal to the boundary oriented inward to the liquid region. Due to Eqs. (13) and (14), only the integration over the plate surface gives a nonzero result; therefore, the reference kinetic energy of the liquid is given by

$$T_L^* = -\frac{1}{2}\rho_L \int_0^{2\pi}\int_0^a \left(\frac{\partial \phi_{mn}}{\partial z}\right)_{z=0} \phi_{mn}(r,0)\cos^2(m\theta) r\, dr\, d\theta$$
$$= \frac{1}{2}\rho_L a^2 \psi_m \int_0^1 W_{mn}(a\rho)\phi_{mn}(a\rho,0)\rho\, d\rho \tag{29}$$
$$= \frac{1}{2}\rho_L a^3 \psi_m \sum_{k=0}^{\infty} \frac{(A_{mn}\beta_{mnk} + C_{mn}\gamma_{mnk})^2}{\alpha_{mk}\varepsilon_{mk}} \tanh\left(\varepsilon_{mk}\frac{H}{a}\right),$$

where $\psi_m = 2\pi$ for $m = 0$ and π for $m > 0$, and T_L^* depends on m and n. The reference kinetic energy of the plate is given by

$$T_P^* = \frac{1}{2}\rho_P h \int_0^a \int_0^{2\pi} W_{mn}^2(r)\cos^2(m\theta) r\, dr d\theta$$
$$= \frac{1}{4}\psi_m \rho_P h a^2 \delta_{P_{mn}}, \tag{30}$$

where

$$\delta_{P_{mn}} = A_{mn}^2\left[J_m'^2(\lambda_{mn}) + \left(1 - \frac{m^2}{\lambda_{mn}^2}\right) J_m^2(\lambda_{mn})\right]$$
$$- C_{mn}^2\left[I_m'^2(\lambda_{mn}) - \left(1 + \frac{m^2}{\lambda_{mn}^2}\right) I_m^2(\lambda_{mn})\right]$$
$$+ \frac{2A_{mn}C_{mn}}{\lambda_{mn}}[J_m(\lambda_{mn})I_{m+1}(\lambda_{mn}) + I_m(\lambda_{mn})J_{m+1}(\lambda_{mn})], \tag{31}$$

and J_m' and I_m' are the derivatives of corresponding Bessel functions in respect to their arguments. Then, by using Eq. (9), the AVMI factor is given by

$$\xi_{mn} = \frac{T_L^*}{T_P^*} = 2\frac{\rho_L a}{\rho_P h \delta_{P_{mn}}} \sum_{k=0}^{\infty} \frac{(A_{mn}\beta_{mnk} + C_{mn}\gamma_{mnk})^2}{\alpha_{mk}\varepsilon_{mk}} \tanh\left(\varepsilon_{mk}\frac{H}{a}\right), \tag{32}$$

and the NAVMI factor is

$$\Gamma_{mn} = \frac{2}{\delta_{P_{mn}}} \sum_{k=0}^{\infty} \frac{(A_{mn}\beta_{mnk} + C_{mn}\gamma_{mnk})^2}{\alpha_{mk}\varepsilon_{mk}} \tanh\left(\varepsilon_{mk}\frac{H}{a}\right). \tag{33}$$

For axisymmetric modes ($m = 0$), the spatial velocity potential of the liquid is described by Eq. (16); for these modes, the boundary condition at the liquid–plate interface, Eq. (15), is satisfied if

$$-K_{0n0} + \sum_{k=1}^{\infty} K_{0nk} J_0\left(\varepsilon_{0k}\frac{r}{a}\right) \frac{\varepsilon_{0k}}{a \cdot \tanh\left(\varepsilon_{0k}\frac{H}{a}\right)} = \left[A_{0n}J_0\left(\frac{\lambda_{0n}r}{a}\right) + C_{0n}I_0\left(\frac{\lambda_{0n}r}{a}\right)\right]. \tag{34}$$

The constant K_{0n0} is given by

$$\frac{K_{0n0}}{2} = -\int_0^1 [A_{0n}J_0(\lambda_{0n}\rho) + C_{0n}I_0(\lambda_{0n}\rho)]\rho\, d\rho = -\tau_{0n}, \tag{35}$$

where (Wheelon, 1968)

$$\tau_{0n} = \left[\frac{A_{0n}}{\lambda_{0n}}J_1(\lambda_{0n}) + \frac{C_{0n}}{\lambda_{0n}}I_1(\lambda_{0n})\right]. \tag{36}$$

The constants K_{0nk}, for $k > 0$, are obtained by Eq. (27) computed for $m = 0$; therefore, for axisymmetric modes, the reference kinetic energy of the liquid is

$$T_L^* = \frac{1}{2}\rho_L a^3 \psi_m \left[2\frac{H}{a}\tau_{0n}^2 + \sum_{k=1}^{\infty} \frac{(A_{0n}\beta_{0nk} + C_{0n}\gamma_{0nk})^2}{\alpha_{0k}\varepsilon_{0k}} \tanh\left(\varepsilon_{0k}\frac{H}{a}\right)\right], \tag{37}$$

and the NAVMI factor is given by

$$\Gamma_{0n} = \frac{2}{\delta_{P_{mn}}}\left[2\frac{H}{a}\tau_{0n}^2 + \sum_{k=1}^{\infty} \frac{(A_{0n}\beta_{0nk} + C_{0n}\gamma_{0nk})^2}{\alpha_{0k}\varepsilon_{0k}} \tanh\left(\varepsilon_{0k}\frac{H}{a}\right)\right]. \tag{38}$$

NUMERICAL RESULTS

NAVMI factors are computed for circular plates with different boundary conditions by using Eqs. (33) and (38) and the software Mathematica (Wolfram, 1991). Data for simply supported plates is listed in Table 1 for $m \le 4$, $n \le 3$ and for different liquid depth ratios H/a. NAVMI factors

Table 1. NAVMI Factors for Simply Supported Circular Plates (ν = 0.3)

H/a	n	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
0.1	0	0.098391	0.096003	0.093042	0.089681	0.086056
0.1	1	0.092046	0.087819	0.083471	0.079132	0.074892
0.1	2	0.082724	0.077819	0.073163	0.068790	0.064727
0.1	3	0.072629	0.067887	0.063568	0.059637	0.056068
0.3	0	0.27307	0.23771	0.20248	0.17190	0.14705
0.3	1	0.19476	0.16204	0.13689	0.11755	0.10259
0.3	2	0.13502	0.11514	0.099871	0.087934	0.078500
0.3	3	0.099569	0.087473	0.077770	0.069908	0.063503
0.5	0	0.42482	0.32286	0.24346	0.19058	0.15548
0.5	1	0.23908	0.18413	0.14705	0.12208	0.10461
0.5	2	0.14999	0.12355	0.10396	0.089811	0.079350
0.5	3	0.10679	0.091918	0.080024	0.070969	0.063990
0.7	0	0.56892	0.37144	0.25685	0.19423	0.15650
0.7	1	0.26783	0.19397	0.14979	0.12283	0.10482
0.7	2	0.16083	0.12770	0.10516	0.090148	0.079445
0.7	3	0.11258	0.094265	0.080725	0.071168	0.064047
1	0	0.78213	0.40489	0.26173	0.19500	0.15663
1	1	0.30528	0.20035	0.15075	0.12298	0.10485
1	2	0.17605	0.13050	0.10560	0.090219	0.079457
1	3	0.12087	0.095863	0.080979	0.071210	0.064055
2	0	1.4909	0.42224	0.26267	0.19507	0.15664
2	1	0.42649	0.20361	0.15093	0.12299	0.10485
2	2	0.22622	0.13194	0.10568	0.090225	0.079458
2	3	0.14825	0.096689	0.081028	0.071214	0.064055
∞	0	—	0.42270	0.26267	0.19507	0.15664
∞	1	—	0.20370	0.15093	0.12299	0.10485
∞	2	—	0.13198	0.10568	0.090225	0.079458
∞	3	—	0.096711	0.081028	0.071214	0.064055

increase with the ratio H/a and decrease with m and n. In particular, the fundamental mode ($m = 0$ and $n = 0$) presents a high value of the NAVMI factor for $H/a \geq 1$, due to the movement of the liquid center of mass during vibrations; on the contrary, no change of the center of mass height is verified for mode shapes having $m > 0$. The limit NAVMI factors, when H/a goes to infinity, are given for asymmetric modes. These limit values cannot be obtained for axisymmetric modes by using the assumed modes approach but can be obtained by the Rayleigh–Ritz method. A similar behavior is shown in Table 2, where the data for clamped plates is reported; however, the factor for the fundamental mode is lower in this case in respect to simply supported plates. NAVMI factors for axisymmetric modes are also plotted in Fig. 2. The case of supported plates with a constant elastic moment constraint at the edge is studied for different torsional distributed stiffness K_t [moment/unit length] values; this constraint simulates well the plate boundary condition when this plate is welded to a circular cylindrical shell. Results are reported in Tables 3 and 4 for axisymmetric and asymmetric modes, respectively, and were obtained by using the modal parameters given in Azimi (1988). The full range 0 (simply supported plate) + ∞ (clamped plate) of the stiffness K_t was studied.

IMPROVED SOLUTION: RAYLEIGH–RITZ METHOD

The Rayleigh–Ritz method (Meirovitch, 1986) is applied to eliminate the restrictive hypothesis that dry and wet modes have the same shape. All the other hypotheses, previously introduced, are retained. The wet mode shapes W, by using the unknown parameters q_n and the admissible functions W_{mn}, can be described by

$$W(r, \theta) = \cos(m\theta) \sum_{n=0}^{\infty} q_n W_{mn}(r), \qquad (39)$$

where W_{mn} is given by Eq. (4). To simplify the

Table 2. NAVMI Factors for Clamped Circular Plates

H/a	n	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
0.1	0	0.097822	0.094943	0.091566	0.087876	0.084007
0.1	1	0.090596	0.086083	0.081542	0.077109	0.072837
0.1	2	0.080858	0.075921	0.071285	0.066976	0.063004
0.1	3	0.070826	0.066204	0.062016	0.058206	0.054765
0.3	0	0.25929	0.21971	0.18523	0.15710	0.13489
0.3	1	0.18245	0.15304	0.12983	0.11178	0.097757
0.3	2	0.13033	0.11226	0.097624	0.085924	0.076620
0.3	3	0.098648	0.087184	0.077504	0.069479	0.062925
0.5	0	0.38260	0.28372	0.21504	0.17050	0.14089
0.5	1	0.22690	0.17674	0.14079	0.11662	0.099885
0.5	2	0.15041	0.12404	0.10333	0.088506	0.077769
0.5	3	0.11001	0.094289	0.081074	0.071131	0.063672
0.7	0	0.49286	0.31845	0.22438	0.17302	0.14159
0.7	1	0.26282	0.18911	0.14414	0.11751	0.10013
0.7	2	0.16724	0.13042	0.10514	0.088997	0.077905
0.7	3	0.11974	0.098195	0.082219	0.071450	0.063761
1	0	0.65320	0.34209	0.22775	0.17354	0.14168
1	1	0.31406	0.19750	0.14535	0.11770	0.10016
1	2	0.19151	0.13478	0.10579	0.089101	0.077923
1	3	0.13381	0.10087	0.082635	0.071517	0.063773
2	0	1.1844	0.35431	0.22840	0.17359	0.14168
2	1	0.48321	0.20184	0.14558	0.11771	0.10016
2	2	0.27178	0.13703	0.10592	0.089110	0.077923
2	3	0.18037	0.10225	0.082715	0.071523	0.063773
∞	0	—	0.35463	0.22840	0.17359	0.14168
∞	1	—	0.20195	0.14558	0.11771	0.10016
∞	2	—	0.13709	0.10592	0.089110	0.077923
∞	3	—	0.10229	0.082715	0.071523	0.063773

computations, the mode shape constants, A_{mn} and C_{mn}, are normalized to have

$$\int_0^1 W_{mn}^2(\rho)\rho\, d\rho = 1. \tag{40}$$

The result of quadrature of Eq. (40) is [see Wheelon eqs. 11.106, 33.10, and 31.101 (1968)]

$$\begin{aligned}\Bigg\{&\frac{A_{mn}^2}{2}\left[(J'_m(\lambda_{mn}))^2+\left(1-\frac{m^2}{\lambda_{mn}^2}\right)J_m^2(\lambda_{mn})\right]\\ &-\frac{C_{mn}^2}{2}\left[(I'_m(\lambda_{mn}))^2-\left(1+\frac{m^2}{\lambda_{mn}^2}\right)I_m^2(\lambda_{mn})\right]\\ &+\frac{A_{mn}C_{mn}}{\lambda_{mn}}[J_m(\lambda_{mn})I_{m+1}(\lambda_{mn})\\ &+I_m(\lambda_{mn})J_{m+1}(\lambda_{mn})]\Bigg\}=1.\end{aligned} \tag{41}$$

In Eq. (39) the eigenfunctions of the plate vibrating in a vacuum are assumed to be admissible functions; in fact, dry mode shapes are quite similar to wet mode shapes. The trial functions W_{mn} are linearly independent and constitute a complete set.

The spatial distribution of the velocity potential of the liquid, Φ, calculated at the liquid–plate interface ($z = 0$), is given by

$$\Phi(r, \theta, 0) = \phi(r, 0)\cos(m\theta), \tag{42}$$

where

$$\phi(r, 0) = \sum_{n=0}^{\infty} q_n \phi_{mn}(r, 0). \tag{43}$$

Using the principle of superposition, considering that the plate deflection is given by the sum of Eq. (39), and Eq. (43), the function ϕ at the liquid–plate interface is given by the following sum:

$$\phi(a\rho, 0) = \sum_{n=0}^{\infty} q_n \sum_{k=0}^{\infty} K_{mnk} J_m(\varepsilon_{mk}\rho). \tag{44}$$

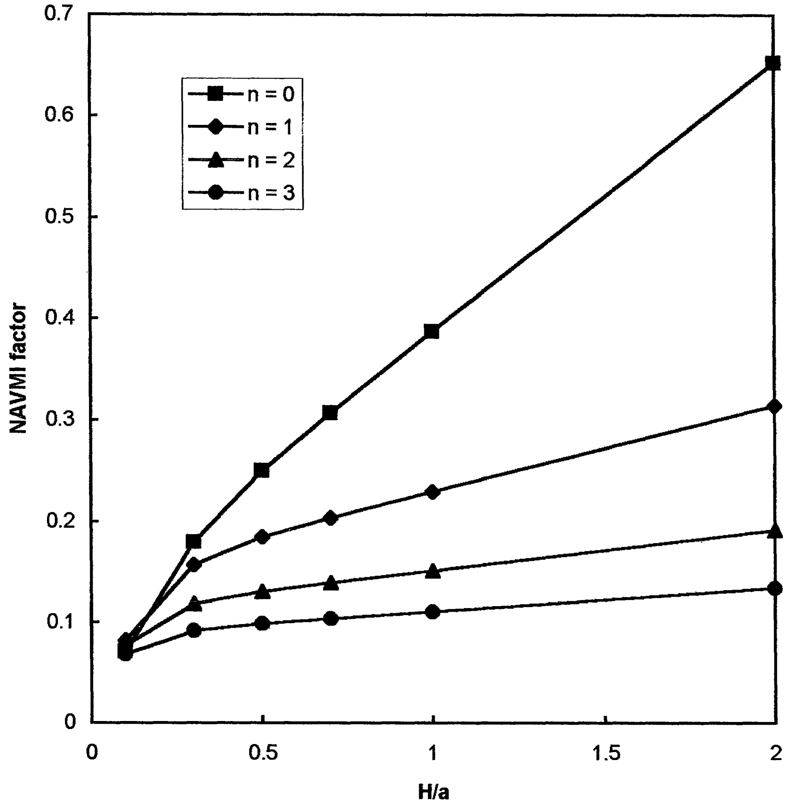

FIGURE 2 NAVMI factors for axisymmetric modes ($m = 0$) of clamped circular plates.

The reference kinetic energy of the liquid is given by

$$T_L^* = -\frac{1}{2}\rho_L \int_0^{2\pi}\int_0^a \left(\frac{\partial \Phi}{\partial z}\right)_{z=0} \Phi(r,0)\, r\, dr\, d\theta. \quad (45)$$

Using the boundary condition at the liquid–plate interface $(\partial\Phi/\partial z)_{z=0} = -W(r, \theta)$, the following expression for the reference kinetic energy of the liquid for asymmetric modes is obtained:

$$T_L^* = \frac{1}{2}\rho_L a^2 \psi_m \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} q_n q_j \sum_{k=0}^{\infty} K_{mnk}(A_{mj}\beta_{mjk} + C_{mj}\gamma_{mjk}). \quad (46)$$

The reference kinetic energy of the liquid for axisymmetric modes ($m = 0$) is then given by

Table 3. NAVMI Factors for Supported Circular Plates with Elastic Moment Edge Constraint, Axisymmetric Modes ($m = 0$), and $\nu = 0.3$

H/a	n	$K_t a/D \to \infty$	$K_t a/D = 100$	$K_t a/D = 10$	$K_t a/D = 1$	$K_t a/D \to 0$
0.1	0	0.097822	0.097864	0.098086	0.098340	0.098391
0.1	1	0.090596	0.090770	0.091474	0.091964	0.092046
0.3	0	0.25929	0.26013	0.26484	0.27139	0.27307
0.3	1	0.18245	0.18415	0.19084	0.19447	0.19476
0.5	0	0.38260	0.38508	0.39915	0.41943	0.42482
0.5	1	0.22690	0.22970	0.23939	0.24036	0.23908
0.7	0	0.49285	0.49726	0.52239	0.55906	0.56892
0.7	1	0.26282	0.26632	0.27663	0.27157	0.26783
1	0	0.65319	0.66061	0.70300	0.76528	0.78213
1	1	0.31406	0.31840	0.32864	0.31311	0.30528
2	0	1.1844	1.2019	1.3022	1.4506	1.4909
2	1	0.48321	0.49028	0.49953	0.44818	0.42649

Table 4. NAVMI Factors for Supported Circular Plates with Elastic Moment Edge Constraint, Asymmetric Modes with No Internal Circular Nodes (n = 0), and ν = 0.3

H/a	m	$K_t a/D \to \infty$	$K_t a/D = 100$	$K_t a/D = 10$	$K_t a/D = 1$	$K_t a/D \to 0$
0.1	1	0.094943	0.095038	0.095490	0.095921	0.096003
0.1	2	0.091567	0.091722	0.092392	0.092942	0.093043
0.3	1	0.21971	0.22112	0.22818	0.23598	0.23771
0.3	2	0.18523	0.18686	0.19426	0.20112	0.20249
0.5	1	0.28372	0.28671	0.30189	0.31901	0.32286
0.5	2	0.21504	0.21769	0.22980	0.24116	0.24346
0.7	1	0.31845	0.32247	0.34295	0.36620	0.37144
0.7	2	0.22438	0.22740	0.24346	0.25422	0.25685
1	1	0.34209	0.34684	0.37108	0.39866	0.40489
1	2	0.22775	0.23090	0.24536	0.25898	0.26173
2	1	0.35432	0.35944	0.38564	0.41549	0.42224
2	2	0.22840	0.23158	0.24616	0.25989	0.26267

$$T_L^* = \frac{1}{2}\rho_L a^2 \psi_m \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} q_n q_j \left[2H\tau_{0n}\tau_{0j} + \sum_{k=1}^{\infty} K_{0nk}(A_{0j}\beta_{0jk} + C_{0j}\gamma_{0jk}) \right]. \tag{47}$$

The reference kinetic energy of the plate, using the normalization introduced in Eq. (40) and the orthogonality of the dry mode shapes, is given by

$$T_P^* = \frac{1}{2}\rho_P a^2 h \psi_m \sum_{n=0}^{\infty} q_n^2. \tag{48}$$

The maximum potential energy of the system, considering an incompressible liquid, coincides with that of the plate and can be computed as a sum of the reference kinetic energies of the dry eigenfunctions,

$$V_P = \frac{1}{2}\frac{\psi_m D}{a^2}\sum_{n=0}^{\infty} q_n^2 \lambda_{mn}^4. \tag{49}$$

To perform numerical computations for each fixed m value, only a finite number N of terms must be considered in all the previous sums. To this purpose, the vector $\mathbf{q}$ of the unknown parameters is introduced

$$\mathbf{q} = \begin{Bmatrix} q_0 \\ q_1 \\ \vdots \\ q_{N-1} \end{Bmatrix}. \tag{50}$$

To make the formulas more compact, the following constant is also given:

$$\sigma = a^2 \psi_m. \tag{51}$$

Using the introduced notation, the kinetic energy of the liquid is given by

$$T_L^* = \frac{1}{2}\sigma \rho_L a \mathbf{q}^T \mathbf{M}_L \mathbf{q}, \tag{52}$$

where the $N \times N$ symmetric NAVMI matrix $\mathbf{M}_L$ for asymmetric modes ($m \geq 1$) is introduced

$$\mathbf{M}_{Lij} = \sum_{k=0}^{\infty} \frac{(A_{mi}\beta_{mik} + C_{mi}\gamma_{mik})}{\alpha_{mk}\varepsilon_{mk}} (A_{mj}\beta_{mjk} + C_{mj}\gamma_{mjk}) \tanh\left(\varepsilon_{mk}\frac{H}{a}\right), \quad i,j = 0, 1, \ldots, N-1, \tag{53}$$

for symmetric modes (m = 0) it is defined by

$$\mathbf{M}_{Lij} = 2\tau_{0i}\tau_{0j}\frac{H}{a} + \sum_{k=1}^{\infty} \frac{(A_{0i}\beta_{0ik} + C_{0i}\gamma_{0ik})}{\alpha_{0k}\varepsilon_{0k}} (A_{0j}\beta_{0jk} + C_{0j}\gamma_{0jk}) \tanh\left(\varepsilon_{0k}\frac{H}{a}\right), \quad i,j = 0, 1, \ldots, N-1. \tag{54}$$

The NAVMI matrix describes the inertial effect of the liquid on the modes. Therefore, this is the extension of the NAVMI factor to the Rayleigh–Ritz approach. Moreover, the NAVMI factors are the diagonal elements of the NAVMI matrix.

Similar to Eq. (52), the reference kinetic energy of the plate is obtained

$$T_{\mathrm{P}}^{*} = \frac{1}{2}\sigma\rho_{\mathrm{P}}h\mathbf{q}^{T}\mathbf{I}\mathbf{q}, \tag{55}$$

where $\mathbf{I}$ is the $N \times N$ identity matrix. Then, the maximum potential energy of the plate takes the following expression:

$$V_{\mathrm{P}} = \frac{1}{2}\sigma\frac{D}{a^4}\mathbf{q}^{T}\mathbf{P}\mathbf{q}, \tag{56}$$

where $\mathbf{P}$ is the $N \times N$ diagonal matrix given by

$$\mathbf{P}_{ij} = \delta_{ij}\lambda_{mi}^{4}, \tag{57}$$

and δ_{ij} is the Kronecker delta. In order to find natural frequencies and wet mode shapes of the plate vibrating in contact with the liquid, the Rayleigh quotient for coupled vibration in an inviscid and incompressible liquid (Zhu, 1994) is used. Minimizing the Rayleigh quotient with respect to the unknown parameters q_n, one gets

$$\frac{D}{a^4}\mathbf{P}\mathbf{q} - \Lambda^2(\rho_{\mathrm{P}}h\mathbf{I} + \rho_{\mathrm{L}}a\mathbf{M}_{\mathrm{L}})\mathbf{q} = 0, \tag{58}$$

where Λ is the circular frequency of the wet plate. Equation (58) is a Galerkin equation and gives an eigenvalue problem. It would be convenient to introduce the following nondimensional constants:

$$\Omega^2 = \Lambda^2\frac{a^4}{D}\rho_{\mathrm{P}}h, \tag{59}$$

$$\mu = \frac{\rho_{\mathrm{L}}a}{\rho_{\mathrm{P}}h}. \tag{60}$$

Ω and μ are called the wet frequency parameter and the density–thickness correction factor, respectively. Then, Eq. (58) can be written in the following nondimensional form:

$$\mathbf{P}\mathbf{q} - \Omega^2(\mathbf{I} + \mu\mathbf{M}_{\mathrm{L}})\mathbf{q} = 0. \tag{61}$$

It is interesting to see that, if the NAVMI matrix $\mathbf{M}_{\mathrm{L}}$ is diagonal, the system of Eq. (61) is uncoupled; in this case, the approximate solutions given in the previous section become exact.

DISCUSSION AND COMPARISON OF NUMERICAL RESULTS

The numerical solution to the Galerkin equation, Eq. (61), is obtained by using the Mathematica

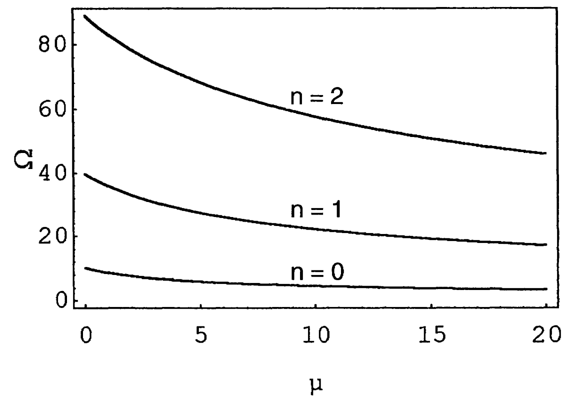

FIGURE 3 The wet frequency parameters Ω for axisymmetric modes ($m = 0$) of clamped plates having a ratio $H/a = 0.5$, as a function of the density–thickness correction factor μ. Curves relative to the first three axisymmetric modes.

(Wolfram, 1991) computer program. The computation of the NAVMI matrix $\mathbf{M}_{\mathrm{L}}$ is performed by using Eq. (53) for asymmetric modes and Eq. (54) for axisymmetric modes. Due to the nondimensional form of these equations and the number of nodal diameters m once fixed, $\mathbf{M}_{\mathrm{L}}$ depends only on the plate boundary conditions at the edge, on the ratio H/a, and on the Poisson ratio ν; clamped plates are independent from ν. The numerical results for three different cases are reported here; these results are also used as a testing bench for the assumed modes approach. The NAVMI matrix for axisymmetric modes ($m = 0$) of clamped plates having a ratio $H/a = 0.5$ is

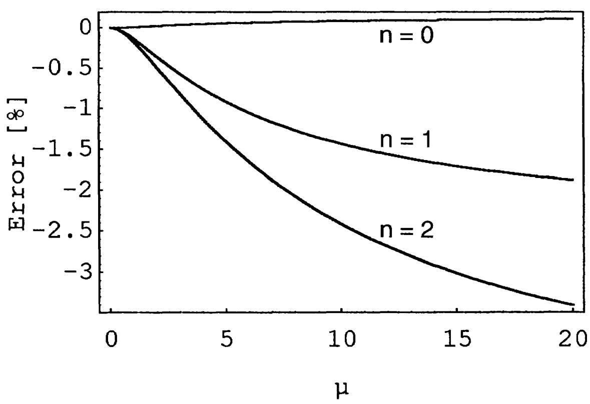

FIGURE 4 The percentage errors of the NAVMI factor solution in respect to the Rayleigh–Ritz results as a function of μ for clamped plates. Axisymmetric modes ($m = 0$) and $H/a = 0.5$.

$$\begin{bmatrix} 0.38260 & -0.072123 & 0.052138 & -0.040194 & 0.032578 & -0.027353 \\ -0.072123 & 0.22690 & -0.053876 & 0.042747 & -0.035069 & 0.029613 \\ 0.052138 & -0.053876 & 0.15041 & -0.036409 & 0.030367 & -0.025878 \\ -0.040194 & 0.042747 & -0.036409 & 0.11001 & -0.025847 & 0.022253 \\ 0.032578 & -0.035069 & 0.030367 & -0.025847 & 0.085817 & -0.019282 \\ -0.027353 & 0.029613 & -0.025878 & 0.022253 & -0.019282 & 0.069921 \end{bmatrix} \tag{62}$$

The NAVMI matrix for axisymmetric modes ($m = 0$) of clamped plates having a ratio $H/a = 2$ is

$$\begin{bmatrix} 1.1844 & -0.51803 & 0.35960 & -0.27439 & 0.22161 & -0.18580 \\ -0.51803 & 0.48321 & -0.23007 & 0.17693 & -0.14337 & 0.12039 \\ 0.35960 & -0.23007 & 0.27178 & -0.12882 & 0.10495 & -0.088390 \\ -0.27439 & 0.17693 & -0.12882 & 0.18037 & -0.082638 & 0.069854 \\ 0.22161 & -0.14337 & 0.10495 & -0.082638 & 0.13165 & -0.057701 \\ -0.18580 & 0.12039 & -0.088390 & 0.069854 & -0.057701 & 0.10212 \end{bmatrix} \tag{63}$$

The NAVMI matrix for modes with a nodal diameter ($m = 1$) of clamped plates having a ratio $H/a = 2$ is

$$\begin{bmatrix} 0.35432 & -0.09636 & 0.071934 & -0.056968 & 0.047049 & -0.040039 \\ -0.09636 & 0.20184 & -0.057211 & 0.046136 & -0.038442 & 0.032866 \\ 0.071934 & -0.057211 & 0.13703 & -0.037619 & 0.031698 & -0.027282 \\ -0.056968 & 0.046136 & -0.037619 & 0.10225 & -0.026719 & 0.023162 \\ 0.047049 & -0.038442 & 0.031698 & -0.026719 & 0.080907 & -0.020026 \\ -0.040039 & 0.032866 & -0.027282 & 0.023162 & -0.020026 & 0.066598 \end{bmatrix} \tag{64}$$

Six terms in the series expansion of mode shapes are used in the cases presented; this choice allows a good evaluation of the first three eigenvalues (Meirovitch, 1986). Observing Eqs. (62)–(64), one can find that the NAVMI factors previously computed are the diagonal elements of the NAVMI matrices. Moreover, the off-diagonal elements can be positive or negative, but generally have a smaller absolute value than the diagonal elements; exceptions are obtained in Eq. (63), where the axisymmetric modes of a clamped plate having a ratio $H/a = 2$ are considered. It is clear that the matrix in Eq. (62), relative to clamped circular plates with a ratio $H/a = 0.5$, is more diagonal than the others reported in Eqs. (63) and (64). In fact, an increment of the off-diagonal elements with the ratio H/a can be observed; on the contrary, the importance of the off-diagonal elements decrease with m, as verified comparing Eqs. (63) and (64).

Figure 3 shows the wet frequency parameters Ω for axisymmetric modes ($m = 0$) of clamped plates for $H/a = 0.5$ as function of the density–thickness correction factor μ. Curves relative to the first three axisymmetric modes are given. The circular frequency of this plate is obtained by using Eq. (59) and Fig. 3. The percentage errors that one commits by using the NAVMI factor solution, instead of the Rayleigh–Ritz solution, is plotted in Fig. 4 for this plate as function of μ. It is very interesting to note that the fundamental mode is accurately estimated by the NAVMI factor solution and that the error increases with the number of nodal circles and with μ. In Figs. 5 and 6 the

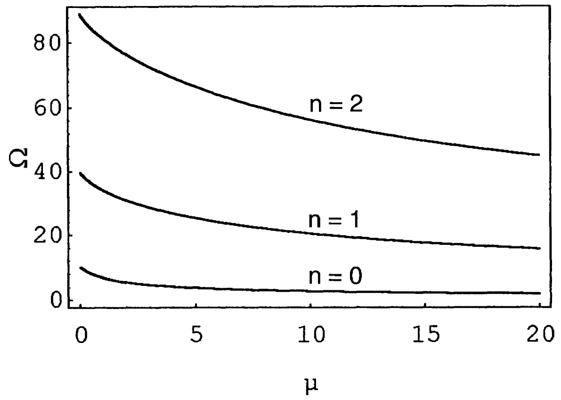

FIGURE 5 The wet frequency parameters Ω for axisymmetric modes ($m = 0$) of clamped plates having a ratio $H/a = 2$, as a function of the density–thickness correction factor μ. Curves relative to the first three axisymmetric modes.

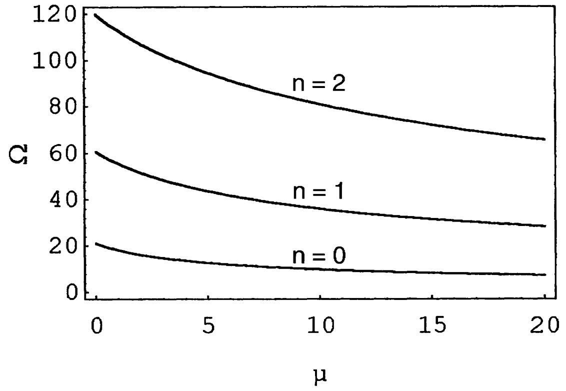

FIGURE 7 The wet frequency parameters Ω for modes with a nodal diameter ($m = 1$) of clamped plates having a ratio $H/a = 2$, as a function of the density–thickness correction factor μ. Curves relative to the first three modes.

wet frequency parameters Ω and the percentage errors are given for axisymmetric modes ($m = 0$) of clamped plates for $H/a = 2$. These figures are given in order to understand the changes due to the different ratio of H/a; it is clearly an increment of the percentage errors that are maximum for modes having one nodal circle ($n = 1$). In Figs. 7 and 8 the wet frequency parameters Ω and the percentage errors are plotted for modes with one nodal diameter ($m = 1$) of clamped plates with a ratio $H/a = 2$. It is interesting to observe that the frequency error for all modes and for all the μ values are lower in this case with respect to the one shown in Fig. 6; therefore, as well as the off-diagonal elements becoming less important for modes having nodal diameters, the percentage errors decrease with m.

The wet mode shapes are investigated by using Eq. (61); data relative to the three is reported. In Fig. 9 the dry and wet mode shapes are plotted along a radius for axisymmetric modes of clamped plates with $H/a = 0.5$ and $\mu = 10$. Modes with up to two nodal circles are considered. It is clear that, as for the natural frequency, the mode shape of the fundamental mode ($m = 0$; $n = 0$) shows little change. Similar results are given in Fig. 10 for axisymmetric modes of clamped plates with $H/a = 2$ and $\mu = 10$ and in Fig. 11 for modes with one nodal diameter of clamped plates with $H/a = 2$ and $\mu = 10$.

The results of the NAVMI factor approach

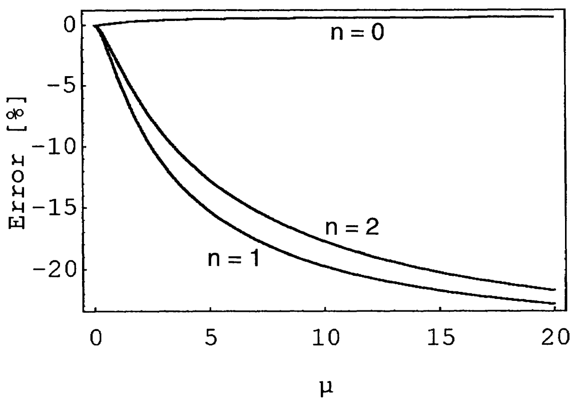

FIGURE 6 The percentage errors of the NAVMI factor solution in respect to the Rayleigh–Ritz results as a function of μ for clamped plates. Axisymmetric modes ($m = 0$) and $H/a = 2$.

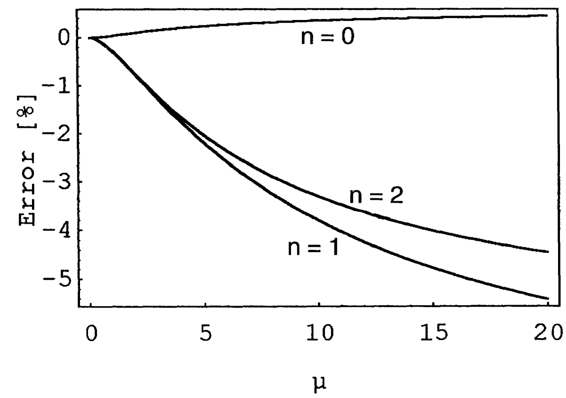

FIGURE 8 The percentage errors of the NAVMI factor solution in respect to the Rayleigh–Ritz results as a function of μ for clamped plates. Modes with a nodal diameter ($m = 1$) and $H/a = 2$.

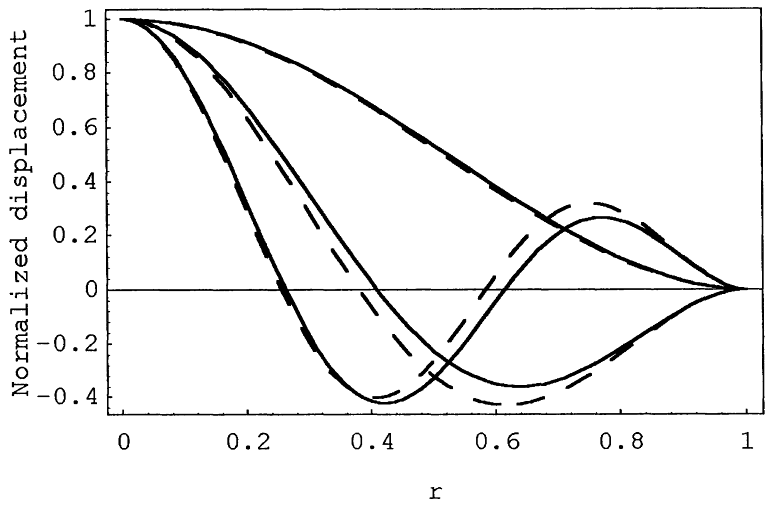

FIGURE 9 Comparison of (---) dry and (—) wet modes for clamped plates; axisymmetric modes ($m = 0$) and $H/a = 0.5$.

and Rayleigh–Ritz methods are compared to those given by Chiba (1993) for bulging modes of a clamped circular bottom plate, where both the effects of the free surface waves on the dynamic pressure and the in-plane stress of the plate are considered. Results of Chiba (1993) are dimensional and refer to a steel plate having radius $a = 0.144$ m, thickness $h = 0.002$ m, Young's modulus $E = 206$ Gpa, Poisson ratio $\nu = 0.25$, mass density $\rho_P = 7850\,\text{kg m}^{-3}$ in contact with water, having $\rho_L = 1000$ kg m^{-3}. In Fig. 12 the natural frequencies (Hz) of the first three axisymmetric modes of the plate investigated by Chiba (1993) are compared to the results obtained by using both the simple NAVMI factor approach and the Rayleigh–Ritz method. It is clear that the results of the Rayleigh–Ritz method always match very well with Chiba's results (see Fig. 12), confirming that the applied free surface condition (zero dynamic pressure at $z = H$) is correct when one is studying bulging modes. Moreover, the simple NAVMI factor approach gives quite good results, especially for the fundamental mode and for small values of

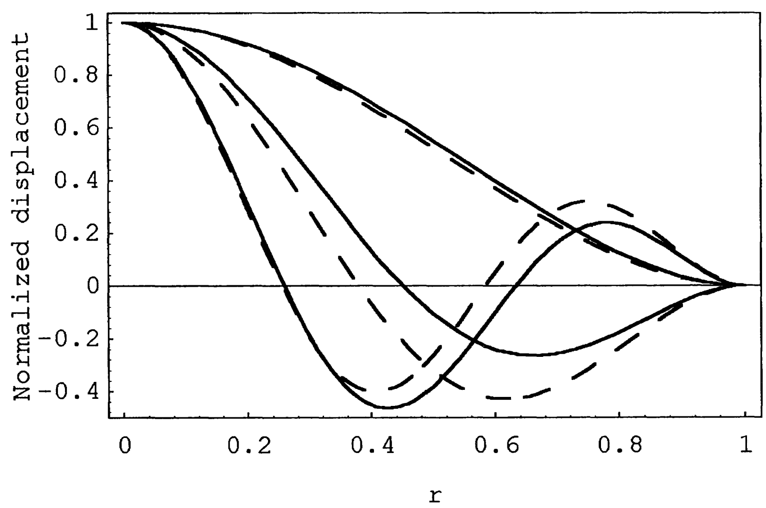

FIGURE 10 Comparison of (---) dry and (—) wet modes for clamped plates; axisymmetric modes ($m = 0$) and $H/a = 2$.

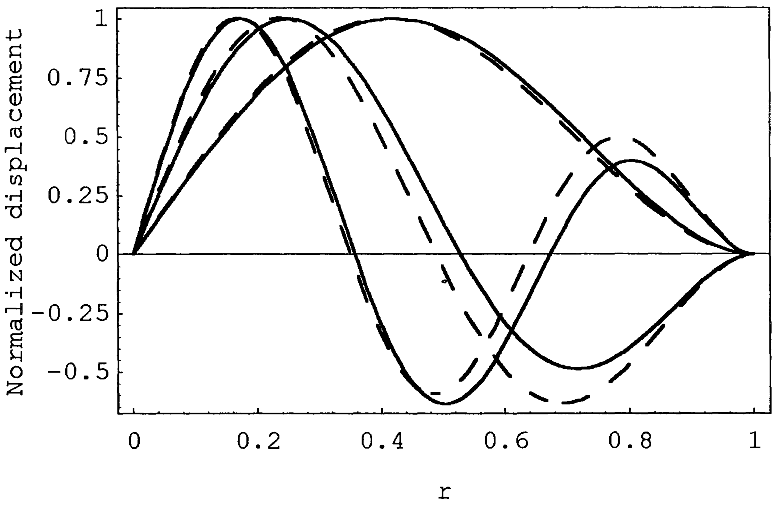

FIGURE 11 Comparison of (---) dry and (—) wet modes for clamped plates; modes with a nodal diameter ($m = 1$) and $H/a = 2$.

the ratio H/a; in fact, it was previously found that, when the ratio H/a increases, the accuracy of the NAVMI factor approach decreases.

CONCLUSIONS

The proposed approach to the problem has the advantage that the natural frequencies are quickly obtained by using the NAVMI factors and the modal properties of dry plates. The NAVMI factors are listed for application to engineering and design; to this end, some common constraints at the plate edge are studied. In spite of the computational simplicity, the present approach gives good results, especially for the fundamental mode ($m = 0$ and $n = 0$) or modes without nodal circles ($n = 0$), and for values of the ratio $H/a \leq 1$; within the aforestated range, mode shapes having no circular nodes present nearly equal wet and dry mode shapes, so that the assumed mode approach gives nearly exact natural frequencies.

When more accurate results are necessary, the Rayleigh–Ritz method, which retains a relative computation simplicity and gives nondimensional results, can be used. This method for bulging modes gives results nearly equal with those obtained by using more complex theories.

NOMENCLATURE

a	plate radius
A_{mn}, C_{mn}	mode shape constants
D	$= Eh^3/[12(1 - \nu^2)]$, flexural rigidity
E	Young's modulus of the plate material
f_V	natural frequency in a vacuum (Hz)
f_L	natural frequency of the plate in contact with the liquid (Hz)
h	plate thickness
H	level of the liquid
m	number of nodal diameters
n	number of nodal circles
r	radial coordinate
t	time
T_L^*	reference kinetic energy of the liquid
T_P^*	reference kinetic energy of the plate
V_P	maximum potential energy of the plate
w	$= w(r, \theta, t)$, deflection of the plate
W_{mn}	$= W_{mn}(r)$, radial mode shape function
z	coordinate along the tank axis
ξ_{mn}	added virtual mass incremental (AVMI) factor
Γ_{mn}	nondimensionalized added virtual mass incremental (NAVMI) factor
θ	angular coordinate
λ_{mn}	plate frequency parameter
Λ	circular frequency of the wet plate
μ	density–thickness correction factor
ν	Poisson ratio
ϕ_{mn}	spatial velocity potential of the liquid in the plane $\theta = 0$
Φ_{mn}	spatial velocity potential of the liquid
ρ_L	mass density of the liquid
ρ_P	mass density of the plate material

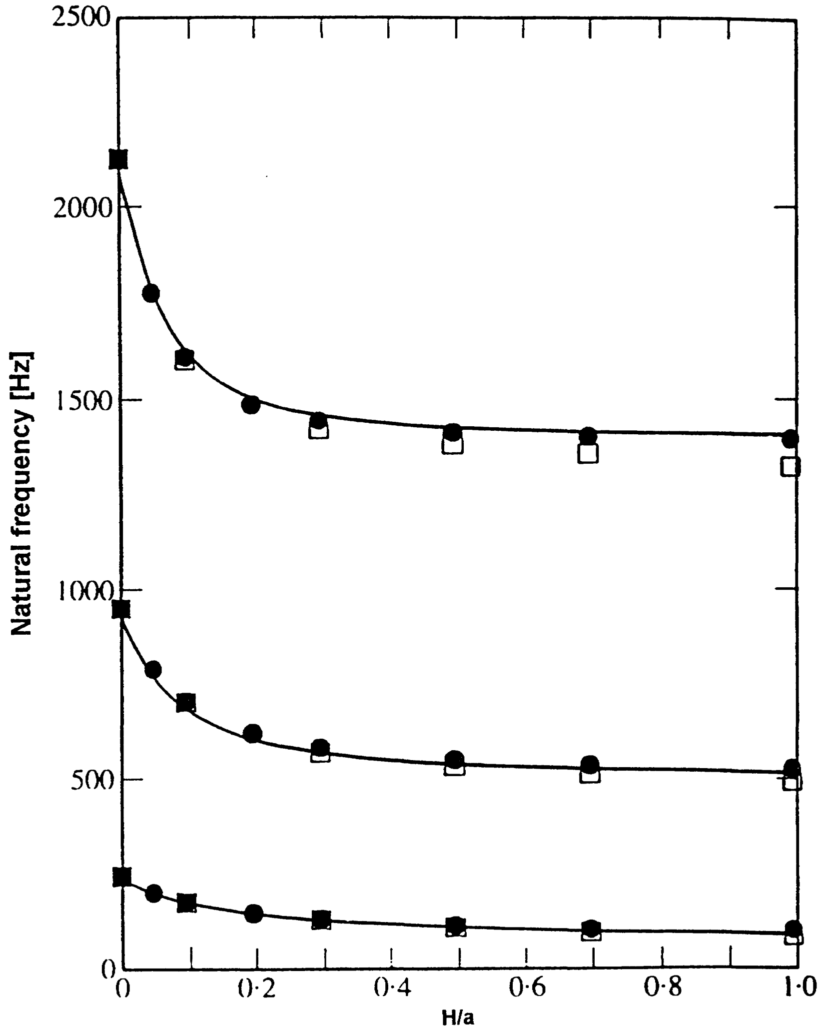

FIGURE 12 Comparison of natural frequencies obtained by using different theories as a function of the ratio *H/a*. (□) NAVMI factors approach; (●) Rayleigh–Ritz method; (—) results of Chiba (1993).

ω	circular frequency of the plate (rad/s)
Ω	frequency parameter of the wet plate
∇^2	Laplace operator
∇^4	$= \nabla^2\nabla^2$, iterated Laplace operator

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