W.Q. Zhu
Department of Mechanics
Zhejiang University
Hangzhou 310027
China

C.D. Huang
Canton/Seinuk Group 219
East 42nd St., New York
NY 10017-5736, USA

T.T. Soong
Department of Civil Engineering
State University of New York
at Buffalo, Buffalo, NY 14260, USA

Narrow-Band Excitation of Hysteretic Systems *

The stationary response of smooth and bilinear hysteretic systems to narrow-band random excitations is investigated by using the quasistatic method and digital simulation. It is shown that the response is qualitatively different in different ranges of values of the ratio of the excitation central frequency to the natural frequency of the system. In the resonant zone, the response is essentially non-Gaussian. For bilinear hysteretic systems with strong yielding, stochastic jumps may occur for a range of values of the ratio between nonresonant and resonant zones.

INTRODUCTION

It is well known that many engineering systems exhibit hysteresis when subjected to severe dynamic loadings. In the last three decades much work has been done on the analysis of the dynamic response of hysteretic systems. The dynamic loadings in all analyses are modeled as either sinusoidal (Caughey, 1960; Iwan, 1966; Jennings, 1964) or wide-band random excitation (Wen, 1989). To the authors' knowledge, no analysis has been made of the response of hysteretic systems to narrow-band excitations, which is also of practical interest.

The response of nonlinear systems to narrow-band excitations has many interesting features and is a relatively underdeveloped area in the theory of nonlinear random vibration. An example attracting many investigators over the last three decades is the stationary response of the Duffing oscillator with a hardening spring to a narrow-band excitation. Under certain conditions, jumps may occur in the response. This phenomenon of a stochastic jump was first ob-

Received August 28 1995; Revised August 28 1995.

*This article was adapted from "Stochastic Response of Hysteretic Systems to Narrow Band Excitations" W. Q. Zhu, C. D. Huang, and T. T. Soong, 1994, *ASME Journal of Applied Mechanics*, Vol. 61, pp. 211–213, and is published with permission from ASME.

Shock and Vibration, Vol. 4, No. 4, pp. 241–250 (1997)
ISSN 1070-9622/97/$8.00 © 1997 IOS Press

served experimentally and was studied theoretically by Lyon et al. (1961). Later it was further examined by a number of authors (Davies and Liu, 1990; Davies and Nandall, 1986; Dimentberg, 1970; Fang and Dowell, 1987; Iyengar, 1988; Lennox and Kuak, 1976; Roberts, 1991; Zhu et al., 1993). There also appears to have been misinterpretations regarding this phenomenon.

Under certain combinations of the parameters of the oscillator and the excitation, the variance of the stationary displacement response of the Duffing oscillator to a narrow-band excitation obtained from equivalent linearization is triple valued. The stability analysis indicates that only two of them are stable and realizable. The jump has been interpreted as the switch between the two stable branches of the stationary displacement variance. However, the variance of the stationary displacement response of the Duffing oscillator to a narrow-band excitation should be unique and it can be easily proven by using the Fokker–Planck equation. Digital simulation also showed (Zhu et al., 1993) that neither of the two stable solutions obtained by equivalent linearization gives the true variance of the stationary displacement response. They may approximately represent the mean-square values of the displacement in a short duration within which no jump occurs (Roberts, 1991). It is also shown (Zhu et al., 1993) that in the case where stochastic jumps occur, the joint probability density of the stationary displacement and velocity responses has both a central peak and a ring of peaks surrounding the origin with a ring of valleys between them. This implies that the response contains two types of more probable motions: one is random vibration with smaller amplitudes corresponding to a central peak and another has larger amplitudes similar to a diffused limit cycle corresponding to a ring of peaks. The stochastic jump is essentially the transition of the response between the two more probable motions.

The purpose of the present article is to study the behavior of the stationary response of smooth and bilinear hysteretic systems subject to narrow-band excitations. The probability density of the displacement and the joint probability density of the displacement and velocity are obtained by using the quasistatic method (Stratonovich, 1967) and digital simulation. Based on these probability densities, it is shown that the response of hysteretic systems to narrow-band excitations can be nonresonant and resonant, depending on the values of the ratio of the excitation central frequency to the natural frequency of the system. In the resonant zone, the response is essentially non-Gaussian. For bilinear hysteretic systems with strong yielding, stochastic jumps may occur for certain values of the ratio between nonresonant and resonant zones.

QUASISTATIC METHOD

Consider a single degree of freedom hysteretic system subject to a narrow-band excitation $\xi(\tau)$. The nondimensional equation of motion can be written as

$$x'' + 2\zeta x' + g(x, x') = \xi(\tau), \qquad (1)$$

where $x = y/\Delta$, $\tau = \omega_0 t$; y is the displacement; Δ is the characteristic displacement; ω_0 is the natural frequency of the preyield system; $g(x, x')$ is the nondimensional restoring force; and a prime represents the derivative with respect to τ. For bilinear hysteretic systems (Fig. 1),

$$g(x, x') = \begin{cases} x, & a \leqslant 1, \\ x + (1-\alpha)(a-1), & a > 1, \; -a \leqslant x \leqslant 2-a, \\ \alpha x + (1-\alpha), & a > 1, \; 2-a \leqslant x \leqslant a, \end{cases} \qquad (2)$$

where α is the ratio of postyield to preyield stiffness and a is the nondimensional amplitude of displacement. For smooth hysteretic systems (Bouc–Wen model, Fig. 2),

$$g(x, x') = \alpha x + (1-\alpha)z, \qquad (3)$$

$$z' = -\gamma |x'| z |z|^{n-1} - \beta x' |z|^n + A x', \qquad (4)$$

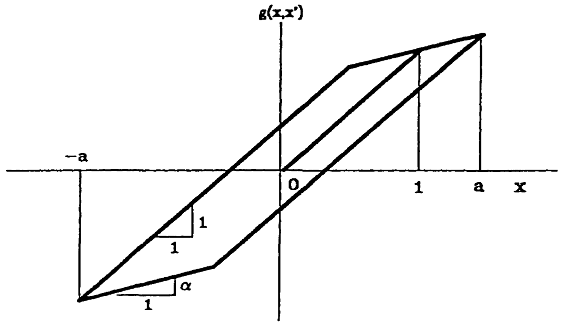

FIGURE 1 Restoring force of a bilinear hysteretic system as a function of displacement.

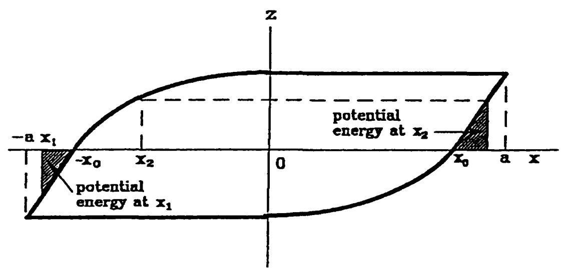

FIGURE 2 Hysteretic component of a smooth hysteretic system as a function of displacement.

where $(1-\alpha)z$ is the nondimensional hysteretic restoring force and β, γ, A, and n are parameters that control the shape of the hysteresis. Equation (4) can be integrated to yield a functional relationship between z and x (Cai and Lin, 1990). For the special case of $A = n = 1$ and $\beta \neq \pm\gamma$,

$$z(x) = \begin{cases} \dfrac{1}{\gamma-\beta}[1-e^{-(\gamma-\beta)(x+x_0)}], & -a \leqslant x \leqslant -x_0, \\ \dfrac{1}{\gamma+\beta}[1-e^{-(\gamma+\beta)(x+x_0)}], & -x_0 \leqslant x \leqslant a, \end{cases} \tag{5}$$

where x_0 is uniquely determined for a given amplitude a by solving $z(-x_0) = 0$. Equations (2) and (5) hold for $x' \geqslant 0$. The values of $g(x, x')$ and $z(x)$ for $x' \leqslant 0$ can be obtained using symmetry properties of these functions about $x = x' = 0$.

Suppose that $\xi(\tau)$ is a narrow-band Gaussian stationary random process. It can be expressed as

$$\xi(\tau) = \rho\cos(\nu\tau + \theta), \tag{6}$$

where $\rho = \rho(\tau)$ and $\theta = \theta(\tau)$ are the slowly varying amplitude and phase, respectively, of the excitation and $\nu\omega_0$ is the central frequency of the excitation. Lin (1976) showed that ρ and θ are independent, ρ has Rayleigh distribution, and θ is uniformly distributed over $[0, 2\pi)$, that is,

$$p_s(\rho) = \frac{\rho}{\sigma_\xi^2}\exp\left(-\frac{\rho^2}{2\sigma_\xi^2}\right), \quad \rho \geqslant 0, \tag{7}$$

$$p_s(\theta) = \frac{1}{2\pi}, \quad 0 \leqslant \theta < 2\pi. \tag{8}$$

Assume that the stationary response of the system described by Eq. (1) exists and it is narrow band. It can be expressed as

$$x = a\cos\Phi, \quad x' = -\nu a\sin\Phi, \tag{9}$$

where $\Phi = \nu\tau + \varphi$, and $a = a(\tau)$ and $\varphi = \varphi(\tau)$ are the slowly varying amplitude and phase, respectively, of the system. A derivation similar to the method of averaging (Bogoliubov and Mitropolsky, 1961) leads to

$$2\nu a' = -2\zeta a\nu + F(a) - \rho\sin(\varphi - \theta), \tag{10}$$

$$2a\nu\varphi' = -a\nu^2 + H(a) - \rho\cos(\varphi - \theta), \tag{11}$$

where

$$F(a) = \frac{1}{\pi}\int_0^{2\pi} g(a\cos\Phi, -\nu a\sin\Phi)\sin\Phi\, d\Phi, \tag{12}$$

$$H(a) = \frac{1}{\pi}\int_0^{2\pi} g(a\cos\Phi, -\nu a\sin\Phi)\cos\Phi\, d\Phi. \tag{13}$$

For bilinear systems, substituting Eq. (2) into Eq. (12) and (13), we obtain

$$F(a) = \begin{cases} 0, & a \leqslant 1, \\ -\frac{4}{\pi a}(1-\alpha)(a-1), & a > 1, \end{cases} \tag{14}$$

$$H(a) = \begin{cases} a, & a \leqslant 1, \\ \frac{a}{\pi}\left[(1-\alpha)\Phi^* + \alpha\pi - \frac{1}{2}(1-\alpha)\sin 2\Phi^*\right], & a > 1, \end{cases} \tag{15}$$

where $\Phi^* = \cos^{-1}[(a-2)/a]$, $0 \leqslant \Phi^* \leqslant \pi$. For the Bouc–Wen model, substituting Eq. (5) into Eq. (3) then into Eqs. (12) and (13), we have

$$F(a) = (1-\alpha)\frac{2}{\pi}\left\{\frac{2(\beta x_0/a - \gamma)}{\gamma^2-\beta^2} - \frac{1}{(\gamma-\beta)^2 a}\left[1 - e^{(\gamma-\beta)(a-x_0)}\right] - \frac{1}{(\gamma+\beta)^2 a}\left[e^{-(\gamma+\beta)(a+x_0)} - 1\right],\right. \tag{16}$$

$$H(a) = \alpha a + (1-\alpha)\frac{2}{\pi} \times \left\{\frac{1}{\gamma-\beta}\int_\pi^{\Phi^{**}}\left[1 - e^{-(\gamma-\beta)(a\cos\Phi + x_0)}\right]\cos\Phi\, d\Phi + \frac{1}{(\gamma+\beta)}\int_{\Phi^{**}}^{2\pi}\left[1 - e^{-(\gamma+\beta)(a\cos\Phi + x_0)}\right]\cos\Phi\, d\Phi,\right. \tag{17}$$

where $\Phi^{**} = \cos^{-1}(-x_0/a)$ and $\pi \leqslant \Phi^{**} \leqslant 2\pi$.

For a stationary response a' and φ' in Eqs. (10) and (11) are assumed to be equal to zero. Solving the resultant equations we obtain

$$\rho = \sqrt{[F(a) - 2\zeta a\nu]^2 + [H(a) - a\nu^2]^2} \tag{18}$$

and

$$\theta = \varphi - \tan^{-1}\frac{F(a) - 2\zeta a\nu}{H(a) - a\nu^2}. \tag{19}$$

Equations (18) and (19) can be regarded as the memoryless nonlinear transformation relating ρ, θ to a, φ. The stationary joint probability density of a and φ can be obtained as follows:

$$p_s(a, \varphi) = p_s(\rho, \theta)\left|\frac{\partial(\rho, \theta)}{\partial(a, \varphi)}\right|, \tag{20}$$

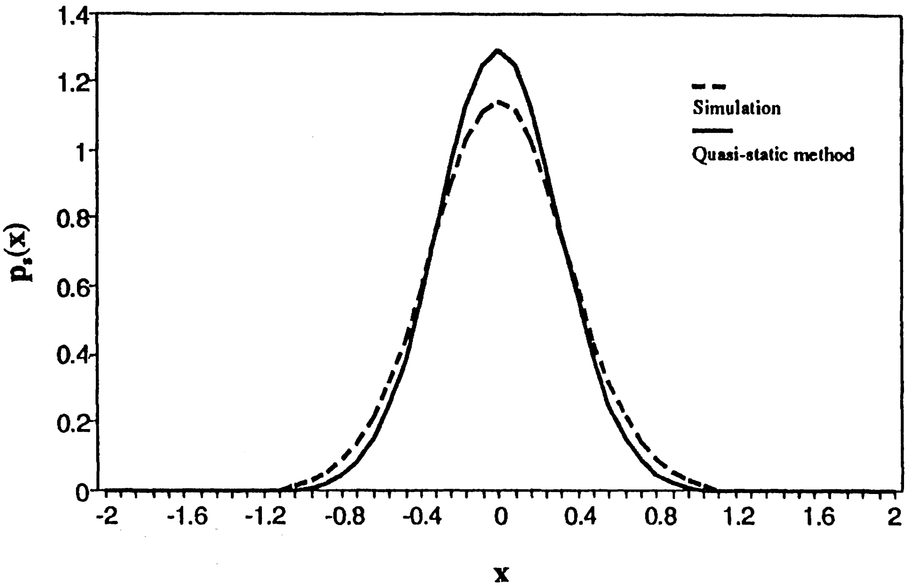

FIGURE 3 Probability density of a stationary displacement response of a bilinear hysteretic system to narrow-band excitation. $\alpha = 0.5$, $\delta = 0.01$, $\zeta = 0.01$, $\nu = 0.7$, $D = 0.05$.

where $\partial(\rho, \theta)/\partial(a, \varphi)$ is the Jacobi determinant. It can be easily shown that

$$p_s(a) = p_s(\rho)\left|\frac{\partial \rho}{\partial a}\right|, \quad a \geqslant 0, \tag{21}$$

$$p_s(\varphi) = \frac{1}{2\pi}, \quad 0 \leqslant \varphi < 2\pi. \tag{22}$$

The stationary joint probability density of the displacement x and velocity x' can be obtained from Eqs. (21) and (22) as follows:

$$\begin{aligned} p_s(x, x') &= p_s(a, \varphi)\left|\frac{\partial(a, \varphi)}{\partial(x, x')}\right| \\ &= \frac{1}{2\pi a \nu} p_s(a)\bigg|_{a=\sqrt{x^2+x'^2/\nu^2}}. \end{aligned} \tag{23}$$

From Eq. (23) we can further obtain stationary marginal probability densities of x and x' and stationary moments of x and x'.

The conditions under which amplitude a and phase φ take "quasistable" values and thus cause a' and φ' in Eqs. (10) and (11) to vanish are

$$\tau_{\text{cor}} \gg \tau_{\text{rel}}[a] \tag{24}$$

and

$$\tau_{\text{cor}} \gg \tau_{\text{rel}}[\varphi], \tag{25}$$

where τ_{cor} is the correlation time of excitation $\xi(\tau)$ and $\tau_{\text{rel}}[a]$ and $\tau_{\text{rel}}[\varphi]$ are the relaxation times of amplitude a and phase φ, respectively. If $\xi(\tau)$ is the output of a linear filter to Gaussian white noise $w(\tau)$ with intensity D, that is,

$$\xi'' + \delta\xi' + \nu^2\xi = \nu\sqrt{\delta}w(\tau), \tag{26}$$

then the noise bandwidth of $\xi(\tau)$ is $\pi\delta/2$. The correlation time τ_{cor} is of the order $2/\pi\delta$.

The relaxation time $t_{\text{rel}}[a]$ of amplitude can be evaluated from Eq. (10) as

$$\tau_{\text{rel}}[a] = \left\langle \frac{1}{\zeta - (dF(a)/da)/2\nu} \right\rangle, \tag{27}$$

where $\langle \ \rangle$ indicates the ensemble average. Because $dF(a)/da \leqslant 0$ for all values of a,

$$\tau_{\text{rel}}[a] \leqslant \frac{1}{\zeta}. \tag{28}$$

Similarly, the relaxation time $\tau_{\text{rel}}[\varphi]$ of phase φ can be evaluated from Eq. (11) as

$$\begin{aligned} \tau_{\text{rel}}[\varphi] &= \left\langle \frac{2a\nu}{\rho} \right\rangle \\ &= \left\langle \frac{1}{\sqrt{[F(a)/2\nu a - \zeta]^2 + [H(a)/2\nu a - \nu/2]^2}} \right\rangle. \end{aligned} \tag{29}$$

Because $F(a) \leqslant 0$ for all values of a,

$$\tau_{\text{rel}}[\varphi] \leqslant \frac{1}{\zeta}. \tag{30}$$

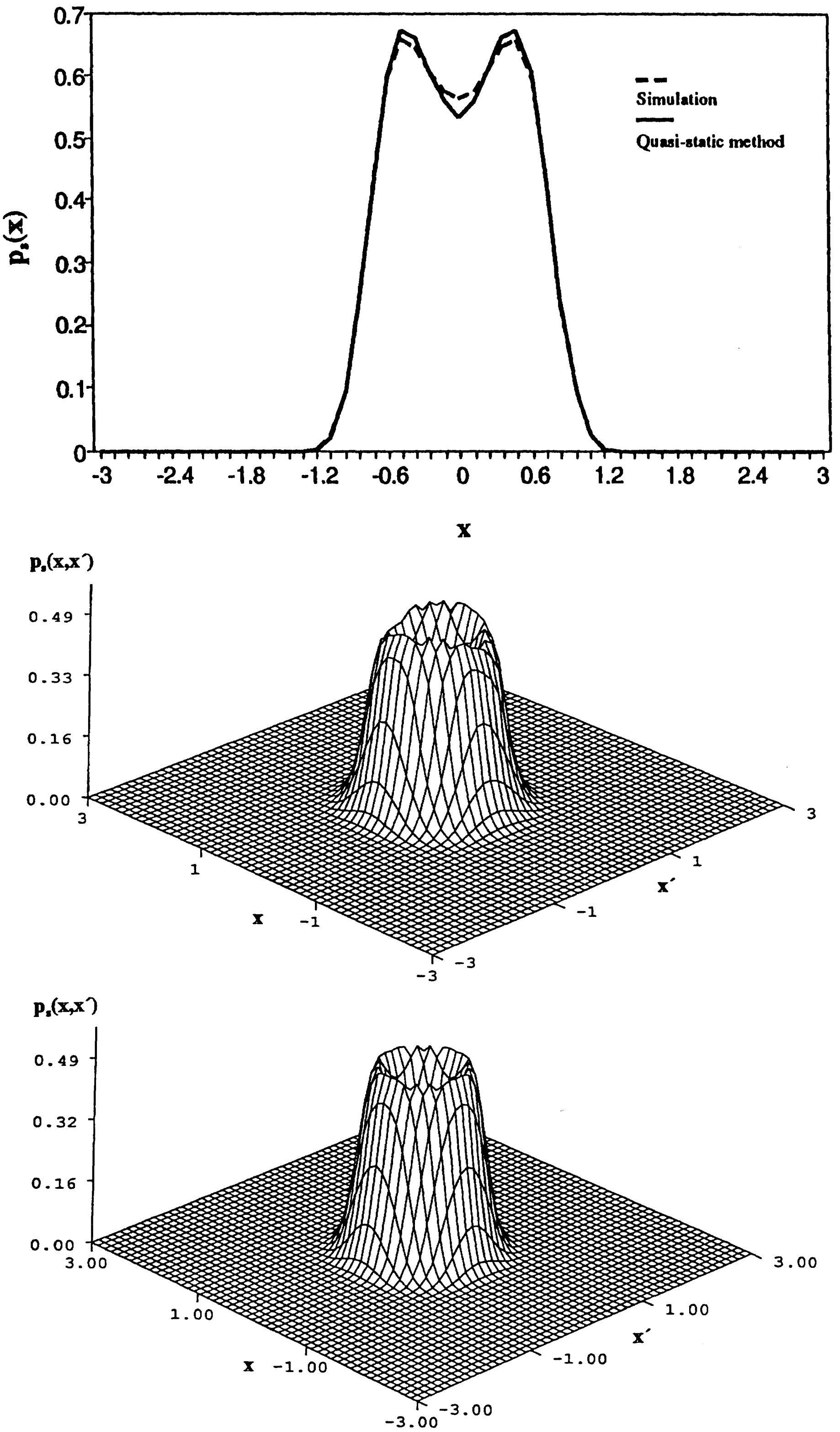

FIGURE 4 Stationary probability densities of (a) displacement, (b) displacement and velocity (from simulation), and (c) displacement and velocity (from the quasistatic method) of a smooth hysteretic system to narrow-band excitation. $\alpha = 0.05$, $\beta = 0.05$, $\gamma = 0.95$, $\delta = 0.01$, $\zeta = 0.01$, $\nu = 1.0$, $D = 0.05$.

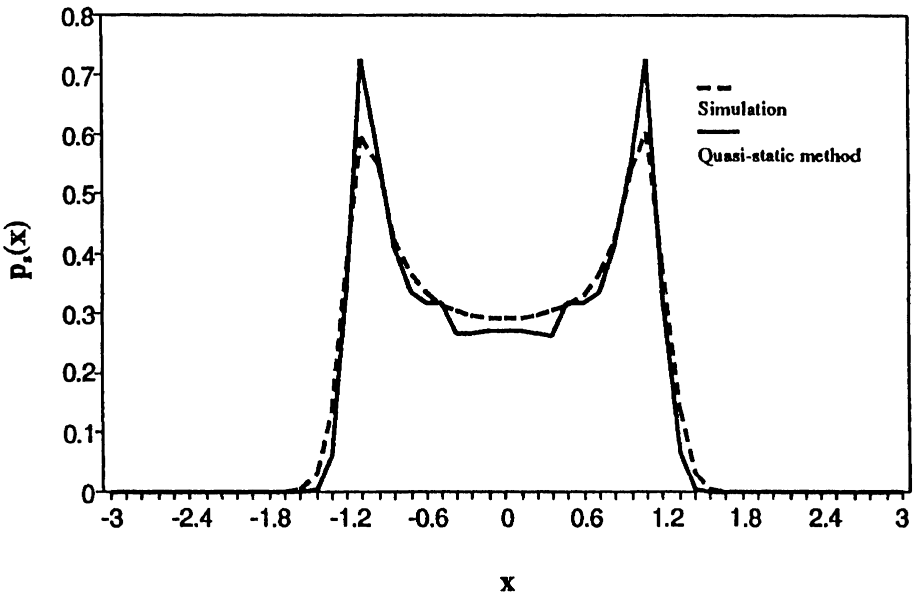

FIGURE 5 Probability density of a stationary displacement response of a bilinear hysteretic system to narrow-band excitation. $\alpha = 0.05,\ \delta = 0.01,\ \zeta = 0.01,\ \nu = 1.0,\ D = 0.05$.

Equations (18) and (19) represent necessary but not sufficient conditions for the existence of stable values of a and φ. The stability of these values must be verified. Let δa and $\delta\varphi$ be some small deviations of a and φ from their stationary values. The linearized equations for δa and $\delta\varphi$ can be obtained from Eqs. (10) and (11), giving

$$\begin{aligned}\delta a' &= a_{11}\delta a + a_{12}\delta\varphi,\\ \delta\varphi' &= a_{21}\delta a + a_{22}\delta\varphi,\end{aligned} \tag{31}$$

where

$$\begin{aligned}a_{11} &= -\zeta + \frac{1}{2\nu}\frac{dF(a)}{da},\\ a_{12} &= -\frac{\rho}{2\nu}\cos(\varphi-\theta) = \frac{1}{2\nu}\big(a\nu^2 - H(a)\big),\\ a_{21} &= \frac{1}{2\nu}\frac{d}{da}\big(H(a)/a\big) + \frac{1}{2\nu a^2}\cos(\varphi-\theta)\\ &= \frac{1}{2\nu a}\frac{dH(a)}{da} - \frac{\nu}{2a},\\ a_{22} &= \frac{\rho}{2\nu a}\sin(\varphi-\theta) = -\zeta + \frac{F(a)}{2\nu a}.\end{aligned} \tag{32}$$

The necessary and sufficient conditions for stability of a and φ are

$$a_{11} + a_{22} < 0 \quad \text{and} \quad a_{11}a_{22} - a_{12}a_{21} > 0.$$

That is,

$$\frac{1}{2\nu}\left(\frac{dF(a)}{da} + \frac{F(a)}{a}\right) - 2\zeta < 0 \tag{33}$$

and

$$\frac{1}{4\nu^2}\frac{d(\rho^2)}{d(a^2)} > 0. \tag{34}$$

Because $F(a) \leqslant 0$ and $dF(a)/da \leqslant 0$ for all a, condition (33) is always fulfilled. Condition (34) implies that a increases as ρ increases. This is the case for hysteretic systems. Therefore, solutions (18) and (19) are always stable.

DIGITAL SIMULATION

In digital simulation, the sample functions of excitation $\xi(\tau)$ is generated by using Eq. (26). When δ is small, $\xi(\tau)$ is a narrow-band Gaussian process with central frequency ν, bandwidth $\pi\delta/2$, and variance $\sigma_\xi^2 = D/2$.

For smooth hysteretic systems, the sample functions of the responses x and x' are computed from Eqs. (1), (3), and (4) by using the Runge–Kutta method of the fourth order. For bilinear hysteretic systems, instead of Eq. (2), we use Eq. (3) for the restoring force $g(x, x')$, where z satisfies the following differential equation (Suzuki and Minai, 1987):

$$z' = x'\big[1 - u(x')u(z-1) - u(-x')u(-z-1)\big], \tag{35}$$

where $u(\ )$ is a unit step function. Then the responses x and x' are computed from Eqs. (1), (3), and (35).

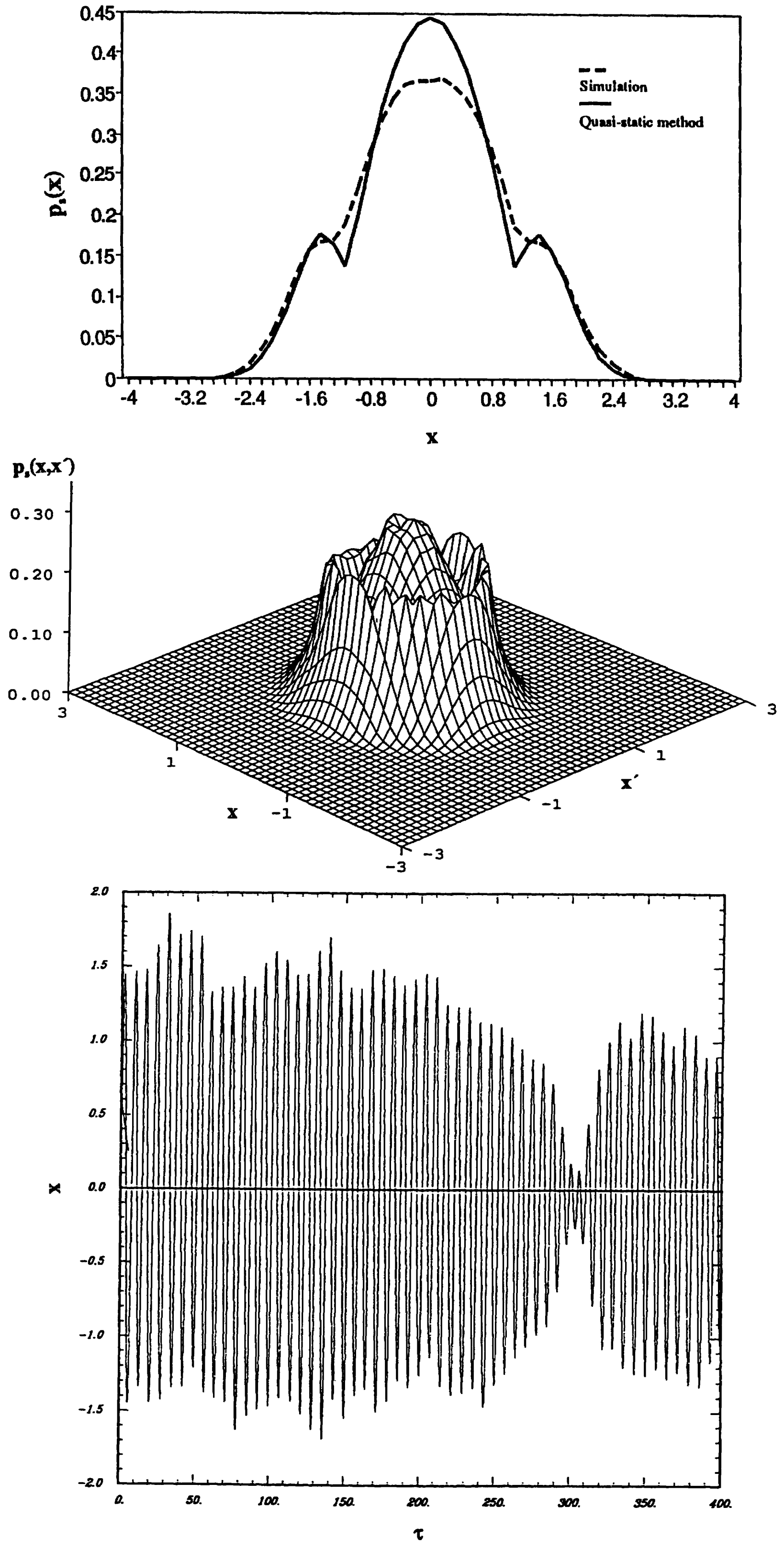

FIGURE 6 Stationary response of a bilinear hysteretic system to narrow-band excitation. $\alpha = 0.05$, $\delta = 0.01$, $\zeta = 0.01$, $\nu = 0.87$, $D = 0.05$. (a) Probability density of displacement, (b) probability density of displacement and velocity, and (c) sample function of displacement.

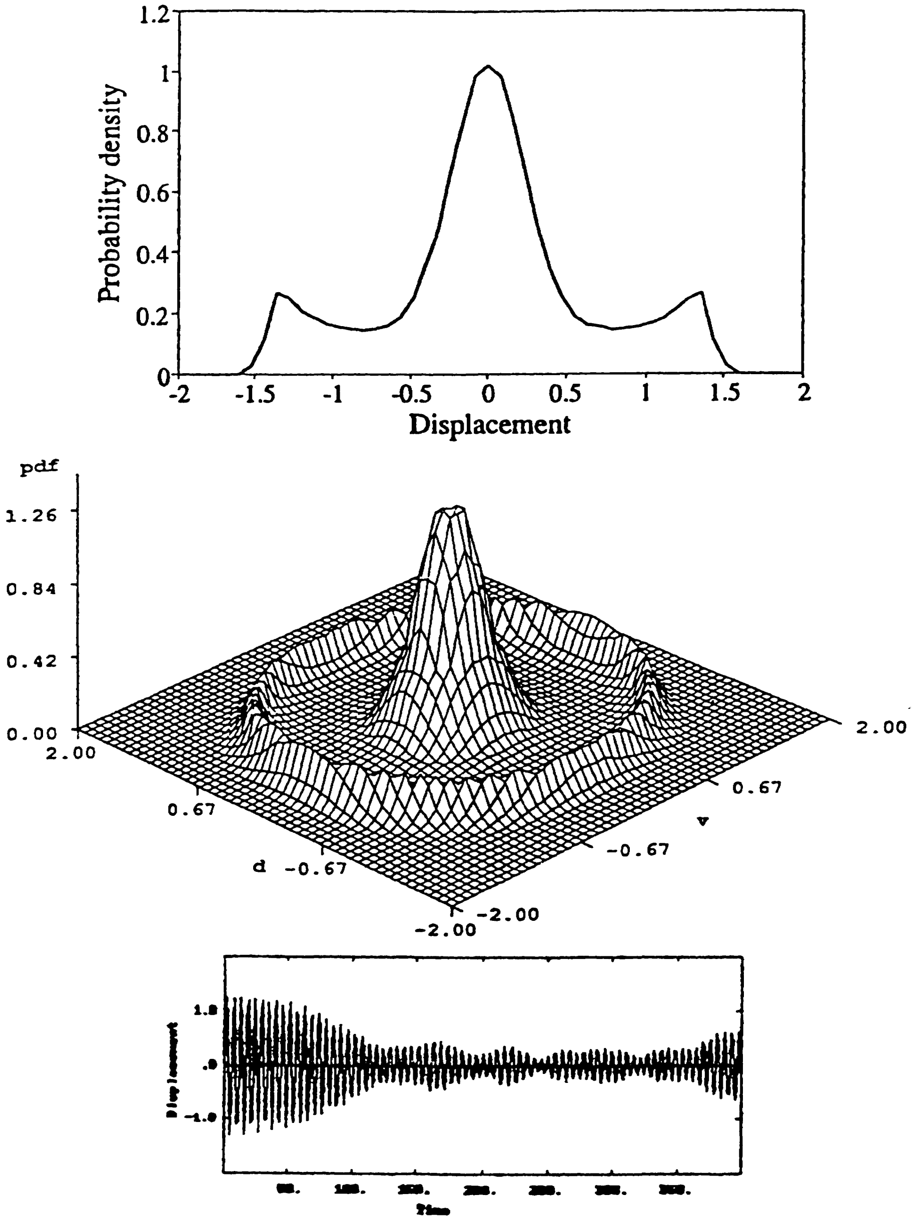

FIGURE 7 Stationary response of Duffing oscillator to narrow-band excitation. $x'' + 2\zeta x' + x + \gamma x^3 = \xi(\tau)$, $\zeta = 0.05$, $\gamma = 0.3$, $\delta = 0.002$, $\nu = 1.15$, $D = 0.02$. (a) Probability density of displacement, (b) probability density of displacement and velocity, and (c) sample function of displacement (from simulation).

RESULTS AND DISCUSSION

The behavior of the stationary response of a hysteretic system to narrow-band random excitations depends mainly on ν, the ratio of the central frequency of the excitation to the natural frequency of the preyielded hysteretic system. Similar to the case of sinusoidal excitations, the value of ν can be divided into nonresonant and resonant zones. The range of ν values for the resonant zone depends upon the intensity of the excitation, α, and the model of the system.

In the nonresonant zone the response of both bilinear and smooth hysteretic systems is relatively simple and similar to that due to wide-band random excitations (Cai and Lin, 1990; Zhu and Lei, 1987). The probability densities of the displacement, velocity, and amplitude are all unimodal (see Fig. 3). However, the displacement and velocity are usually non-Gaussian.

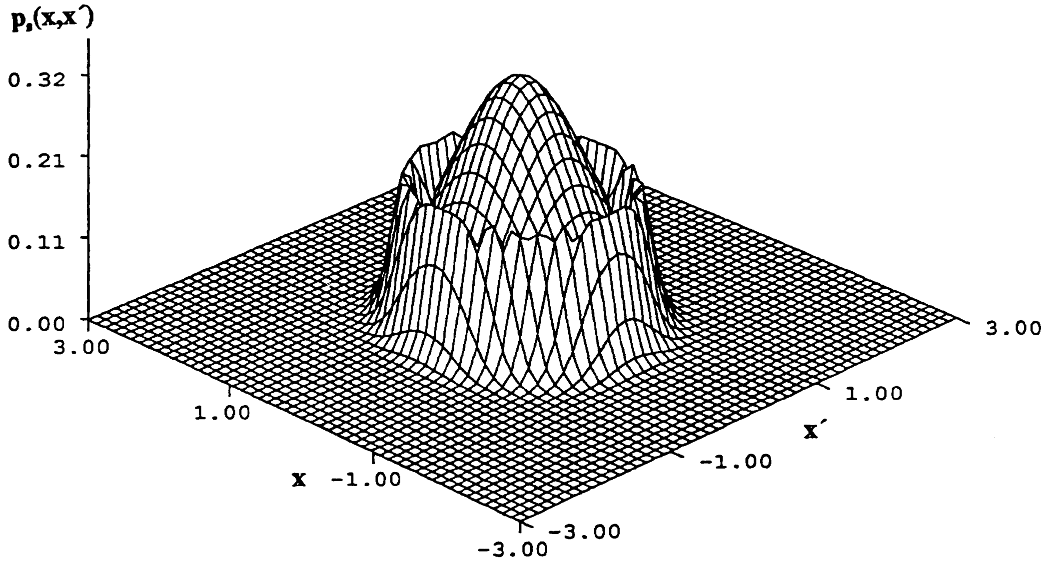

FIGURE 8 Stationary probability density of displacement and velocity of a bilinear hysteretic system to narrow-band excitation (from the quasistatic method). $\alpha = 0.05$, $\delta = 0.01$, $\zeta = 0.01$, $\nu = 0.89$, $D = 0.05$.

In the resonant zone the stationary displacement and velocity responses of both smooth and bilinear hysteretic systems are essentially non-Gaussian. The probability densities of the displacement and velocity are bimodal while the joint probability density of the displacement and velocity has a ring of peaks (see Figs. 4, 5). This implies that the stationary response behaves like a diffused limit cycle.

An interesting observation is that, for a range of ν values between nonresonant and resonant zones, the joint probability density of the displacement and the velocity of bilinear hysteretic systems with strong yielding (small α) has both a central peak and a ring of peaks with a ring of valleys between them [see Fig. 6(b)]. It is basically similar to that of the Duffing oscillator subject to a narrow-band excitation when stochastic jumps may occur [see Fig. 7(b)]. In this case, the response contains two types of more probable motions: one is a random vibration with smaller amplitudes similar to nonresonant vibration and another has larger amplitudes similar to resonant vibration.

This implies that jumps may occur between the two types of more probable motions and this phenomenon of stochastic jump has been observed in the sample functions of the displacement generated from digital simulation [see Fig. 6(c)]. It is noted that the jump occurs simultaneously in displacement and velocity and that even in this case the stationary variances of the displacement and velocity are unique.

Although the stochastic jumps in the responses of the Duffing oscillator and bilinear hysteretic systems to narrow-band excitations are essentially the same (see Figs. 6, 7), it is noted that the mechanisms leading to the jumps in these two systems are different. It is well known that jumps may occur in the response of the Duffing oscillator to sinusoidal excitation. When the quasistatic method is applied to the Duffing oscillator, for each value of ρ there are three values of a among which two are stable and the other is unstable. The two peaks in the probability density of the amplitude of the Duffing oscillator subject to a narrow-band excitation correspond to these two stable solutions of a. For a bilinear hysteretic system under sinusoidal excitation, Caughey (1960) showed rigorously that no jump can occur. The solution of a in Eq. (18) is simple valued. However, for certain combinations of the parameters, the function $\partial\rho/\partial a$ can be such that $p_s(a)$ in Eq. (21) is bimodal.

A comparison of the results obtained from the quasistatic method and digital simulation shows that the quasistatic method predicts the response of a smooth hysteretic system to a narrow-band excitation quite well. For example, for the case depicted in Fig. 4, the relative errors in the variances of the displacement and velocity are less than 1%. For bilinear hysteretic systems, the quasistatic method can predict the qualitative behavior of the response but with less accuracy. For the cases depicted in Figs. 3, 5, and 6, the relative errors in displacements are 21, 6 and 13%, respectively. It is interesting to note that the results obtained from digital simulation for $\nu = 0.87$ (Fig. 6) well match the corresponding results obtained from the quasistatic method for $\nu = 0.89$ (Fig. 8).

CONCLUSIONS

The present study examines the stationary response of smooth and bilinear hysteretic systems to narrow-band excitations in some detail. The response is qualitatively different in the nonresonant zone and in the resonant zone. In the nonresonant zone, the probability densities of the displacement and velocity are unimodal although the response is generally non-Gaussian. In the resonant zone, the response is essentially non-Gaussian and behaves like a diffused limit cycle.

For a range of ν values between nonresonant and resonant zones, the response of bilinear hysteretic systems with strong yielding contains two types of more probable motions, nonresonant and resonant vibrations, and stochastic jumps may occur between the two motions.

The quasistatic method provides a quite good prediction of the response of smooth hysteretic systems to narrow-band excitations but provides a less accurate prediction for that of bilinear hysteretic systems.

REFERENCES

Bogoliubov, N. V., and Mitropolsky, Y. A., 1961, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Gordon and Breach, New York.

Cai, G. Q., and Lin, Y. K., 1990, "On Randomly Excited Hysteretic Structures," *Journal of Applied Mechanics*, Vol. 57, pp. 442–448.

Caughey, T. K., 1960, "Sinusoidal Excitation of a System with Bilinear Hysteresis," *Journal of Applied Mechanics*, Vol. 27, pp. 640–644.

Davies, H. G., and Liu, Q., 1990, "The Response of Probability Density Function of a Duffing Oscillator with Random Narrow Band Excitation," *Journal of Sound and Vibration*, Vol. 139, pp. 1–8.

Davies, H. G., and Nandall, D., 1986, "Phase Plane for Narrow Band Random Excitation of a Duffing Oscillator," *Journal of Sound and Vibration*, Vol. 104, pp. 277–283.

Dimentberg, M. F., 1970, "Oscillations of a System with a Nonlinear Cubic Characteristic under Narrow-Band Excitation," *Mechanica Tverdogo Tela*, Vol. 6, pp. 150–154.

Fang, T., and Dowell, E. H., 1987, "Numerical Simulation of Jump Phenomena in Stable Duffing Systems," *International Journal of Non-Linear Mechanics*, Vol. 22, pp. 267–274.

Iwan, W. D., 1966, "A Distributed-Element Model for Hysteresis and its Steady-State Dynamic Response," *Journal of Applied Mechanics*, Vol. 33, pp. 893–900.

Iyengar, R. N., 1988, "Stochastic Response and Stability of the Duffing Oscillator under Narrow-Band Excitation," *Journal of Sound and Vibration*, Vol. 126, pp. 355–263.

Jennings, P. C., 1964, "Periodic Response of General Yielding Structure," *Journal of Engineering Mechanics Division, ASCE*, Vol. 90, pp. 131–165.

Lennox, W. C., and Kuak, Y.C., 1976, "Narrow Band Excitation of a Nonlinear Oscillator," *Journal of Applied Mechanics*, Vol. 43, pp. 340–344.

Lin, Y. K., 1976, *Probabilistic Theory of Structural Dynamics*, Robert E. Krieger Publishing Company, Melbourne, FL.

Lyon, R. H., Heckl, M., and Hazelgrove, C. B., 1961, "Response of Hard-Spring Oscillator to Narrow-Band Excitation," *Journal of the Acoustical Society of America*, Vol. 33, pp. 1404–1411.

Roberts, J. B., 1991, "Multiple Solutions Generated by Statistical Linearization and Their Physical Significance," *International Journal of Non-Linear Mechanics*, Vol. 26, pp. 945–960.

Stratonovich, R. L., 1967, *Topics in the Theory of Random Noise*, Vol. 2, Gordon and Breach, New York.

Suzuki, Y., and Minai, R., 1987, "Application of Stochastic Differential Equation to Seismic Reliability Analysis of Hysteretic Structure," *Proceedings of the US–Japan Joint Seminar on Stochastic Approaches in Earthquake Engineering*, Boca Raton, FL, pp. 334–356.

Wen, Y. K., 1989, "Methods of Random Vibration for Inelastic Structure," *Applied Mechanics Reviews*, Vol. 42, No. 2, pp. 39–52.

Zhu, W. Q., and Lei, Y., "Stochastic Averaging of Energy Envelope of Bilinear Hysteretic Systems," in F. Ziegler and G. I. Schuëller, Eds., *Nonlinear Stochastic Dynamic Engineering Systems, Proc. of the IUTAM Symposium*, 1988, Springer–Verlag, Berlin.

Zhu, W. Q., Lu, M. Q., and Wu, Q. T., 1993, "Stochastic Jump and Bifurcation of a Duffing Oscillator under Narrow-Band Excitation," *Journal of Sound and Vibration*, Vol. 165, pp. 285–304.


Journal of Engineering
The Scientific World Journal
International Journal of Rotating Machinery
Journal of Sensors
International Journal of Distributed Sensor Networks
Advances in Civil Engineering
Hindawi
Submit your manuscripts at http://www.hindawi.com
Journal of Control Science and Engineering
Journal of Robotics
Journal of Electrical and Computer Engineering
Advances in OptoElectronics
VLSI Design
International Journal of Navigation and Observation
Modelling & Simulation in Engineering
International Journal of Aerospace Engineering
International Journal of Chemical Engineering
International Journal of Antennas and Propagation
Active and Passive Electronic Components
Shock and Vibration
Advances in Acoustics and Vibration