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A combination of modal synthesis and subspace iteration for an efficient algorithm for modal analysis within a FE-code

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Abstract. Various well-known modal synthesis methods exist in the literature, which are all based upon certain assumptions for the relation of generalised modal co-ordinates with internal modal co-ordinates. If employed in a dynamical FE substructure/superelement technique the generalised modal co-ordinates are represented by the master degrees of freedom (DOF) of the master nodes of the substructure. To conduct FE modal analysis the modal synthesis method can be integrated to reduce the number of necessary master nodes or to ease the process of defining additional master points within the structure. The paper presents such a combined method, which can be integrated very efficiently and seamless into a special subspace eigenvalue problem solver with no need to alter the FE system matrices within the FE code. Accordingly, the merits of using the new algorithm are the easy implementation into a FE code, the less effort to carry out modal synthesis, and the versatility in dealing with superelements. The paper presents examples to illustrate the proper work of the algorithm proposed.

1. Introduction

If complex real-world structures have to be modelled to conduct FE modal analysis or model updating it is for various reasons advantageous to subdivide the structure into substructures and condense them to macro or superelements. Another reason to consider subdivision might be that measurements have been taken only from parts of the structure. The assembled superelements represent on a so-called hyperstructure level the structure's dynamical behaviour. The advantage is obvious: we can get a reduced system for the modelling and on the other hand the substructure can be parameterised and employed in conjunction with model updating procedures. Accordingly, it remains a fruitful area for research and development in both academia and industry. But the merits of using simple static condensation method based on the assumption of a minimum of the potential energy vanish if higher modes are needed and the substructures are rather complex. To improve the situation it would require a selection of additional DOF beside the boundary DOF of the substructures, which are in most cases difficult to choose or need a great deal of computation to get them. Modal synthesis can help to get a better approximation within the substructure and can include measured modal data if appropriate. Various well-known substructure synthesis methods exist, pioneered, e.g., by Hurty [2], Gladwell [3], Rubin [4], Benfield-Hurda [5], Craig-Bampton [6], Meirovitch [7], and others. They are commonplace in structural dynamics now, but nonetheless there is ongoing work in that field [14–17] and there are of course other substructure representations as those used in this paper, based on "co-ordinate transformations". All this covers a wide range of problems, but in this article the emphasis will be more on the algorithmically side. The references given so far are only meant to serve as starting point and by no means exhaustive. The paper presents a very efficient algorithm that combines modal synthesis on the basis of the Craig-Bampton method, superelement technique [8,10–12], and McCormick-Noe simultaneous iteration [9,13]. The essence of the

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idea is to bring seamless Craig-Bampton method into the subspace iteration, there are only a few changes in the iteration scheme necessary, and it requires a minimum additional calculation effort. The algorithm enables to combine in a structure statically condensed superelements, superelements improved by modal syntheses with inclusion of computed interior modal data, and superelements improved by modal syntheses with inclusion of measured modal data, alternatively. The algorithm proposed is an ideal basis for modal analysis of complex structures in particular if higher modes have to be included. It provides an interesting operational ground for model updating and efficient model reduction. For the algorithm proposed we can regard the substructured systems on either way as an assemblage of components (more guided by design) or substructures obtained by pure domain decomposition (with optional parallel processing of substructures). The paper is going to present two examples that will illustrate the algorithm and its abilities.

2. Basic equations

2.1. Substructure/superelemt technique

After FE discretisation the linear dynamic vibration problem is represented by a system of linear differential equations and the associated general matrix eigenvalue problem

$$\begin{aligned} &\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{D}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{f}(t)\\ &(\mathbf{K}-\lambda\mathbf{M})\boldsymbol{\phi}=\mathbf{0}\\ &\Phi=[\boldsymbol{\phi}_1,\ldots,\boldsymbol{\phi}_i,\ldots\boldsymbol{\phi}_m]\\ &\Lambda=\mathrm{diag}(\lambda_i)\\ &\mathbf{M},\mathbf{D},\mathbf{K}\in\mathcal{R}^{n\times n}\\ &\Phi\in\mathcal{R}^{n\times m}\ \Lambda\in\mathcal{R}^{n\times n} \end{aligned} \tag{1}$$

where $\mathbf{M},\mathbf{D},\mathbf{K},\Lambda$, and Φ are the mass, damping, stiffness, spectral, and modal matrix, respectively. n is the number of model DOF and m is the number of calculated eigenvalues and eigenvectors. Typically, Φ is ortho-normalized with the relation

$$\begin{aligned} &\Phi^T\mathbf{M}\Phi=\mathbf{I}=\mathrm{diag}(1)\\ &\Phi^T\mathbf{K}\Phi=\Lambda=\mathrm{diag}(\lambda_i) \end{aligned} \tag{2}$$

The time for the solution of dynamic problems can be reduced tremendously if not the full-size system matrices are used. To this end a wide variety of reduction methods exist. For a structure the relationship between internal (slave) DOF $\mathbf{x}_i$ and selected external (master) DOF $\mathbf{x}_e$ can be written as

$$\mathbf{x}=\begin{bmatrix}\mathbf{x}_e\\ \mathbf{x}_i\end{bmatrix}=\begin{bmatrix}\mathbf{I}\\ \mathbf{T}\end{bmatrix}\mathbf{x}_e=\overline{\mathbf{T}}\mathbf{x}_e \tag{3}$$

According to Eq. (3) the eigenvalue problem can be subdivided into the following submatrix form

$$\begin{aligned} &\left(\begin{bmatrix}\mathbf{K}_{ii}\ \mathbf{K}_{ie}\\ \mathbf{K}_{ei}\ \mathbf{K}_{ee}\end{bmatrix}-\lambda\begin{bmatrix}\mathbf{M}_{ii}\ \mathbf{M}_{ie}\\ \mathbf{M}_{ei}\ \mathbf{M}_{ee}\end{bmatrix}\right)\begin{bmatrix}\boldsymbol{\phi}_i\\ \boldsymbol{\phi}_e\end{bmatrix}=\mathbf{0}\\ &\mathbf{M}_{ii},\ \mathbf{K}_{ii}\in\mathcal{R}^{n_i\times n_i}\\ &\mathbf{M}_{ee},\mathbf{K}_{ee}\in\mathcal{R}^{n_e\times n_e} \end{aligned} \tag{4}$$

Using Eq. (3), the system of differential equations shown in Eq. (1) can now be written with reduced dimension as

$$\begin{aligned} &\overline{\mathbf{T}}^T\mathbf{M}\overline{\mathbf{T}}\ddot{\mathbf{x}}_e+\overline{\mathbf{T}}^T\mathbf{D}\overline{\mathbf{T}}\dot{\mathbf{x}}_e+\overline{\mathbf{T}}^T\mathbf{K}\overline{\mathbf{T}}\mathbf{x}_e=\overline{\mathbf{T}}^T\mathbf{f}\\ &\overline{\mathbf{M}}\ddot{\mathbf{x}}_e+\overline{\mathbf{D}}\dot{\mathbf{x}}_e+\overline{\mathbf{K}}\mathbf{x}_e=\overline{\mathbf{f}}_e \end{aligned} \tag{5}$$

The degree of reduction that still produces sensible solutions depends upon the reduction method deployed (and the skills and experiences of the analyst, of course). The calculation of the transformation matrix $\overline{\mathbf{T}}$ and the reduction itself can be very time consuming, if we perform the reduction on the whole system. In this case the condensation of parts of the structure (so called substructures) is more advantageous. Accordingly, each substructure $j(j=1,\ldots,n_{str})$ will be reduced with its own transformation matrix $\overline{\mathbf{T}}^j$. The pattern of the substructuring can follow structural members or be guided by design, FE modelling, or computational related (e.g., parallel processing). Special techniques and procedures for substructuring are beyond the scope of this paper. Letting $\mathbf{m}^j,\mathbf{d}^j$, and $\mathbf{k}^j$ be the matrices of the substructures and $\overline{\mathbf{m}}^j,\overline{\mathbf{d}}^j$, and $\overline{\mathbf{k}}^j$ the reduced substructure matrices the so-called macro- or superelements, which will be assembled to the condensed or hyper-structure matrices

$$\overline{\mathbf{M}},\overline{\mathbf{D}},\overline{\mathbf{K}},\overline{\mathbf{f}}=\bigcup_{j=1}^{n_{str}}\left(\overline{\mathbf{m}}^j,\overline{\mathbf{d}}^j,\overline{\mathbf{k}}^j,\overline{\mathbf{f}}^j\right) \tag{6}$$

The problem is to determine the transformation matrices $\overline{\mathbf{T}}^j$. For static problems each transformation matrix can always be calculated exactly. The exact calculation of the transformation matrix for dynamic problems depends upon the modal data of the structures. But it is our aim to solve the eigenvalue problem for the whole structure without using the complete system matrices.

This aim places a demand upon approximations for the transformation matrix. The implication is that model condensation becomes a delicate matter too, for the reduction usually limits the frequency range the reduced model is capable to cope with.

Since FE damping terms for large models are usually not known with any confidence (more mathematical model than physically based) the transformation of the damping matrix with $\overline{\mathbf{T}}$ would be merely formal and in most cases not very useful. Consequently, $\overline{\mathbf{C}}$ is often constructed after condensation with $\overline{\mathbf{K}}$ and $\overline{\mathbf{M}}$ by assuming a proportional damping model, or by using measured modal damping data, etc.

Zehn [10] has shown that most condensation methods, which can be employed for the complete system, could also be used for each substructure. Using for the eigenvalue problem of a substructure Eq. (4) accordingly, we can get a relationship between the internal (slave) and external (master) DOF for the jth substructure given by

$$\begin{aligned}\mathbf{x}_i^j &= -\left(\mathbf{k}_{ii}^j - \hat{\lambda}^j \mathbf{m}_{ii}^j\right)^{-1}\left(\mathbf{k}_{ie}^j - \hat{\lambda}^j \mathbf{m}_{ie}^j\right)\mathbf{x}_e^j \\ &= \mathbf{T}^j\left(\hat{\lambda}^j\right)\mathbf{x}_e^j\end{aligned} \tag{7}$$

The superscripts for substructures and superelements will, henceforth, be dropped for the sake of brevity of notation, but in all further formulas we still consider a substructure j. The expansion of Eq. (7) in a Taylor series yields

$$\begin{aligned}\mathbf{T}\left(\hat{\lambda}\right) &= \sum_{k=0}^{\infty}\hat{\lambda}^k(\mathbf{k}_{ii}^{-1}\mathbf{m}_{ii})^k\mathbf{k}_{ii}^{-1}\left(\hat{\lambda}\mathbf{m}_{ie} - \mathbf{k}_{ie}\right) \\ &= -\mathbf{k}_{ii}^{-1}\mathbf{k}_{ie} + \hat{\lambda}\mathbf{k}_{ii}^{-1}\mathbf{m}_{ie} \\ &\quad + \sum_{k=1}^{\infty}\hat{\lambda}^k(\mathbf{k}_{ii}^{-1}\mathbf{m}_{ii})^k\mathbf{k}_{ii}^{-1}\left(\hat{\lambda}\mathbf{m}_{ie} - \mathbf{k}_{ie}\right) \\ &= \mathbf{T}_1 + \mathbf{B}\left(\hat{\lambda}\right)\end{aligned} \tag{8}$$

The latter is a Neumann series with the convergence condition $\|\mathbf{k}_{ii}^{-1}\mathbf{m}_{ii}\| \leqslant c < 1$. Different approximations of $\mathbf{T}$ (e.g., truncations of the series) lead to a number of procedures (see Zehn [11]). The row expansion of $\mathbf{T}$ in Eq. (8) truncated after the first term gives the simplest form of an approximation for $\mathbf{T}$, namely the static condensation $\mathbf{T}_1 = -\mathbf{k}_{ii}^{-1}\mathbf{k}_{ie}$, which is independent of any modal data. Static condensation can be derived also from the idea that the potential energy has a minimum with respect to slave displacements (Irons static approach: no applied forces including inertia forces associated with the slave displacements). From the cut off condition (convergence condition of the Neumann series) for the Taylor series in Eq. (8), follows an error criterion for a superelement

$$(\mathbf{k}_{ii} - \vartheta\mathbf{m}_{ii})\Xi = \mathbf{0} \quad \frac{\hat{\lambda}_k}{\vartheta_1} = \mu_{\max} < 1 \tag{9}$$

where ϑ_1 is the cut off frequency for each substructure, Ξ are the m_j normal modes of the slave DOF of the structure fixed at the boundary (master) DOF, and $\hat{\lambda}_k$ is the kth eigenvalue of the jth substructure. For the static condensation method one has to choose additional internal points, if the inequality condition in Eq. (9) has not been satisfied sufficiently. Physically, this implies that the master nodes (or DOF) should be chosen from areas of low stiffness and high inertia. $\mu_{\max}$ is a sort of quality measure for each substructure. So the lowest value for ϑ_1 defines a lower frequency bound beyond which the structure becomes unfit to fulfil its purpose. That is important for simulation, hence in that case the reduced model is not capable to work in sampling respectively integration time steps close or above the period of the cut-off frequency. Neither can we conduct modal analysis for frequencies close to or above the cut off frequency. The selection of external DOF relies mostly on the substructuring of the model, experience, and experimental results. The degree of reduction, which still produces sensible solutions, depends a great deal upon the reduction method used.

2.2. Modal synthesis methods

Now we turn our attention to modal synthesis methods that exist in various forms and are widely used in structural dynamic analysis. In the literature we can find various methods as aforementioned. A survey and the theoretical background are given in more detail in Meirovitch [1]. The governing idea of the method is to set up the structure model as an assemblage of substructures, each represented by a finite set of suitable modal data of the substructure.

$$\begin{aligned}\mathbf{x}_i &= \Psi(x,y,z)\mathbf{r}(t) \\ &= [\Psi^R,\ \Psi^C,\ \Psi^N]\mathbf{r}(t)\end{aligned} \tag{10}$$

Meirovitch [7] extended the method to substructure synthesis by assuming that the modal data describing the substructures (where Ψ^R accommodates the "rigid-body displacement" part, Ψ^C "constraint displacement" part, and Ψ^N "displacement relative to constraints" part) are only one form of sets of suitable admissible functions. The method is essentially a kind

of Rayleigh-Ritz type method and $\Psi(x,y,z)$ has therefore only to belong to a set of admissible functions that represent the motion of the substructure. While with the Meirovitch approach the time consuming calculation of internal modal data can be avoided by using sets of low-order polynomials, we will follow the Craig-Bampton idea. On one hand the method allows us to operate with fixed boundaries in the master DOF and can be very advantageously employed in an FE environment within a special subspace iteration eigenproblem solver. By using the algorithm shown in the next paragraph, we can avoid alterations in the stiffness and mass matrices for the substructures and superelements in the FE code. And the orthogonality relations permit us to construct an algorithm with only minor extra computational effort. That is why we have confined ourselves to the Craig-Bampton approach. In that case the motion of the internal DOF of the substructures is described by a combination of static condensation as shown in paragraph 2.1. and the eigenvectors Ξ of the internal eigenvalue problem Eq. (9). For a substructure the transformation is now given by

$$\begin{aligned}
&\mathbf{r} = \begin{bmatrix} \mathbf{x}_e \\ \mathbf{a} \end{bmatrix} \qquad \Psi^R = \mathbf{T}_1 \\
&\Psi^C = \mathbf{0} \qquad \Psi^N = \Xi \\
&\mathbf{x} = \begin{bmatrix} \mathbf{x}_e \\ \mathbf{x}_i \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{T}_i & \Xi \end{bmatrix} \begin{bmatrix} \mathbf{x}_e \\ \mathbf{a} \end{bmatrix} = \overline{\mathbf{T}}\mathbf{r} \\
&\mathbf{x}_e \in \mathcal{R}^{n_e},\ \mathbf{a} \in \mathcal{R}^{m_j}
\end{aligned} \tag{11}$$

Clearly, an important role plays the selection and truncation of the eigenvalues of each substructure that we have to compute from the eigenvalue problem Eq. (9). But this paper does not cover theory and application problems of the Craig-Bampton method that can be found in the literature to a great extent. Nor, incidentally, is it the intention of the author to deal with eigenvalue problem solvers in detail.

2.3. Special subspace iteration

Based on the theory of McCormick-Noe [13] a subspace iteration algorithm has been developed and firstly realised and employed in a FE code by Zehn [9]. The algorithm has proven to be very effective and we and others have conducted thousands of modal analyses for industrial applications with it. Nevertheless, the algorithm is still widely unknown. It can be shown that all classes of subspace iteration algorithms have the same convergence rate, but the algorithm based on the theory of McCormick-Noe is the one with least computational effort. To explain the iteration scheme for such FE related general eigenvalue problems, let us consider the following algorithm

1) Cholesky decomposition of the stiffness matrix

$$\overline{\mathbf{K}} = \overline{\mathbf{R}}^T\overline{\mathbf{R}} \tag{12}$$

2) Setting of start vectors $\mathbf{X}^{(0)}$
3) Multiplication with the mass matrix

$$\tilde{\mathbf{Y}}^{(i)} = \overline{\mathbf{M}}\mathbf{X}^{(i-1)} \tag{13}$$

4) Solve linear equation by forward and back substitution and calculate subspace matrix $\tilde{\mathbf{K}}^{(i)}$ in between

$$\begin{aligned}
&\overline{\mathbf{R}}^T\tilde{\tilde{\mathbf{X}}}^{(i)} = \tilde{\mathbf{Y}}^{(i)} \\
&\tilde{\mathbf{K}}^{(i)} := \tilde{\mathbf{X}}^{(i)^T}\overline{\mathbf{K}}\tilde{\mathbf{X}}^{(i)} = \tilde{\tilde{\mathbf{X}}}^{(i)^T}\tilde{\tilde{\mathbf{X}}}^{(i)} \\
&\overline{\mathbf{R}}\tilde{\mathbf{X}}^{(i)} = \tilde{\tilde{\mathbf{X}}}^{(i)}
\end{aligned} \tag{14}$$

5) Solving subspace standard eigenvalue problem and calculation of matrix $\mathbf{L}^{(i)}$ for improving the eigenvectors in the subspace spanned by the eigenvector approximations $\tilde{\mathbf{X}}^{(i)}$

$$\begin{aligned}
&\tilde{\mathbf{K}}^{(i)}\mathbf{Q}^{(i)} = \mathbf{I}\mathbf{Q}^{(i)}\Gamma^{(i)} \\
&\text{with} \quad \Gamma^{(i)} = \text{diag}\left(\gamma_k^{(i)^2}\right) \\
&\mathbf{L}^{(i)} = \mathbf{Q}^{(i)}\text{diag}\left(\gamma_k^{(i)^{-1}}\right) \\
&\mathbf{X}^{(i)} = \tilde{\mathbf{X}}^{(i)}\mathbf{L}^{(i)}, \\
&\mathbf{X}^{(i)} \to \overline{\Phi}, \quad \gamma_k^{(i)^{-1}} = \tilde{\lambda}_k \to \lambda_k
\end{aligned} \tag{15}$$

6) Check for convergence of eigenvalues, if $\left(\tilde{\lambda}_k^{(i)} - \tilde{\lambda}_k^{(i-1)}\right)/\tilde{\lambda}_k^{(i-1)} \leqslant \varepsilon_E$ stop else go to 3)

The convergence rate depends upon the subspace chosen. If m_{ei} is the number of natural vectors and eigenfrequencies wanted and m_s the dimension of the subspace spanned by $\mathbf{X}^{(i)}$ the convergence order is determined by the relation $O\left(\frac{\lambda_{m_s}}{\lambda_{m_s+1}}\right)$, thus m_s should be somewhat greater to match that relation. Multiple roots within the subspace do not influence the convergence at all. The main advantage above other subspace iteration methods is that only the stiffness matrix has to be transformed into the subspace. That is carried out in a very efficient way as shown in step 4) Eq. (14). So we can avoid the time consuming transformation of the mass matrix. Consequently, the resulting eigenvectors are obtained as ortho-normal towards the stiffness

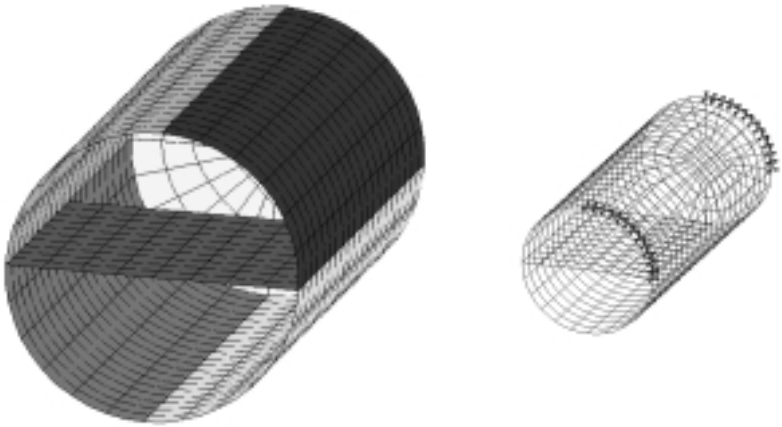

Fig. 1. Cylinder with end-cap and floor panel substructured into six substructures (left) boundary conditions (right).

matrix

$$\begin{aligned} \overline{\Phi}^T \overline{\mathbf{K}}\, \overline{\Phi} &= \mathbf{I} = \text{diag}(1) \\ \overline{\Phi}^T \overline{\mathbf{M}}\, \overline{\Phi} &= \Lambda^{-1} = \text{diag}(\lambda_k^{-1}) \end{aligned} \tag{16}$$

3. Combined algorithm

The method of substructure/superelement-technique based on static condensation does not perform very well when the master DOF in the substructure are unable to describe the modes and if Eq. (9) is not satisfied sufficiently. This is particular true for higher modes with frequencies close to $\mu_{\max}$. To this end as mentioned above, we could increase the number of master DOF. This can prove troublesome for real industrial parts and would consequently diminish the degree of reduction of the structure. To improve or at least to alleviate the poor static condensation of substructures we are going to employ modal synthesis methods. From our preceding discussion of the Craig-Bampton method we know that some eigenvectors of the internal eigenvalue problem can be included in the substructure condensation, namely those modes, which are deemed to occur as parts of the modes of the overall structure. The number of modes included for each structure can be different or even zero (for pure static condensation). Therefore, the algorithm developed takes for the substructures either automatically a selection of modal data from the solution of the internal eigenvalue problem Eq. (9) or can process measured modal data taken from substructures fixed at the master DOF (boundaries). In general, special algorithms are necessary to integrate the method into a FE code. The paper presents an algorithm for addressing seamless integration into the subspace iteration method as shown in paragraph 2.2. with least possible additional effort in computation.

From the preceding discussion it follows that we have to employ the transformation as shown in Eq. (4) in conjunction with Eq. (11) upon the substructures selected. Using the ortho-normalisation in the form of Eq. (16) for the internal eigenvectors Ξ obtained from the eigenvalue problem Eq. (9) yields

$$\overline{\mathbf{k}}^* = \begin{bmatrix} \overline{\mathbf{k}}\, \mathbf{0} \\ \mathbf{0}\ \mathbf{I} \end{bmatrix} \quad \overline{\mathbf{m}}^* = \begin{bmatrix} \overline{\mathbf{m}} & \mathbf{p} \\ \mathbf{p}^T & \boldsymbol{\theta}^{-1} \end{bmatrix} \tag{17}$$

where $\boldsymbol{\theta} = \text{diag}(\vartheta_k),\ k = 1, \ldots, m_{jCB}$, denotes the spectral matrix of the eigenvalues of the internal eigenvalue problem, Eq. (9), or the measured angular frequency for the substructure grounded in the master DOF of each substructure selected for a Craig-Bampton transformation. $\overline{\mathbf{m}}$ and $\overline{\mathbf{k}}$ are the superelement matrices obtained by static condensation. Matrix $\mathbf{p}$ can be expressed for each substructure concerned as

$$\begin{aligned} \mathbf{p} &= \mathbf{m}_{ie}^T \cdot \Xi - \mathbf{k}_{ie}^T \cdot \Xi \cdot \boldsymbol{\theta}^{-1} \\ \mathbf{p} &\in \mathcal{R}^{n_e \times m_{jCB}} \end{aligned} \tag{18}$$

Matrices of all structures, superelements (with and without Craig-Bampton transformation) can be assembled separately form the condensed substructure matrices

$$\begin{aligned} &\overline{\mathbf{M}}, \overline{\mathbf{D}}, \overline{\mathbf{K}}, \overline{\mathbf{f}}, \overline{\boldsymbol{\theta}}, \overline{\mathbf{P}} = \\ &\bigcup_{j=1}^{nstr} \left\{ \overline{\mathbf{m}}^j, \overline{\mathbf{d}}^j, \overline{\mathbf{k}}^j, \overline{\mathbf{f}}^j, \boldsymbol{\theta}^j, \mathbf{p}^j : \Xi^j \neq \mathbf{0} \right\} \end{aligned} \tag{19}$$

Table 1
Eigenfrequencies

No	not substructured [Hz]	substructured static condensation [Hz]	Substructured Craig-Bampton (new algorithm) [Hz]
1	21.6	23.1	21.6
2	27.4	30.4	27.5
3	28.9	–	29.2
4	40.5	37.3	40.5
5	45.6	46.5	45.4
6	61.2	62.3	61.7
7	66.7	–	66.9
8	66.9	–	67.0
9	71.4	76.9	71.6
10	87.9	92.1	88.0
11	93.6	104.1	93.7
12	94.0	99.5	94.5
13	96.6	106.4	95.7
14	108.5	–	108.6
15	115.7	–	118.8

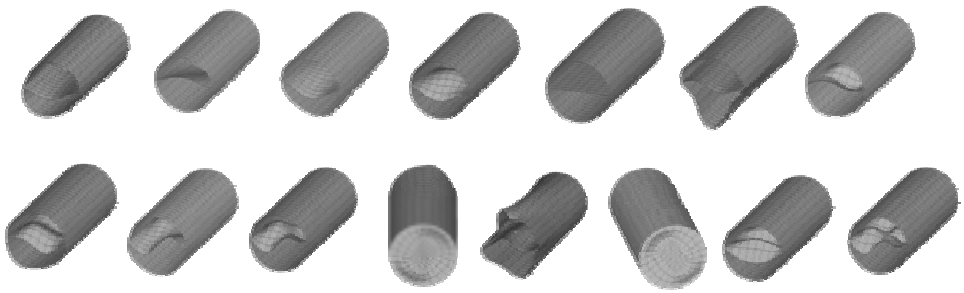

Fig. 2. Eigenmodes.

Recalling, we have split up the vector $\mathbf{r}$ in Eq. (11); we can write now for the condensed eigenvalue problem of the structure

$$\left(\begin{bmatrix} \overline{\mathbf{K}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} - \lambda \begin{bmatrix} \overline{\mathbf{M}} & \overline{\mathbf{P}} \\ \overline{\mathbf{P}}^T & \overline{\boldsymbol{\theta}}^{-1} \end{bmatrix} \right) \begin{bmatrix} \overline{\Phi} \\ \mathbf{a} \end{bmatrix} = \mathbf{0}$$
$$\Phi \in \mathcal{R}^{n \times m},\ \mathbf{a} \in \mathcal{R}^{\overline{m} \times m},\ \overline{\boldsymbol{\theta}} \in \mathcal{R}^{\overline{m} \times \overline{m}} \qquad (20)$$
$$n = \bigcup_{j=1}^{n_{str}} n_e,\ \overline{m} = \bigcup_{j=1}^{n_{CB}} m_{jCB}$$

We now turn our attention to the solution of the eigenvalue problem Eq. (20) with the aforementioned subspace algorithm. To this end it is not necessary to solve the eigenvalue problem of order $n + \overline{m}$ directly. The integration into the subspace algorithm as shown in paragraph 2.3. is a particularly attractive idea, if we observe that we can separate the matrices. Thus, the additional equations are readily accommodated through the use of the separation and keep the FE procedure nearly unchanged. Then the new algorithm for the subspace iteration is given by

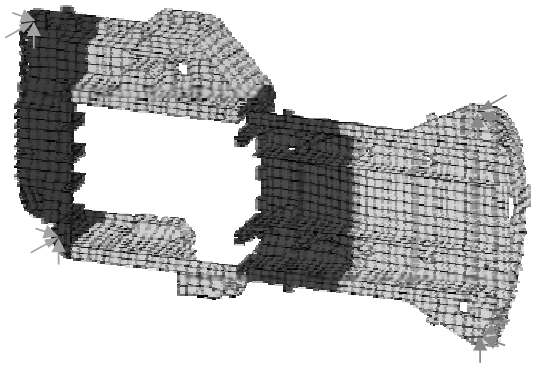

Fig. 3. Sump of a car engine, five substructures.

1) For the Cholesky decomposition Eq. (12) we can now write

$$\begin{bmatrix} \overline{\mathbf{R}}^T \overline{\mathbf{R}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \qquad (21)$$

2) Setting of start vectors in $\mathbf{X}^{(0)}$ and $\mathbf{a}^{(0)}$
3) Multiplication

$$\tilde{\mathbf{Y}}^{(i)} = \left(\overline{\mathbf{M}} \cdot \mathbf{X}^{(i-1)} + \overline{\mathbf{P}} \cdot \mathbf{a}^{(i-1)} \right)$$

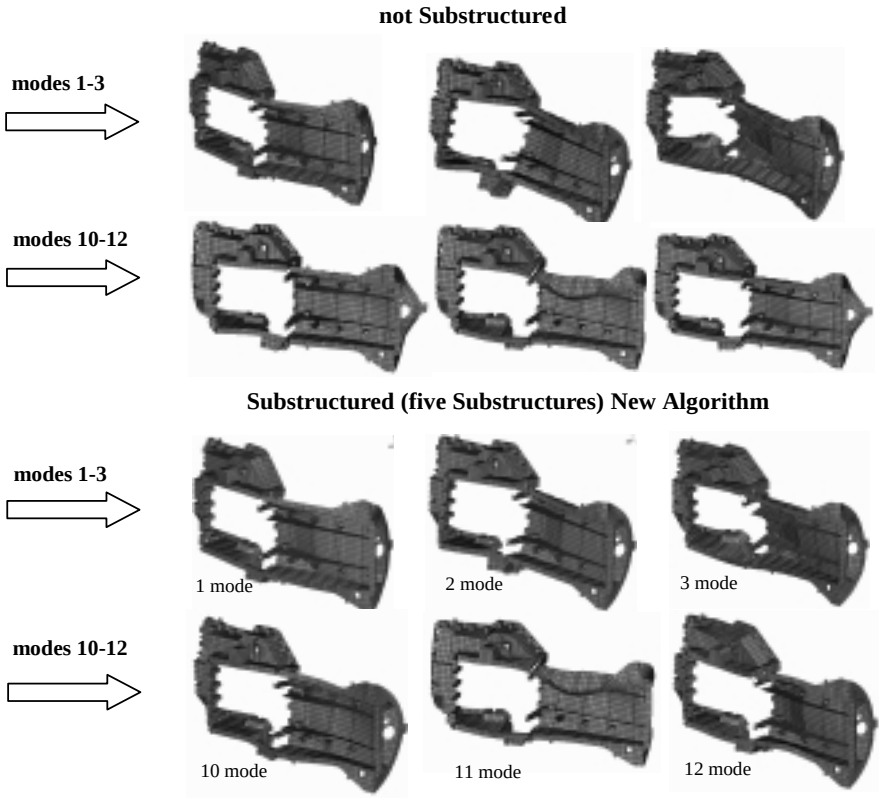

Fig. 4. Eigenmodes.

$$\tilde{\mathbf{a}}^{(i)} = \left(\overline{\mathbf{P}}^T \cdot \mathbf{X}^{(i-1)} + \overline{\boldsymbol{\theta}}^{-1} \cdot \mathbf{a}^{(i-1)}\right) \tag{22}$$

4) Solve linear equation by forward and back substitution and calculate subspace matrix $\tilde{\mathbf{K}}^{(i)}$ in between

$$\begin{aligned} &\overline{\mathbf{R}}^T \tilde{\tilde{\mathbf{X}}}^{(i)} = \tilde{\mathbf{Y}}^{(i)} \\ &\tilde{\mathbf{K}}^{(i)} := \tilde{\mathbf{X}}^{(i)^T} \overline{\mathbf{K}} \tilde{\mathbf{X}}^{(i)} = \tilde{\tilde{\mathbf{X}}}^{(i)^T} \tilde{\tilde{\mathbf{X}}}^{(i)} \\ &\tilde{\mathbf{K}}^{(i)} := \tilde{\mathbf{K}}^{(i)} + \tilde{\mathbf{a}}^{(i)^T} \tilde{\mathbf{a}}^{(i)} \\ &\overline{\mathbf{R}} \tilde{\mathbf{X}}^{(i)} := \tilde{\tilde{\mathbf{X}}}^{(i)} \end{aligned} \tag{23}$$

5) Solving subspace standard eigenvalue problem and calculation of matrix $\mathbf{L}^{(i)}$ for improving the eigenvectors in the subspace spanned by the eigenvector approximations $\tilde{\mathbf{X}}^{(i)}$ and $\tilde{\mathbf{a}}^{(i)}$,

$$\begin{aligned} &\tilde{\mathbf{K}}^{(i)} \mathbf{Q}^{(i)} = \mathbf{I} \mathbf{Q}^{(i)} \Gamma^{(i)} \\ &\text{with} \quad \Gamma^{(i)} = \text{diag}\left(\gamma_k^{(i)^2}\right) \\ &\mathbf{L}^{(i)} = \mathbf{Q}^{(i)} \text{diag}\left(\gamma_k^{(i)^{-1}}\right) \end{aligned} \tag{24}$$

$$\begin{aligned} &\mathbf{X}^{(i)} = \tilde{\mathbf{X}}^{(i)} \mathbf{L}^{(i)}, \quad \mathbf{a}^{(i)} = \tilde{\mathbf{a}}^{(i)} \mathbf{L}^{(i)} \\ &\mathbf{X}^{(i)} \to \overline{\Phi}, \quad \gamma_k^{(i)^{-1}} = \tilde{\lambda}_k \to \lambda_k \end{aligned}$$

6) Check for convergence of eigenvalues, if $\left(\tilde{\lambda}_k^{(i)} - \tilde{\lambda}_k^{(i-1)}\right) / \tilde{\lambda}_k^{(i-1)} \leqslant \varepsilon_E$ stop else go to 3)

Form the algorithm we get the following orthogonality relation

$$\begin{aligned} &\begin{bmatrix} \overline{\Phi} \\ \mathbf{a} \end{bmatrix}^T \begin{bmatrix} \overline{\mathbf{K}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \overline{\Phi} \\ \mathbf{a} \end{bmatrix} = \text{diag}(1) \\ &\begin{bmatrix} \overline{\Phi} \\ \mathbf{a} \end{bmatrix}^T \begin{bmatrix} \overline{\mathbf{M}} & \overline{\mathbf{P}} \\ \overline{\mathbf{P}}^T & \overline{\boldsymbol{\theta}}^{-1} \end{bmatrix} \begin{bmatrix} \overline{\Phi} \\ \mathbf{a} \end{bmatrix} = \Lambda^{-1} \\ &= \text{diag}(\lambda_k^{-1}) \end{aligned} \tag{25}$$

If we compare Eq. (25) with the orthogonality relation in Eq. (16), we observe that the orthogonality for holds true even though the Craig-Bampton transformation has been employed.

Table 2
Eigenfrequencies

No	not substructured [Hz]	substructured static condensation [Hz]	Substructured Craig-Bampton (new algorithm) [Hz]
1	140.6	137.6	137.4
2	215.1	224.1	223.2
3	293.2	292.5	292.0
4	359.2	367.3	353.6
5	459.3	448.4	441.3
6	478.0	487.6	460.9
7	565.7	571.5	537.3
8	576.6	590.4	553.7
9	650.1	715.2	624.5
0	739.1	825.1	715.9
11	820.4	870.5	802.1
12	841.8	978.8	835.0
13	911.6	1072.0	899.6
14	943.3	1156.0	919.0
15	990.1	1192.0	986.1

4. Examples

To exemplify the power of the new algorithm, which has been integrated into the COSAR©[1] FE code let us consider the following two examples. First for a cylinder-like structure, modelled with semiloof shell elements, and subdivided into 6 substructures (depicted in Fig. 1) the 15 first eigenfrequencies and corresponding eigenvectors have been calculated. This example follows the idea of pure domain decomposition for substructures 1 to 6. The first column in Table 1 contains the results obtained for the not substructured uncondensed model corresponding to the model in Fig. 1 that we will refer to as the "exact" results. The unreduced full structure has 8764 DOF and the matrices have a half bandwidth of 661. The reduced model assembled from the six superelements consists of 755 DOF with a half bandwidth of 675.

With the pure static condensation we cannot obtain all of the 15 modes as column 2 in Table 1 shows. The third column in Table 1 shows the results of the new algorithm with the aforementioned substructuring, but for each structure the first ten internal modes and eigenfrequencies have been included to form Craig-Bampton-superelements. The improvements are obvious.

In the next example, we will apply the new algorithm to a more real-world application the sump of a car engine. The FE-model has been verified by measurements (experimental modal analysis) and calculation conducted under free suspension conditions. The sump in our case has been supported at four nodal points as depicted in Fig. 3. The FE model of the sump consists of 3539 semiloof shell and solid elements and the material is aluminium. For the substructuring it has been subdivided in the sense of domain decomposition into five substructures, see Fig. 3. With the same mesh a model with no substructuring has been used for the modal analysis as reference and we will refer to the solutions gathered from this uncondensed model as the "exact" results. The unreduced full structure has 46825 DOF and the matrices have a half bandwidth of 2530. The reduced model assembled from the five superelements consist of 1079 DOF with a half bandwidth of 640.

In the substructured model no additional master nodes have been introduced to improve the model. That means we have master nodes in the adjacent (boundary) sections of the substructures only.

The results obtained and displayed in Table 2 show again that the algorithm proposed apparently improves the higher modes a good deal. Other experiments we carried out to place additional master points onto the substructures (10 up to 20 for each substructure), deploying well-known rules of thumb for the placement, did not lead per se to better results.

5. Conclusions

Although with the algorithm proposed and realised in a FE code it is not necessary to interfere in the system matrices in the FE code the Craig-Bampton method can be carried out in a very efficient way with a minimum of additional computational effort. The power of the new algorithm that integrates Craig-Bampton method

[1] Registered trade mark of FEMCOS mbH Germany.

and subspace iteration has been exemplified with the two examples shown. So far the new algorithm has been regarded for the free vibration problem the idea and framework could perhaps be applied to the buckling problem as well. In the latter case, it is very difficult to select additional DOF within a substructure, if static condensation is used for the superelement transformation. Usually, the implication is that the substructure/superelement technique leads to completely faulty results. Even for the lowest eigenvalue the buckling mode could exhibit small buckling wavelength. In view of this, the inclusion of internal modes appears as sensible idea improving the reduced model and makes the solution of the buckling problem more tractable. But so far we have not gathered experience in that area.

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