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A state space method for modal identification of mechanical systems from time domain responses

Xiaobo Liu
General Motors Corporation, Noise & Vibration Performance & Center, 3300 GM Road, Milford, MI 48380, USA
E-mail: xiaobo.liu@gm.com

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Abstract. A new state space method is presented for modal identification of a mechanical system from its time domain impulse or initial condition responses. A key step in this method is the identification of the characteristic polynomial coefficients of an adjoint system. Once these coefficients are determined, a canonical state space realization of the adjoint system and the system's modal parameters are formulated straightforwardly. This method is conceptually and mathematically simple and is easy to be implemented. Detailed mathematical treatments are demonstrated and numerical examples are provided to illustrate the use and effectiveness of the method.

1. Introduction

Modal identification is a process to obtain the modal parameters of a mechanical system from measured data. The past three decades have witnessed substantial progresses in modal identification methods. Overviews and comparisons of the methods can be found in [1,2]. The similarities in all the methods arise from a common theoretical basis. Basically each method starts with a system with its physics being governed by linear second order dynamic equations, it then seeks to fit the measurement data into a mathematical model, and finally derives the desired modal parameters from the identified mathematical model. The differences in the methods lie in the measurement databases being used, and the mathematical models being employed for data fitting. In general, measurement data can be frequency responses, impulse responses, forced responses, free decays, etc. Depending upon the nature of the measurement database to be used, the mathematical model can be nonparametric, parametric or state space.

In this paper, we present a new state space method for modal identification. The data employed for identification are the system's impulse responses or its responses to arbitrary initial conditions. Compared to the existing methods, the main advantage of the new method is in its conceptual and mathematical simplicities. In this method, the identification of a dynamic system is first translated into the identification of an adjoint system. The method then determines the order and the characteristic polynomial coefficients of the adjoint system. Once these coefficients are determined, the state space matrices and the modal parameters are formulated in a straightforward fashion. The new method only involves convenient matrix and algebraic operations and is easy to be implemented. A couple of examples are provided to illustrate its use and effectiveness.

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2. Dynamic equations

The governing dynamic equations for a linear finite dimensional mechanical system are a set of second order differential equations, which can be expressed in matrix form as

$$\bar{\mathbf{M}}\ddot{\mathbf{q}}(t)+\bar{\mathbf{C}}\dot{\mathbf{q}}(t)+\bar{\mathbf{K}}\mathbf{q}(t)=\mathbf{f}(t) \tag{1}$$

where the matrices $\bar{\mathbf{M}}$, $\bar{\mathbf{C}}$, $\bar{\mathbf{K}}$ are the mass, damping and stiffness matrices, respectively, $\mathbf{q}$ is a vector of generalized coordinates, and $\mathbf{f}$ is a vector of generalized forces. By using the Laplace transform, we can relate the generalized coordinates and the forces in the form

$$\left(\bar{\mathbf{M}}s^2+\bar{\mathbf{C}}s+\bar{\mathbf{K}}\right)\mathbf{q}(s)=\mathbf{f}(s) \tag{2}$$

The complex eigenvalues of the system are the roots of the determinant of the matrix $\bar{\mathbf{M}}s^2+\bar{\mathbf{C}}s+\bar{\mathbf{K}}$. For stable systems, the eigenvalues will have negative real parts. For each eigenvalue λ_j, the corresponding eigenvector Φ_j is obtained by solving the matrix equation

$$\left(\bar{\mathbf{M}}\lambda_j^2+\bar{\mathbf{C}}\lambda_j+\bar{\mathbf{K}}\right)\Phi_j=\mathbf{0} \tag{3}$$

The eigenvalues and eigenvectors are characteristics of the system. The eigenvectors are also known as modal vectors, and the eigenvalues will give the corresponding modal damping rates and damped natural frequencies as

$$\zeta_j=-Re(\lambda_j) \tag{4a}$$

$$f_j=\frac{Im(\lambda_j)}{2\pi} \tag{4b}$$

The governing dynamic Eq. (1) can be reformulated into state space form [6–8] by different means. One commonly used reformulation is to let

$$\mathbf{x}=\begin{Bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{Bmatrix} \tag{5a}$$

$$\mathbf{f}=\bar{\mathbf{B}}\mathbf{u} \tag{5b}$$

and

$$\mathbf{A}=\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\bar{\mathbf{M}}^{-1}\bar{\mathbf{K}} & -\bar{\mathbf{M}}^{-1}\bar{\mathbf{C}}\end{bmatrix} \tag{6a}$$

$$\mathbf{B}=\begin{bmatrix}\mathbf{0}\\ \bar{\mathbf{M}}^{-1}\bar{\mathbf{B}}\end{bmatrix} \tag{6b}$$

Then the state space dynamic equations are

$$\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u} \tag{7}$$

In addition, if the output or measurement vector is $\mathbf{y}=\Theta\mathbf{q}$, then by letting $\mathbf{C}=\begin{bmatrix}\Theta & \mathbf{0}\end{bmatrix}$, we obtain the output equations

$$\mathbf{y}=\mathbf{C}\mathbf{x} \tag{8}$$

Equations (7) and (8) constitute a state space model of the dynamic system. The triplet $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is called a state space realization. The transfer function matrix of the system is

$$\begin{aligned}\mathbf{H}(s) &= \mathbf{C}\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B} \\ &= \sum_{k=0}^{\infty}\mathbf{C}\mathbf{A}^{k}\mathbf{B}s^{-(k+1)}\end{aligned} \tag{9}$$

The coefficient matrices $\mathbf{C}\mathbf{A}^{k}\mathbf{B}$ in Eq. (9) are called the system's Markov parameters. In the case the realization is minimal, the eigenvalues of the system are equivalent to those of the matrix $\mathbf{A}$. If $\mathbf{A}$ has an eigen-decomposition

$$\mathbf{A} = \Psi\Lambda\Psi^{-1} \tag{10}$$

then $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ can be transformed into an equivalent realization $\left(\Lambda, \Psi^{-1}\mathbf{B}, \mathbf{C}\Psi\right)$ in modal space. The diagonal matrix Λ contains the eigenvalues of the system, $\mathbf{C}\Psi$ will be the modal vector matrix, and $\Psi^{-1}\mathbf{B}$ will be the modal participation factor matrix. Next we will present a method to obtain a minimal realization of a system from its time domain responses. We will first consider the case with impulse responses, and then extend the method to cover the case with initial condition responses.

3. Modal identification from impulse responses

We consider the single input case where the matrix $\mathbf{B}$ is replaced by a column vector $\mathbf{b}$. Let $\{\mathbf{h}(k), k = 0, 1, \cdots\}$ be the system's impulse response set which can be obtained as the inverse Fourier transforms of the frequency response functions. The impulse responses can be expressed as

$$\mathbf{h}(k) = \mathbf{C}e^{\mathbf{A}k\ddot{A}t}\mathbf{b} \tag{11}$$

where Δt is the sampling period. Let

$$\tilde{\mathbf{A}} = \exp(\mathbf{A}\Delta t) \tag{12}$$

then Eq. (11) is equivalent to

$$\mathbf{h}(k) = \mathbf{C}\tilde{\mathbf{A}}^{k}\mathbf{b} \tag{13}$$

The eigenvalues of $\tilde{\mathbf{A}}$ will be the exponentials of those of $\mathbf{A}$, and both matrices share the same set of eigenvectors. Let $\omega_{\max}$ be the maximum of the absolute values of the imaginary parts of all the eigenvalues of $\mathbf{A}$. If the condition $0 \leqslant \omega_{\max}\Delta t \leqslant \pi$ is satisfied, then there is a one-to-one correspondence between the eigenvalues of the two matrices $\tilde{\mathbf{A}}$ and $\mathbf{A}$. This fact is just a reiteration of Shannon's sampling theorem. In such a case, the eigenstructure of the system matrix $\mathbf{A}$ can be uniquely recovered from that of $\tilde{\mathbf{A}}$. For such a reason, we can focus on the identification of the adjoint system $\left(\tilde{\mathbf{A}}, \mathbf{B}, \mathbf{C}\right)$ and return to the original state space system as needed.

The first step in the identification process is to determine the order of the minimal realization. This can be done by examining the rank of the Hankel matrix [3,5,6]

$$\begin{aligned}\mathbf{H} &= \begin{bmatrix}\mathbf{C} \\ \mathbf{C}\tilde{\mathbf{A}} \\ \vdots \\ \mathbf{C}\tilde{\mathbf{A}}^{\alpha}\end{bmatrix}\begin{bmatrix}\mathbf{b} & \tilde{\mathbf{A}}\mathbf{b} & \cdots & \tilde{\mathbf{A}}^{\beta}\mathbf{b}\end{bmatrix} \\ &= \begin{bmatrix}\mathbf{h}(0) & \mathbf{h}(1) & \cdots & \mathbf{h}(\beta) \\ \mathbf{h}(1) & \cdots & \cdots & \mathbf{h}(1+\beta) \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{h}(\alpha) & \cdots & \cdots & \mathbf{h}(\alpha+\beta)\end{bmatrix}\end{aligned} \tag{14}$$

for large enough parameters α, β. In the absence of measurement noise, the rank of the Hankel matrix is equal to the order of the system. When measurement noises are presented in the data, we can perform singular value decomposition (SVD) of the matrix $\mathbf{H}$ and determine its rank by neglecting the singular values below a threshold level.

The minimal realization is not unique. The triplet $\left(\tilde{\mathbf{A}}, \mathbf{b}, \mathbf{C}\right)$ is equivalent to $\left(\mathbf{P}^{-1}\tilde{\mathbf{A}}\mathbf{P}, \mathbf{P}^{-1}\mathbf{b}, \mathbf{CP}\right)$ for any non-singular matrix $\mathbf{P}$. Suppose the system order is n, then we have

$$\text{rank}\left[\,\mathbf{b}\ \tilde{\mathbf{A}}\mathbf{b}\cdots\tilde{\mathbf{A}}^{n-1}\mathbf{b}\,\right] = n \tag{15}$$

If we assume the characteristic equation of $\tilde{\mathbf{A}}$ takes the form

$$\begin{aligned}\det(s\mathbf{I}-\tilde{\mathbf{A}}) &= s^n + a_{n-1}s^{n-1} + \cdots + a_0\\ &= 0\end{aligned} \tag{16}$$

then we can construct a non-singular transformation matrix

$$\mathbf{P} = \left[\,\tilde{\mathbf{A}}^{n-1}\mathbf{b}\ \cdots\ \tilde{\mathbf{A}}\mathbf{b}\ \ \mathbf{b}\,\right]\begin{bmatrix} 1 & 0 & \cdots & 0\\ a_{n-1} & 1 & \ddots & 0\\ \vdots & \ddots & \ddots & 0\\ a_1 & \cdots & a_{n-1} & 1\end{bmatrix} \tag{17}$$

and arrive at the state space realization in controllable canonical form as [8]

$$\tilde{\mathbf{A}}_1 = \mathbf{P}^{-1}\tilde{\mathbf{A}}\mathbf{P} = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1\\ -a_0 & -a_1 & \cdots & -a_{n-1}\end{bmatrix} \tag{18a}$$

$$\mathbf{b}_1 = \mathbf{P}^{-1}\mathbf{b} = \left[0\cdots 0\ 1\right]^T \tag{18b}$$

$$\mathbf{C}_1 = \mathbf{CP} = \left[\,a_1\mathbf{I}\ \cdots\ a_{n-1}\mathbf{I}\ \ \mathbf{I}\,\right]\begin{bmatrix} \mathbf{Cb} & 0 & \cdots & 0\\ \mathbf{C}\tilde{\mathbf{A}}\mathbf{b} & \mathbf{Cb} & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ \mathbf{C}\tilde{\mathbf{A}}^{n-1}\mathbf{b} & \cdots & \mathbf{C}\tilde{\mathbf{A}}\mathbf{b} & \mathbf{Cb}\end{bmatrix} \tag{18c}$$

By noticing Eq. (13), we can rewrite the matrix $\mathbf{C}_1$ as

$$\mathbf{C}_1 = \left[\,a_1\mathbf{I}\ \cdots\ a_{n-1}\mathbf{I}\ \ \mathbf{I}\,\right]\begin{bmatrix} \mathbf{h}(0) & 0 & \cdots & 0\\ \mathbf{h}(1) & \mathbf{h}(0) & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ \mathbf{h}(n-1) & \cdots & \mathbf{h}(1) & \mathbf{h}(0)\end{bmatrix} \tag{19}$$

It is obvious that a key step is to determine the characteristic polynomial coefficients $a_{n-1},\ \cdots,\ a_0$ as in Eq. (16). Once these coefficients are determined, a canonical system realization can be formulated as in Eqs (18) and (19). The determination of the coefficients is explained next.

We begin with the characteristic Eq. (16). By using the Caley-Hamilton theorem we have

$$\tilde{\mathbf{A}}^n + a_{n-1}\tilde{\mathbf{A}}^{n-1} + \cdots + a_0\mathbf{I} = 0 \tag{20}$$

Furthermore, through consecutively multiplying Eq. (20) by $\tilde{\mathbf{A}}$ we get

$$\begin{cases} \tilde{\mathbf{A}}^n + a_{n-1}\tilde{\mathbf{A}}^{n-1} + \cdots + a_0\mathbf{I} = \mathbf{0}\\ \tilde{\mathbf{A}}^{n+1} + a_{n-1}\tilde{\mathbf{A}}^{n} + \cdots + a_0\tilde{\mathbf{A}} = \mathbf{0}\\ \qquad\vdots\\ \tilde{\mathbf{A}}^{2n-1} + a_{n-1}\tilde{\mathbf{A}}^{2n-2} + \cdots + a_0\tilde{\mathbf{A}}^{n-1} = \mathbf{0}\\ \qquad\vdots \end{cases} \tag{21}$$

Pre-multiplying the above equations by $\mathbf{C}$ and post-multiplying by $\mathbf{b}$, and noticing Eq. (13), we obtain

$$\begin{cases} \mathbf{h}(n) + a_{n-1}\mathbf{h}(n-1) + \cdots + a_0\mathbf{h}(0) = \mathbf{0} \\ \mathbf{h}(n+1) + a_{n-1}\mathbf{h}(n) + \cdots + a_0\mathbf{h}(1) = \mathbf{0} \\ \vdots \\ \mathbf{h}(2n-1) + a_{n-1}\mathbf{h}(2n-2) + \cdots + a_0\mathbf{h}(n-1) = \mathbf{0} \\ \vdots \end{cases} \tag{22}$$

or in matrix form

$$\begin{bmatrix} \mathbf{h}(0) & \cdots & \mathbf{h}(n-1) \\ \mathbf{h}(1) & \cdots & \mathbf{h}(n) \\ \vdots & \ddots & \vdots \\ \mathbf{h}(n-1) & \cdots & \mathbf{h}(2n-2) \\ \vdots & \ddots & \vdots \end{bmatrix} \begin{Bmatrix} a_0 \\ \vdots \\ a_{n-1} \end{Bmatrix} = - \begin{Bmatrix} \mathbf{h}(n) \\ \mathbf{h}(n+1) \\ \vdots \\ \mathbf{h}(2n-1) \\ \vdots \end{Bmatrix} \tag{23}$$

The characteristic polynomial coefficients $a_{n-1},\ \cdots,\ a_0$ can then be determined by solving Eq. (23) using the least square method.

In summary, the modal identification method consists of three major steps:

1) Determine the system order by examining the rank of the Hankel matrix in Eq. (14), and obtain the characteristic polynomial coefficients of the adjoint system by solving Eq. (23) using the least square method.
2) Form the adjoint system $\left(\tilde{\mathbf{A}}_1,\ \mathbf{b}_1,\ \mathbf{C}_1\right)$ as in (18)-(19). Solve the eigenvalue problem of $\tilde{\mathbf{A}}_1$ to obtain the eigenvalues, $\tilde{\lambda}_j$, and the eigenvector matrix Ψ.
3) Determine the system eigenvalues as $\lambda_j = \ln(\tilde{\lambda}_j)/\Delta t$. Determine the modal damping rates and damped natural frequencies as in (4). Form the modal vector matrix as $\mathbf{C}_1\Psi$, and the modal participation factor matrix as $\Psi^{-1}\mathbf{b}_1$.

4. Modal identification from initial condition responses

The method presented in Section 3 can be extended for modal identification from initial condition responses, where only the system's responses to arbitrary initial conditions are used to identify the modal parameters [4]. The response data can be in terms of displacements, velocities or accelerations. Here we discuss the case with acceleration responses since they are the most commonly available from experimental measurements.

For the state space system $(\mathbf{A},\ \mathbf{B},\ \mathbf{C})$, the acceleration responses of the system due to an arbitrary initial state condition $\mathbf{x}_0$ is given by

$$\mathbf{a}(t) = \mathbf{C}\mathrm{e}^{\mathbf{A}t}\tilde{\mathbf{x}}_0 \tag{24}$$

where

$$\tilde{\mathbf{x}}_0 = \mathbf{A}^2\mathbf{x}_0 \tag{25}$$

Equation (24) suggests that the acceleration responses are equivalent to the impulse responses of the system $(\mathbf{A},\ \tilde{\mathbf{x}}_0,\ \mathbf{C})$, which will have the same eigenvalues and modal vectors as the original system. Therefore, we can use the method in Section 3 to obtain the modal parameters, with the impulse responses there being replaced by discrete samples of acceleration

$$\mathbf{a}(k) = \mathbf{C}e^{\mathbf{A}k\ddot{A}t}\tilde{\mathbf{x}}_0 \tag{26}$$

for $k = 0,\ 1,\ \cdots$, where Δt is the sampling period. The modes so obtained will be the modes excited by the initial state condition $\mathbf{x}_0$. No modal participation factors will be determined since there's no information on the excitations available in such a case.

$$m_1 = m_2 = m_3 = 0.1\,kg$$
$$k_1 = k_2 = k_3 = 30000\ N/m$$
$$c_1 = c_2 = c_3 = 3\ N \cdot s/m$$

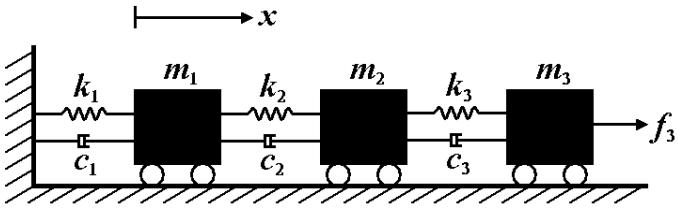

Fig. 1. A Mass-Spring-Damper System.

5. Numerical examples

We now give a couple of examples to illustrate the use of the proposed method. The first example will validate the method for modal identification from impulse responses, the second one will validate the method for modal identification from initial condition responses.

Example 1

A mass-spring-damper system is shown in Fig. 1. The following physical parameters are assumed

$$m_1 = m_2 = m_3 = 0.1\ kg$$

$$k_1 = k_2 = k_3 = 30000\ N/m$$

$$c_1 = c_2 = c_3 = 3\ N \cdot s/m$$

The system has eigenvalues, modal vectors, and modal participation factors as below

$$\lambda_{1,2} = -2.9709 \pm j243.7414, \quad \phi_{1,2} = \begin{Bmatrix} 1 \\ 1.8019 \\ 2.2470 \end{Bmatrix}, \quad l_{1,2} = \mp j0.005 \tag{27a}$$

$$\lambda_{3,4} = -23.3244 \pm j682.6005, \quad \phi_{3,4} = \begin{Bmatrix} 1 \\ 0.4450 \\ -0.8019 \end{Bmatrix}, \quad l_{3,4} = \pm j0.0032 \tag{27b}$$

$$\lambda_{5,6} = -48.7047 \pm j985.7595, \quad \phi_{5,6} = \begin{Bmatrix} 1 \\ -1.2470 \\ 0.5550 \end{Bmatrix}, \quad l_{5,6} = \mp j0.001 \tag{27c}$$

Suppose the system's impulse responses from excitation at m_3 have been obtained. We now use the method developed in Section 3 to identify the modal parameters from the impulse responses.

The mass, damping and stiffness matrices are

$$\bar{\mathbf{M}} = \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix} \tag{28}$$

$$\bar{\mathbf{C}} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \tag{29}$$

$$\bar{\mathbf{K}} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \tag{30}$$

We define the state vector to be

$$\mathbf{x} = \begin{bmatrix} x_1 \; x_2 \; x_3 \; \dot{x}_1 \; \dot{x}_2 \; \dot{x}_3 \end{bmatrix}^T \tag{31}$$

and the measurement vector

$$\mathbf{y} = \begin{bmatrix} x_1 \; x_2 \; x_3 \end{bmatrix}^T \tag{32}$$

Then the state space matrices are

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} \\ -\bar{\mathbf{M}}^{-1}\bar{\mathbf{K}} & -\bar{\mathbf{M}}^{-1}\bar{\mathbf{C}} \end{bmatrix} \tag{33}$$

$$\mathbf{b} = \begin{bmatrix} \mathbf{0} \\ \bar{\mathbf{M}}^{-1}\bar{\mathbf{b}} \end{bmatrix} \quad \text{where } \bar{\mathbf{b}} = \begin{bmatrix} 0 \; 0 \; 1 \end{bmatrix}^T \tag{34}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{I}_{3\times3} \; \mathbf{0}_{3\times3} \end{bmatrix} \tag{35}$$

For simulation purpose, we use Eq. (11) to regenerate the impulse responses. The sampling period is chosen to be $\Delta t = 0.001$ second. By checking the rank of the Hankel matrix in Eq. (14), we find the system order to be 6. Solving Eq. (23), we obtain the characteristic polynomial coefficients of the adjoint system as

$$\begin{aligned} a_0 &= 0.8607, \quad a_1 = -4.0408, \quad a_2 = 8.9062 \\ a_3 &= -11.6209, \quad a_4 = 9.4200, \quad a_5 = -4.5032 \end{aligned} \tag{36}$$

Use Eqs (18)–(19), we can form a minimal realization of the adjoint system

$$\tilde{\mathbf{A}}_1 = \begin{bmatrix} \mathbf{0}_{5\times1} & \mathbf{I}_{5\times5} \\ -0.8607 \; 4.0408 \; -8.9062 \; 11.6209 \; -9.4200 \; 4.5032 \end{bmatrix} \tag{37}$$

$$\mathbf{b}_1 = \begin{bmatrix} 0 \; 0 \; 0 \; 0 \; 0 \; 1 \end{bmatrix}^T \tag{38}$$

$$\mathbf{C}_1 = \begin{bmatrix} 1.3e-6 & 0.0001 & 0.0004 & 0.0002 & 1.5e-5 & 0 \\ 0.0003 & 0.0012 & -0.0015 & 0.0009 & 0.0006 & 0 \\ 0.0083 & -0.0245 & 0.0352 & -0.0262 & 0.0094 & 0 \end{bmatrix} \tag{39}$$

The eigenvalues of $\tilde{\mathbf{A}}_1$ are found to be

$$\tilde{\lambda}_{1,2} = 0.9676 \pm j0.2406 \tag{40a}$$

$$\tilde{\lambda}_{3,4} = 0.7580 \pm j0.6163 \tag{40b}$$

$$\tilde{\lambda}_{5,6} = 0.5260 \pm j0.7941 \tag{40c}$$

and the eigenvalues of the original system are determined as

$$\lambda_{1,2} = \ln(\tilde{\lambda}_{1,2}) \Big/ \Delta t = -\,2.9709 \pm j243.7414 \tag{41a}$$

$$\lambda_{3,4} = \ln(\tilde{\lambda}_{3,4}) \Big/ \Delta t = -\,23.3244 \pm j682.6005 \tag{41b}$$

$$\lambda_{5,6} = \ln\left(\tilde{\lambda}_{5,6}\right) \Big/ \Delta t = -48.7047 \pm j985.7595 \tag{41c}$$

The modal vector and modal participation factor matrices are determined as $\mathbf{C}_1\Psi$ and $\Psi^{-1}\mathbf{b}_1$, with Ψ being the eigenvector matrix of $\tilde{\mathbf{A}}_1$. After normalization with respect to the first elements of the modal vectors, the modal vectors are

$$\phi_{1,2} = \begin{Bmatrix} 1 \\ 1.8019 \\ 2.2470 \end{Bmatrix} \tag{42a}$$

$$\phi_{3,4} = \begin{Bmatrix} 1 \\ 0.4450 \\ -0.8019 \end{Bmatrix} \tag{42b}$$

$$\phi_{5,6} = \begin{Bmatrix} 1 \\ -1.2470 \\ 0.5550 \end{Bmatrix} \tag{42c}$$

and the corresponding modal participation factors are

$$l_{1,2} = \mp j0.005 \tag{43a}$$

$$l_{3,4} = \pm j0.0032 \tag{43b}$$

$$l_{5,6} = \mp j0.001 \tag{43c}$$

Comparing these with Eq. (27), we see that the proposed method precisely identifies the modal parameters of the system.

Example 2

We consider the same system as in Fig. 1. We now use the method in Section 4 to identify the system's modal parameters from its acceleration responses to initial conditions.

Suppose we have available the system's acceleration responses to arbitrary initial state conditions

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_0 = \begin{Bmatrix} 0.0008 \\ 0.0005 \\ 0.0002 \end{Bmatrix} \tag{44a}$$

and

$$\left.\frac{d}{dt}\right|_0 \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 6.7214 \\ 8.3812 \\ 0.1964 \end{Bmatrix} \tag{44b}$$

The system order is found to be 6 by examining the rank of the matrix

$$\mathbf{H} = \begin{bmatrix} \mathbf{a}(0) & \mathbf{a}(1) & \cdots & \mathbf{a}(\beta) \\ \mathbf{a}(1) & \cdots & \cdots & \mathbf{a}(1+\beta) \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{a}(\alpha) & \cdots & \cdots & \mathbf{a}(\alpha+\beta) \end{bmatrix} \tag{45}$$

for large enough parameters α, β. By solving the equations

$$\begin{bmatrix} \mathbf{a}(0) & \cdots & \mathbf{a}(5) \\ \mathbf{a}(1) & \cdots & \mathbf{a}(6) \\ \vdots & \ddots & \vdots \\ \mathbf{a}(5) & \cdots & \mathbf{a}(10) \\ \vdots & \ddots & \vdots \end{bmatrix} \begin{Bmatrix} a_0 \\ \vdots \\ a_5 \end{Bmatrix} = - \begin{Bmatrix} \mathbf{a}(6) \\ \mathbf{a}(7) \\ \vdots \\ \mathbf{a}(11) \\ \vdots \end{Bmatrix} \tag{46}$$

using the least square method, we obtain the adjoint system's characteristic polynomial coefficients

$$\begin{aligned} &a_0 = 0.8607, \quad a_1 = -4.0408, \quad a_2 = 8.9062 \\ &a_3 = -11.6209, \quad a_4 = 9.4200, \quad a_5 = -4.5032 \end{aligned} \tag{47}$$

Form state matrices $\tilde{\mathbf{A}}_1$ as in (18), and $\mathbf{C}_1$ as in Eq. (19) except the impulse responses there being replaced by acceleration responses, we obtain

$$\tilde{\mathbf{A}}_1 = \begin{bmatrix} \mathbf{0}_{5\times 1} & & & \mathbf{I}_{5\times 5} & & \\ -0.8607 & 4.0408 & -8.9062 & 11.6209 & -9.4200 & 4.5032 \end{bmatrix} \tag{48}$$

$$\mathbf{C}_1 = \begin{bmatrix} -956.96 & 2107.07 & -2092.95 & 974.38 & 299.22 & -481.85 \\ -2129.12 & 6780.50 & -9308.23 & 6199.01 & -1429.82 & -295.34 \\ 1727.31 & -5359.77 & 7050.71 & -4600.74 & 841.78 & 335.54 \end{bmatrix} \tag{49}$$

The eigenvalues of the matrix $\tilde{\mathbf{A}}_1$ are

$$\tilde{\lambda}_{1,2} = 0.9676 \pm j0.2406 \tag{50a}$$

$$\tilde{\lambda}_{3,4} = 0.7580 \pm j0.6163 \tag{50b}$$

$$\tilde{\lambda}_{5,6} = 0.5260 \pm j0.7941 \tag{50c}$$

and the eigenvalues of the original system are obtained as

$$\lambda_{1,2} = \ln(\tilde{\lambda}_{1,2}) \Big/ \Delta t = -\,2.9709 \pm j243.7414 \tag{51a}$$

$$\lambda_{3,4} = \ln(\tilde{\lambda}_{3,4}) \Big/ \Delta t = -\,23.3244 \pm j682.6005 \tag{51b}$$

$$\lambda_{5,6} = \ln\left(\tilde{\lambda}_{5,6}\right) \Big/ \Delta t = -48.7047 \pm j985.7595 \tag{51c}$$

The modal vector matrix is $\mathbf{C}_1\Psi$, where Ψ is the eigenvector matrix of $\tilde{\mathbf{A}}_1$. After normalization with respect to the first elements of the modal vectors, the modal vectors are

$$\phi_{1,2} = \begin{Bmatrix} 1 \\ 1.8019 \\ 2.2470 \end{Bmatrix} \tag{52a}$$

$$\phi_{3,4} = \begin{Bmatrix} 1 \\ 0.4450 \\ -0.8019 \end{Bmatrix} \tag{52b}$$

$$\phi_{5,6} = \begin{Bmatrix} 1 \\ -1.2470 \\ 0.5550 \end{Bmatrix} \tag{52c}$$

Once again, the system's modal parameters are accurately identified.

6. Conclusions

We have presented a state space method for modal identification of a mechanical system. This method employs the system's time domain impulse responses or initial condition responses for identification. A key step of this method is the identification of the characteristic polynomial coefficients of an adjoint system. Once these coefficients are determined, the modal parameters can be formulated straightforwardly. This method is conceptually and mathematically simple. We have illustrated through examples the use and effectiveness of the method.

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