Dynamic response of shear-flexible cylindrical isotropic shells with clamped edges

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Abstract. It is fundamental to obtain the natural frequencies and the corresponding mode shapes for cylindrical shells in order to determine their response to different dynamic loading. In this paper an analytical investigation to the free vibration response of moderately-thick shear flexible isotropic cylindrical shells with all edges clamped is presented. The Sander's kinematic relations for moderately thick cylindrical shell panels are utilized to develop the governing partial differential equations in conjunction with the boundary conditions. A recently developed generalized Navier's approach, based on a boundary continuous double Fourier series expansion, is used as a solution methodology. A parametric study is presented with respect to various thicknesses, length and radius of curvature of the shell panel. The convergence of the solution method is established numerically for various parametric properties. The present results are compared with the results obtained from finite element method using a four-node isoparametric shell element. The results thus presented should serve as bench-mark solutions for future comparisons with numerical and approximate methods for calculation of free vibration parameters of moderately-thick isotropic cylindrical shells.

Keywords: Vibration analysis, cylindrical isotropic shell, frequency response, shear-flexible

1. Introduction

Shells have been recognized and used in construction of monumental and landmark structures since the dawn of civilization. Various shell forms such as spherical dome shells, cylindrical shells and other curved surfaces are routinely used in civil, mechanical and marine structures. Shells or curved surfaces have become more popular lately in modern advanced industries, such as aerospace, aircraft, hydrospace, automobile, oil and LNG tanks, pipelines, tunnels and other land based applications. These industries are continuously experiencing development of new materials that are regularly replacing the conventional materials. The advent of these materials is due to their unique and favorable properties such as high strength-to-weight ratio, high stiffness-to-weight ratio, fatigue strength and high performance in maintaining structural integrity at extreme loads and temperatures. These new materials have low shear rigidities in comparison to the conventional ones. The low rigidity response coupled with the curvature effects in shell formulation brings complications in developing analytical solutions.

Over the years, static and dynamic equilibrium equations that govern the behavior of cylindrical shells have been developed by several researchers including Love [1], Donnell [2], Sanders [3], Timoshenko [4] and Flügge [5]. As summarized and investigated in details by Leissa [6], there are several well-known shell theories, with various

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assumptions, that include Love – Timoshenko, Reissner – Naghdi – Berry, Sanders, Goldenveizer – Novozhilov, Flugge – Lur’ye-Byrne, Vlasov and Donnell – Mushtari, to name a few. Updated review and bibliography of recently developed shell theories and solution methodologies is given by Noor [7], Liew [8], Qatu [9] and Soedel [10]. Many approximate and exact closed-form solution methods for governing dynamic equilibrium equations have been proposed in the literature to study and predict the free vibration characteristics – frequencies and mode shapes – of cylindrical shells. As the thickness of the shell increases, first-order shear deformations as well as second-order shear deformations have to be taken into consideration. In addition, there is coupling between longitudinal, radial and circumferential displacements in the dynamic equilibrium equations of cylindrical shells that need to be considered. It is clear that closed-form solution for shells, with all these conditions taken into consideration, is a formidable task. Therefore, several assumptions have been introduced in the literature to simplify the formulation in order to come up with closed-form solutions. Similar assumptions were also made in approximate methods such as the finite element method and large number of shell finite elements was developed by several researchers. Recent advances on development of shell finite elements are surveyed by Yang et al. [11].

Wang and Lia [12] introduced a wave approach based on Love’s equations to predict the natural frequencies of finite length circular cylindrical shells with different boundary conditions. Callahan and Baruh [13] introduced a systematic procedure for obtaining a closed-form eigen solution for the equations of motion of thin circular cylindrical shells and the results they obtained are very comparable to that of experimental results. Xiang et al. [14] provided exact solution for vibration of thin circular cylindrical shells with intermediate ring support based on thin elastic shell theory. In their proposed method, the state-space technique was employed to derive the homogenous differential equations system for a shell segment and a domain decomposition approach was developed to cater for the continuity requirements between shell segments. Stanley and Ganesan [15] presented a semi-analytical finite element solution for vibration of stiffened cylindrical short and long shells with clamped-clamped boundary condition. Chen et al. [16] and Chen and Ding [17] studied the coupled free vibration of transversely isotropic cylindrical shells based on three dimensional elasticity. Bessel functions solution, which includes complex arguments, are used for the case of complex eigenvalues. Ng et al. [18] used an extension of Donnell’s shell theory with first-order shear deformation to investigate the dynamic stability of simply-supported, isotropic cylindrical panels under combined static and dynamic axial forces. A system of equations were generated via normal-mode expansion and the response is obtained. Kurpa et al. [19] proposed an algorithm based on the R-function theory and variational Ritz method for the free vibration of arbitrary plan-form shallow shell with different types of curvatures. Reasonable agreement was observed between the results obtained and those available in the literature. Jansen [20] and Ambili [21] tackled the nonlinear vibrations of cylindrical shells.

The shell formulation considered in this paper is based on Sander’s kinematics relations, where the effect of shear deformations and rotational moment of inertia are accounted for. The inclusion of shear deformation into the shell formulation accurately predicts the shell behavior for a moderately thick situation as presented by Bert and Kumar [22] and Reddy [23]. Vibrations of thick cylindrical shells are also addressed by Vanderpool and Bert [24] and Liew, Lim and Kitipornchai [25]. A recently developed analytical solution by Chaudhuri and Kabir [26] and Kabir and Chaudhuri [27] based on boundary continuous double Fourier series expansion is utilized. Therefore, the main objective of the present work is to obtain an exact solution for free vibration response of a moderately-thick cylindrical shell panel with all edges clamped using conventional isotropic materials with low shear rigidities. Analytical solutions of such shells are absent in the literature.

Results from three finite element programs are compared with those of the analytical solution results. It is evident that the results of this analytical solution can serve as bench-mark solutions for future comparisons with numerical and approximate methods for calculation of free vibration parameters of moderately-thick isotropic cylindrical shells.

2. Dynamic equilibrium equations

A typical cylindrical panel of thickness h and radius R and length b is shown in Fig. 1. Reference axes (x_1, x_2, x_3) are placed at middle surface ($h/2$) of the shell thickness. Axes x_1, x_2 and x_3 represent an orthogonal curvilinear coordinate system along circumferential (tangential), longitudinal (axial) and radial directions of the

Fig. 1. A cylindrical shell panel showing dimensions, coordinate system and displacements.

shell, respectively. Axis x_3 is perpendicular to the surface of the panel, a represent the circumferential length along x_1 axis and b represent the longitudinal length along x_2 axis.

The displacement fields expressed in terms of reference surface (middle surface) displacements are:

$$u_1 = \left(1 + \frac{x_3}{R}\right) \bar{u}_1 + x_3 \phi_1 \quad (1a)$$

$$u_2 = \bar{u}_2 + x_3 \phi_2 \quad (1b)$$

$$u_3 = \bar{u}_3 \quad (1c)$$

where $u_i (i = 1, 2, 3)$ are the displacements of a point $x_i (i = 1, 2, 3)$, $\bar{u}_i$ denotes the displacements of the corresponding point at the reference surface, and ϕ_1 & ϕ_2 are the rotations of the normal to the reference surface about x_2 and x_1 coordinate axes, respectively.

The strain-displacement relations (kinematic relations) based on Sander's [3] shell theory are:

$$\varepsilon_1 = \varepsilon_1^0 + x_3 \kappa_1 \quad (2a)$$

$$\varepsilon_2 = \varepsilon_2^0 + x_3 \kappa_2 \quad (2b)$$

$$\varepsilon_4 = \varepsilon_4^0 \quad (2c)$$

$$\varepsilon_5 = \varepsilon_5^0 \quad (2d)$$

$$\varepsilon_6 = \varepsilon_6^0 + x_3 \kappa_6 \quad (2e)$$

where ε_i and ε_i^0 for $i = 1, 2, 6$ at coordinates (x_1, x_2, x_3) and $(x_1, x_2, 0)$ represent the surface parallel strain, the normal strain and the shearing strain components, respectively. While ε_4 and ε_5 are the transverse shear strain components along the $x_2 - x_3$ and $x_1 - x_3$ planes, respectively; and κ_i for $i = 1, 2, 6$ represent the changes in the curvature and the twist at reference surface. The parameters ε_i^0 and κ_i are defined as follows:

$$\varepsilon_1^0 = \frac{\partial u_1}{\partial x_1} + \frac{u_3}{R} \quad (3a)$$

$$\varepsilon_2^0 = \frac{\partial u_2}{\partial x_2} \quad (3b)$$

$$\varepsilon_4^0 = \frac{\partial u_3}{\partial x_2} + \phi_2 \quad (3c)$$

$$\varepsilon_5^0 = \frac{\partial u_3}{\partial x_1} + \phi_1 - \frac{u_1}{R} \quad (3d)$$

$$\varepsilon_6^0 = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \quad (3e)$$

$$\kappa_1 = \frac{\partial \phi_1}{\partial x_1} \quad (3f)$$

$$\kappa_2 = \frac{\partial \phi_2}{\partial x_2} \quad (3g)$$

$$\kappa_6 = \frac{\partial \phi_2}{\partial x_1} + \frac{\partial \phi_1}{\partial x_2} - C_o \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \quad (3h)$$

where C_o is defined as $C_o = \frac{1}{2R}$

The dynamic equilibrium equations for a cylindrical panel are:

$$\frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} + C_o \frac{\partial M_6}{\partial x_2} + \frac{Q_1}{R} = C_1 \quad (4a)$$

$$\frac{\partial N_6}{\partial x_1} + \frac{\partial N_2}{\partial x_2} - C_o \frac{\partial M_6}{\partial x_1} = C_2 \quad (4b)$$

$$\frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} - \frac{N_1}{R} = C_3 \quad (4c)$$

$$\frac{\partial M_1}{\partial x_1} + \frac{\partial M_6}{\partial x_2} - Q_1 = C_4 \quad (4d)$$

$$\frac{\partial M_6}{\partial x_1} + \frac{\partial M_2}{\partial x_2} - Q_2 = C_5 \quad (4e)$$

Where N_i ($i = 1, 2, 6$) are the in-plane (axial) stress resultants, M_i ($i = 1, 2, 6$) are the bending stress resultants, and Q_i ($i = 1, 2$) are the transverse shear stress resultants. C_i ($i = 1, 2, 3, 4, 5$) are defined as follows:

$$C_1 = \left(P_1 + \frac{2P_2}{R} \right) \frac{\partial^2 u_1}{\partial t^2} + \left(P_2 + \frac{P_3}{R} \right) \frac{\partial^2 \phi_1}{\partial t^2} \quad (5a)$$

$$C_2 = P_1 \frac{\partial^2 u_2}{\partial t^2} + P_2 \frac{\partial^2 \phi_2}{\partial t^2} \quad (5b)$$

$$C_3 = P_1 \frac{\partial^2 u_3}{\partial t^2} \quad (5c)$$

$$C_4 = \left(P_2 + \frac{P_3}{R} \right) \frac{\partial^2 u_1}{\partial t^2} + P_3 \frac{\partial^2 \phi_1}{\partial t^2} \quad (5d)$$

$$C_5 = P_2 \frac{\partial^2 u_2}{\partial t^2} + P_3 \frac{\partial^2 \phi_2}{\partial t^2} \quad (5e)$$

where

$$P_1 = \int_0^h \rho dx_3 \quad (6a)$$

$$P_2 = \int_0^h \rho x_3 dx_3 \quad (6b)$$

$$P_3 = \int_0^h \rho x_3^2 dx_3 \quad (6c)$$

where ρ is the density of the material.

The in-plane stress, bending stress and shear stress resultants can be expressed in terms of the displacements and their derivatives as follows:

$$N_1 = A \left(\frac{\partial u_1}{\partial x_1} + \frac{u_3}{R} \right) + \nu A \left(\frac{\partial u_2}{\partial x_2} \right) \quad (7a)$$

$$N_2 = \nu A \left(\frac{\partial u_1}{\partial x_1} + \frac{u_3}{R} \right) + A \left(\frac{\partial u_2}{\partial x_2} \right) \quad (7b)$$

$$N_6 = \frac{1+\nu}{2} A \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \quad (7c)$$

$$M_1 = D \frac{\partial \phi_1}{\partial x_1} + \nu D \frac{\partial \phi_2}{\partial x_2} \quad (7d)$$

$$M_2 = \nu D \frac{\partial \phi_1}{\partial x_1} + D \frac{\partial \phi_2}{\partial x_2} \quad (7e)$$

$$M_6 = \frac{Gh^3}{12} \left(\frac{\partial \phi_2}{\partial x_1} + \frac{\partial \phi_1}{\partial x_2} - C_o \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \right) \quad (7f)$$

$$Q_1 = k_1^2 \frac{Gh}{2} \left(\frac{\partial u_3}{\partial x_1} + \phi_1 - \frac{u_1}{R} \right) \quad (7g)$$

$$Q_2 = k_1^2 \frac{Gh}{2} \left(\frac{\partial u_3}{\partial x_2} + \phi_2 \right) \quad (7h)$$

where ν is the Poisson's ratio, k_1^2 is the shear correction factor (usually taken as 5/6), E is Young's modulus, and A , D and G are extensional, bending and shear rigidities, respectively. A , D and G are expressed in the following form:

$$A = \frac{Eh}{1+\nu^2}; D = \frac{Eh^3}{12(1-\nu^2)} \text{ and } G = \frac{E}{2(1+\nu)}$$

The substitution of Eqs (2a–e), (3a–h) and (7a–h) into Eqs (4a–e) yields the following five highly coupled second-order partial differential equations with constant coefficients:

$$\begin{aligned} & \left(\frac{1+\nu}{2} A + C_o^2 \frac{Gh^3}{12} \right) \frac{\partial^2 u_1}{\partial x_2^2} + \left(\nu A + \frac{1+\nu}{2} A - C_o^2 \frac{Gh^3}{12} \right) \frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \left(\frac{A}{R_1} + k_1^2 \frac{Gh}{2R} \right) \frac{\partial u_3}{\partial x_1} \\ & + k_1^2 \frac{Gh}{2R} \phi_1 + C_o \frac{Gh^3}{12} \left(\frac{\partial^2 \phi_1}{\partial x_2^2} + \frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} \right) + A \frac{\partial^2 u_1}{\partial x_1^2} - k_1^2 \frac{Gh}{2R^2} u_1 = C_1 \end{aligned} \quad (8a)$$

$$\begin{aligned} & \left(\frac{1+\nu}{2}A - C_o \frac{Gh^3}{12} + \nu A \right) \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \left(\frac{1+\nu}{2}A + C_o \frac{Gh^3}{12} \right) \frac{\partial u_2}{\partial x_1} + A \frac{\partial^2 u_2}{\partial x_2^2} \\ & + \left(\frac{\nu A}{R} \right) \frac{\partial u_3}{\partial x_2} - C_o \frac{Gh^3}{12} \left(\frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} + \frac{\partial^2 \phi_1}{\partial x_1^2} \right) = C_2 \end{aligned} \quad (8b)$$

$$\begin{aligned} & \left(-k_1^2 \frac{Gh}{2R} - \frac{\nu A}{R} - \frac{A}{R} \right) \frac{\partial u_1}{\partial x_1} - \frac{\nu A}{R_1} \frac{\partial u_2}{\partial x_2} - \left(\frac{A}{R^2} \right) u_3 + \left(k_1^2 \frac{Gh}{2} \right) \\ & \left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial \phi_1}{\partial x_1} + \frac{\partial \phi_2}{\partial x_2} \right) = C_3 \end{aligned} \quad (8c)$$

$$\begin{aligned} & k_1^2 \frac{Gh}{2R} u_1 + C_o \frac{Gh^3}{12} \left(\frac{\partial^2 u_1}{\partial x_2^2} - \frac{\partial^2 u_2}{\partial x_1 \partial x_2} \right) - k_1^2 \frac{Gh}{2} \left(\frac{\partial u_3}{\partial x_1} + \phi_1 \right) + D \frac{\partial^2 \phi_1}{\partial x_1^2} + \frac{Gh^3}{12} \frac{\partial^2 \phi_1}{\partial x_2^2} \\ & + \left(\nu D + \frac{Gh^3}{12} \right) \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} = C_4 \end{aligned} \quad (8d)$$

$$C_o \frac{Gh^3}{12} \left(\frac{\partial^2 u_1}{\partial x_1 \partial x_2} - \frac{\partial^2 u_2}{\partial x_1^2} \right) - k_1^2 \frac{Gh}{2} \left(\frac{\partial u_3}{\partial x_2} + \phi_2 \right) + \left(\frac{Gh^3}{12} + \nu D \right) \frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} + D \frac{\partial^2 \phi_2}{\partial x_2^2} = C_5 \quad (8e)$$

The boundary conditions all-edges clamped panel are:

$$u_1 = u_2 = u_3 = \phi_1 = \phi_2 = 0 \text{ at all edges} \quad (9)$$

3. Solution method

The geometric boundary conditions Eq. (9) for a cylindrical panel are exactly satisfied by the following set of displacement functions as shown in Kabir [28], Chaudhuri and Kabir [26,29,30]:

$$u_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn} \sin(\alpha_m x_1) \sin(\beta_n x_2) e^{i\omega t} \quad (10a)$$

$$u_2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn} \sin(\alpha_m x_1) \sin(\beta_n x_2) e^{i\omega t} \quad (10b)$$

$$u_3 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin(\alpha_m x_1) \sin(\beta_n x_2) e^{i\omega t} \quad (10c)$$

$$\phi_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} X_{mn} \sin(\alpha_m x_1) \sin(\beta_n x_2) e^{i\omega t} \quad (10d)$$

$$\phi_2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn} \sin(\alpha_m x_1) \sin(\beta_n x_2) e^{i\omega t} \quad (10e)$$

where $\alpha_m = m\pi/a$ and $\beta_n = n\pi/b$.

The substitution of Eqs (10a–e) and their derivatives into Eqs (8a–e) will yield $15 \times m \times n$ algebraic equations with $5 \times m \times n$ unknowns, thus failing to provide a unique solution to this physical problem. Therefore, since the displacement functions and their derivatives before their substitution into the governing equations satisfy the boundary conditions without invoking any kind of discontinuities, the following series expansion of $\cos(\alpha_m x_1) \cos(\beta_n x_2)$, $\sin(\alpha_m x_1) \cos(\beta_n x_2)$ and $\cos(\alpha_m x_1) \sin(\beta_n x_2)$ are suggested for the displacement derivatives as shown in

Fig. 2. Normalized natural frequencies for isotropic cylindrical panel with $b/a = 1$.

Green [31], Kabir and Chaudhuri [27], Kabir [28] and Chaudhuri and Kabir [26,29,30].

$$\cos(\alpha_m x_1) \sin(\beta_n x_2) = \sum_{\bar{m}}^{\infty} f_{\bar{m}m} \sin(\gamma_{\bar{m}} x_1) \sin(\beta_n x_2) \quad (11a)$$

$$\sin(\alpha_m x_1) \cos(\beta_n x_2) = \sum_{\bar{n}}^{\infty} g_{\bar{n}n} \sin(\alpha_m x_1) \sin(\psi_{\bar{n}} x_2) \quad (11b)$$

$$\cos(\alpha_m x_1) \cos(\beta_n x_2) = \sum_{\bar{n}}^{\infty} \sum_{\bar{m}}^{\infty} f_{\bar{m}m} g_{\bar{n}n} \sin(\gamma_{\bar{m}} x_1) \sin(\psi_{\bar{n}} x_2) \quad (11c)$$

where:

$$f_{\bar{m}m} = \frac{4}{m\pi \left(1 - \frac{\bar{m}^2}{m^2}\right)}; \frac{\bar{m}}{m} \neq 1 \quad g_{\bar{n}n} = \frac{4}{n\pi \left(1 - \frac{\bar{n}^2}{n^2}\right)}; \frac{\bar{n}}{n} \neq 1; \quad \gamma_{\bar{m}} = \frac{\pi \bar{m}}{a}; \psi_{\bar{n}} = \frac{\pi \bar{n}}{b} \quad (12)$$

Introducing the above series expansions Eqs (11a–c) and (12) into Eqs (10a–e) will provide a unique solution to the all-edge clamped boundary condition.

Differentiating Eqs (10a–e) with respect to time and substituting in Eqs (5a–e) gives:

$$C_1 = - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(\alpha_m x_1) \sin(\beta_n x_2) e^{i\omega t} \left[\left(P_1 + \frac{2P_2}{R_1} \right) U_{mn} + \left(P_2 + \frac{P_3}{R_1} \right) X_{mn} \right] \omega^2 \quad (13a)$$

$$C_2 = - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(\alpha_m x_1) \sin(\beta_n x_2) e^{i\omega t} [(P_1) V_{mn} + (P_2) Y_{mn}] \omega^2 \quad (13b)$$

$$C_3 = - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(\alpha_m x_1) \sin(\beta_n x_2) e^{i\omega t} [P_1 W_{mn}] \omega^2 \quad (13c)$$

Fig. 3. Normalized natural frequencies for isotropic cylindrical panel with $b/a = 2$.

$$C_4 = - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(\alpha_m x_1) \sin(\beta_n x_2) e^{i\omega t} \left[\left(P_2 + \frac{P_3}{R_1} \right) U_{mn} + (P_3) X_{mn} \right] \omega^2 \quad (13d)$$

$$C_5 = - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(\alpha_m x_1) \sin(\beta_n x_2) e^{i\omega t} [(P_2) V_{mn} + (P_3) Y_{mn}] \omega^2 \quad (13e)$$

A computer program is developed to solve for the eigenvalues (the frequencies of vibrations of the cylindrical shell) and the corresponding eigenvectors (the mode shapes of vibrations of the cylindrical shell).

4. Finite element formulation

Finite element analyses, using three different shell elements, were used for comparison with and validation of the analytical solutions. The three shell elements used are: (1) A four-node isoparametric element, developed by Bathe and Dvorkin [32], that uses reduced one-point integration approach as well as non-isoparametric form of shape functions for transverse shear deformations. This four-node shell element is implemented in NISA [33] finite element program. The element is denoted as NKPT32, NODRD = 1 in NISA; (2) A four-node quadrilateral shell element that is implemented in SAP2000 [34] finite element program; (3) An eight-node quadrilateral shell element (SHELL93) that is implemented in ANSYS [35] finite element program. All these shell elements have six degrees of freedom per each node (i.e., three translations along x , y and z coordinates and three rotations of the normal about x , y and z coordinates). The finite element analyses results are very comparable to the analytical solution results as will be shown in the subsequent section.

5. Numerical results and discussion

To illustrate the numerical results of the free vibration analysis of all edges clamped cylindrical panel with isotropic materials, cylindrical panels with square and rectangular plan projections are considered with the following values for modulus of elasticity, Poisson's ratio and modulus of rigidity:

Fig. 4. Normalized natural frequencies for isotropic cylindrical panel ($b/a = 1$) with different a/h and R/a .

$$E_1 = E_2 = 10000 \text{ MPa}, \nu_{12} = \nu_{21} = 0.25 \text{ and } G_{13} = G_{23} = G_{12} = 4000 \text{ MPa}$$

The eigenvalues are normalized for the sake of convenience in the following form:

$$\omega_{\text{Norm}} = \omega a^2 \sqrt{\frac{\rho}{E_2}} / h \quad (14)$$

Figure 2 shows the convergence of the analytical normalized natural frequencies ($\omega'_{\text{Norm}} s$) for square isotropic cylindrical panels with $b/a = 1$ and various parametric values. It can be seen that the convergence of ω_1 to ω_7 is obtained at $m = n = 5$. Figure 3 shows the convergence of the analytical normalized natural frequencies ($\omega'_{\text{Norm}} s$) for rectangular isotropic cylindrical panels with $b/a = 2$ and various parametric values. It can be concluded that the convergence of ω_1 to ω_5 is obtained at $m = n = 5$, while ω_6 and ω_7 converge at $m = n = 7$. Figure 4 demonstrates that the frequency of vibration of square cylindrical panels increase with the increase in a/h ratio while the frequencies of these panels are not affected by the ratio of R/a . Figure 5(a) and (b) show sample mode shapes of the cylindrical panels with square plans with different parametric values. Figure 6(a) and (b) show selected mode shapes of the cylindrical panels with rectangular plans with different parametric values.

Tables 1 and 2 present a comparison between the results of the finite element solutions using NISA [33], SAP2000 [34] and ANSYS [35] – with 20×20 mesh, and the analytical solutions for isotropic cylindrical panels with $b/a = 1$, $R/a = 10$, $a/h = 10$ and $b/a = 1$, $R/a = 10$, $a/h = 20$, respectively. Tables 3 and 4 present a comparison between the results of the finite element solutions using NISA [33], SAP2000 [34] and ANSYS [35] – with 20×40 mesh and the analytical solutions for isotropic cylindrical panels with $b/a = 2$, $R/a = 10$, $a/h = 10$ and $b/a = 2$, $R/a = 10$, $a/h = 20$, respectively. Figure 7(a) through Fig. 7(d) show the finite element frequencies' ratios to that of the analytical solutions and their deviation from unity for square and rectangular panels, with different shell thicknesses.

It can be concluded from the above results that, in general, the deviation between the analytical solution and the finite element solution, however small, but it increases as the b/a ratio increases and as the shell thickness decreases (a/h ratio increases). It is also observed that the finite element solutions are normally underestimate the

Table 1
Comparison of normalized frequencies between analytical and FEM solutions for panels with $b/a = 1$, $R/a = 10$, $a/h = 10$

Frequency	Analytical	FEM-NISA [33]	FEM-SAP [34]	FEM-ANSYS [35]
ω_1	9.897736	9.85779	9.8192	9.840725
ω_2	18.924930	18.99485	18.9490	18.940662
ω_3	18.947020	19.01682	18.9810	18.980246
ω_4	26.589800	26.65741	26.2690	26.761971
ω_5	31.330910	32.03936	31.0500	31.640865
ω_6	31.627800	32.34349	31.7320	31.922980
ω_7	37.179730	37.24426	36.7660	37.200855

(a) Mode shape of U3(1,1) for an isotropic shell panel with $b/a = 1$, $R/a = 10$, $a/h = 10$

(b) Mode shape of U3(1,2) for an isotropic shell panel with $b/a = 1$, $R/a = 10$, $a/h = 10$

Fig. 5. (a). Mode shape of U3(1,1) for an isotropic shell panel with $b/a = 1$, $R/a = 10$, $a/h = 10$. (b). Mode shape of U3(1,2) for an isotropic shell panel with $b/a = 1$, $R/a = 10$, $a/h = 10$.

natural frequencies for rectangular panels ($b/a = 2$) and for thin shells ($a/h = 20$) as shown in Fig. 7(d). These discrepancies may be due to the reduced integration schemes and/or exclusion of shell curvature in finite element formulations of shell elements.

It is also observed that the finite element analyses showed similar mode shapes to those obtained from the analytical solution and presented in Figs 5 and 6.

Table 2
Comparison of normalized frequencies between analytical and FEM solution for panels with $b/a = 1$, $R/a = 10$, $a/h = 20$

Frequency	Analytical	FEM-NISA [33]	FEM-SAP [34]	FEM-ANSYS [35]
ω_1	10.90719	10.66991	10.7156	10.492165
ω_2	21.60532	21.34325	21.4720	21.014742
ω_3	21.66954	21.40822	21.5440	21.063750
ω_4	31.32558	30.90899	30.9100	30.557643
ω_5	37.59598	38.13888	38.4400	36.843342
ω_6	37.84910	38.38602	38.6940	37.095926
ω_7	46.43315	46.67562	46.5060	45.603359

(a) Mode shape of U3(3,1) for an isotropic shell panel with $b/a = 2$, $R/a = 10$, $a/h = 10$

(b) Mode shape of U3(3,3) for an isotropic shell panel with $b/a = 2$, $R/a = 10$, $a/h = 10$

Fig. 6. (a). Mode shape of U3(3,1) for an isotropic shell panel with $b/a = 2$, $R/a = 10$, $a/h = 10$. (b). Mode shape of U3(3,3) for an isotropic shell panel with $b/a = 2$, $R/a = 10$, $a/h = 10$.

6. Conclusions and recommendations

The free vibration response of shear-flexible cylindrical panels fabricated with isotropic materials, with all edges clamped boundary conditions was studied using a recently developed analytical solution approach and a well established four-node finite element model. The results of the analytical solution for various parametric ratios were compared with those of the finite element solution and the results were very comparable.

Table 3
Comparison of normalized frequencies between analytical and FEM solution for panels
with $b/a = 2$, $R/a = 10$, $a/h = 10$

Frequency	Analytical	FEM-NISA [33]	FEM-SAP [34]	FEM-ANSYS [35]
ω_1	6.966109	6.863061	6.9404	6.881345
ω_2	8.941903	8.766849	8.9229	8.813424
ω_3	12.351940	12.118350	12.3410	12.198804
ω_4	16.776780	16.768830	16.8940	16.793069
ω_5	17.176630	16.771000	17.0430	16.879777
ω_6	18.493520	18.403500	18.7200	18.497069
ω_7	21.362400	21.198580	21.6650	21.389219

Fig. 7. a–d. Comparison of ratios of finite element solutions to analytical solutions.

The following conclusions are drawn from this study:

- The convergence rate of a cylindrical shell panel with square plan ($b/a = 1$) is faster than that of a cylindrical shell panel with rectangular plan with an aspect ratio $b/a = 2$.
- The values of normalized natural frequencies increase with the increase of the ratio of segment length over the shell thickness, i.e., with the increase of a/h ratio or decrease in shell thickness.
- The fundamental frequency (first mode) is not significantly affected by the ratio of radius of curvature for $R/a > 15$ for $a/h = 10$ to 30 as studied here. In higher frequency, the influence reduces further as R/a increases. This is due to the use of shallow shell theory of Sander's.
- The deviations between the analytical solutions and the finite element solutions are more apparent as b/a and a/h increases.
- The analytical approach presented in this paper can be extended further to study the buckling, thermal and impact loading response of cylindrical panels.

Table 4
Comparison of normalized frequencies between analytical and FEM solution for panels
with $b/a = 2$, $R/a = 10$, $a/h = 20$

Frequency	Analytical	FEM-NISA [33]	FEM-SAP*[34]	FEM-ANSYS [35]
ω_1	7.645855	7.233375	7.5106	7.225412
ω_2	9.842625	9.318339	9.6092	9.317587
ω_3	13.866130	13.052530	13.3990	13.035096
ω_4	18.982900	18.397500	18.8700	18.314229
ω_5	19.480500	18.634570	19.0328	18.467538
ω_6	21.091580	20.572960	21.0980	20.445485
ω_7	24.712600	23.920320	24.6480	23.834635

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