Shock and Vibration 15 (2008) 639–654
IOS Press

Suppression of vibration using passive receptance method with constrained minimization

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Received 6 April 2007
Revised 30 July 2007

Abstract. Structural vibration analysis is used to suppress unwanted vibrations in many areas such as aerospace engineering, manufacturing, defense, automotive, etc. As a result of suppressing the unwanted vibration, the quality of the product is improved. Focusing on the minimization of the vibration amplitudes via a concept of receptance, a new and efficient method for calculating the receptance of a translational mass-spring-damper system with N masses and M absorbers (where N and M are any positive integer) is developed. The receptance of the combined system, in terms of the parameters of the main and absorber systems is derived, separately. The optimal parameters of the absorbers are then found. A methodology is derived using dynamic stiffness and linear graph representation in order to verify the dynamic stiffness, i.e., the inverse of the receptance, of the system.

Keywords: Structural vibration suppression, receptance theory, impedance method, Sherman–Morrison formula, constrained minimization

1. Introduction

Vibration absorbers are efficient and attractive for reducing vibration and noise. Scientists focus on a "vibration absorbers" methodology to minimize the amplitude of the vibration. They use many alternative approaches and validate their methods with mass–spring–damper systems. Vibration absorbers are attached to main systems like plates, shell structures, frames, etc. In general, the vibration absorber approach is preferred as it is cheap, robust, and flexible enough to adapt to the main system that is being considered.

During the design stage, the crucial point is to minimize the amplitude of the vibration depending on the user's wishes. One may need to minimize the vibration at the resonance frequency, while another may need to minimize the amplitudes throughout the working frequency range. When the main system starts to present high amplitudes of vibration, the secondary system, namely the "vibration absorbers", reacts in the opposite direction in order to excite a force on the main system to initiate the minimization of unwanted vibration. The parameters of the absorbers are designed to achieve this goal. Generally, they are found by an iterative process. Grissom et al. [1] have proposed a reduced eigenvalue approach where the additional mass and stiffness matrices are coupled to the already available modal response of the base structure. First, the eigenvalues and eigenvectors of the unmodified structure are calculated. The impedance method (impedance is the relation between input and output, in vibration cases it relates forces with velocities, respectively) is then used to analyze the modified structure.

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ISSN 1070-9622/08/$17.00

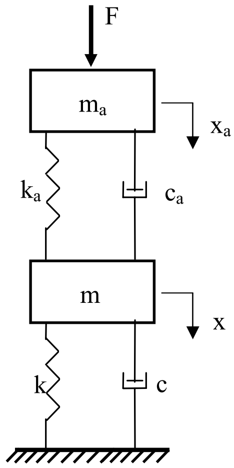

Fig. 1. Mass–spring–damper system.

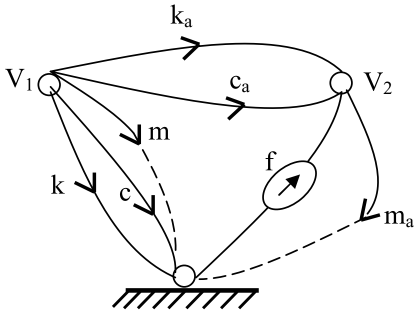

Fig. 2. Linear graph of mass–spring–damper system.

Pennestri [2] has defined an application of Chebyshev's criterion in order to allow for an optimal choice of the parameters of the damped dynamic vibration absorbers. The aim is to minimize the maximum displacement of the primary mass. Defining the basic requirements of the system, the design problem is solved by utilizing the mini-max Chebyshev criterion. Aida et al. [3] have analyzed the vibration absorber techniques on a shallow shell structure. Their aim was to suppress several modes of bending vibration of a main system (a shallow shell). The dynamic absorbing shell, under the same boundary condition and with the same shape as the main shell, activates the vibration absorbers. It acts like a plate-type dynamic vibration absorber. There are uniformly distributed springs and dampers in the vertical direction between the main shell and the dynamic absorbing shells. An approximate tuning method for the shell-type dynamic absorber has been developed.

Some scientists focus on minimizing the vibration amplitudes by using receptance theory. The main disadvantage of this method is taking the inverse of the matrices. Since the expression is becoming complex, the receptance expression is hard to solve from the inverse of the dynamic stiffness matrix. Karakas and Gurgoze [4] have analyzed the problem of solving the receptance matrix. An exact method to evaluate the receptances of non-proportionally damped dynamic systems has been developed by Yang [5]. This is an iterative method for the calculation of the receptance matrix when the damping matrix is decomposed into the sum of the dyadic products. In this method, there is no need to apply an inverse matrix operation. The iteration process starts with the receptance matrix of the undamped system. Iteration, using as many iterations as the number of dyadic products in the damping matrix, then gives the receptance matrix of the damped system. Gurgoze [6] has focused on obtaining the receptance matrix directly without using the iterations that were used in Yang's studies. He used the Sherman–Morrison formula to obtain the inverse of a regular matrix and only one dyadic product (a matrix of rank 1). However, the Woodbury formula gives the inverse of the sum of a regular matrix and a matrix product whose rank can be greater than 1.

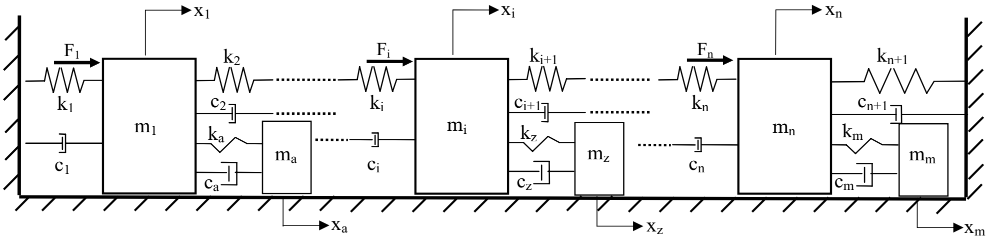

Fig. 3. System of N masses with M absorbers.

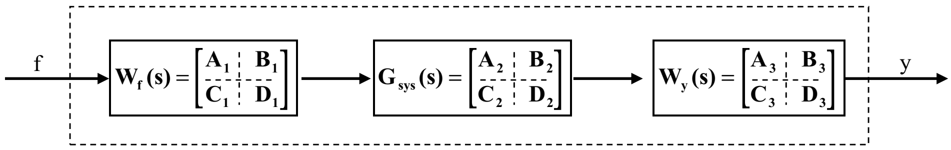

Fig. 4. Transfer function diagram.

Using the Woodbury formula (to decompose the matrices) and a dyadic product formulation, Gurgoze derived the receptance matrix. Gurgoze has also studied the variations of the damping mechanism and the eigen characteristics of the system. Eigenvalues and eigenvectors can be calculated and the receptance matrix of the added system can also be obtained. Lin and Lim [7] have used another method to calculate the receptance, eigenvalues and eigenvector sensitivities using vibration test data. They considered the derivation of the receptance sensitivities to given structural modifications. The relationship between the receptance sensitivities, eigenvalues and eigenvectors is then established. A method to determine the sensitivities of the eigenvalues and eigenvectors can then be developed.

Ozer and Royston [8] have proposed a semi-analytic method to determine the parameters of a damped vibration absorber attached to a damped multi-degree-of-freedom main system. They used the Sherman–Morrison formula to obtain the receptance matrix. Due to the advantage of using an absorber attached to the main system, the Sherman–Morrison formula is only used once. They derived not only the main system's equation of motion but also the absorber's equation of motion and defined the matrices that had been added to the main system. Finally, the parameters of the absorber were found. There are many application areas for the Sherman–Morrison formula. Riedel [9] has investigated a special solution by using the Sherman–Morrison formula. Instead of using a second term consisting of a combination of two vectors, he analyzed and decomposed three matrices as a second term and then displayed this in his third theorem.

Another investigation has been carried out by Gurgoze [10] to combine the receptance approach with constrained equations. Gurgoze aimed to establish the receptance matrix of the constrained system in terms of the receptance matrix of the unconstrained system and the coefficient vectors of the constraint equations. The coordinates of the system are assumed to be subject to several linear constraint equations. He displayed the receptance matrix of the unconstrained system and combined it with the generalized coordinates by using the Lagrange multiplier approach.

Shankar and Keane [11] have displayed a new method based on receptance theory. Substructures are modeled by using finite element analysis, each substructure separately analyzed for its eigenvalues and eigenvectors. Energy balance is also used in this method. Substructure vibrational energy levels are also calculated by evaluating the net energy between input and dissipated energies and the energy transfers through coupling nodes. It is beneficial to carry out this energy into statistical energy analysis. The formulation developed uses a finite elements analysis package program and the response of the total structure is predicted by using substructure modal information with the help of this package program.

In order to reach desired dynamic behaviour, Kyprianou et al. [12] have made changes to mass, stiffness and damping properties. It can be achieved by adding or removing mass, spring and damper elements. By adding mass and spring into the system (two cases; modification by a simple spring mass oscillator and modification by a mass connected by two springs), they investigate the effect on the receptance function and also the response function.

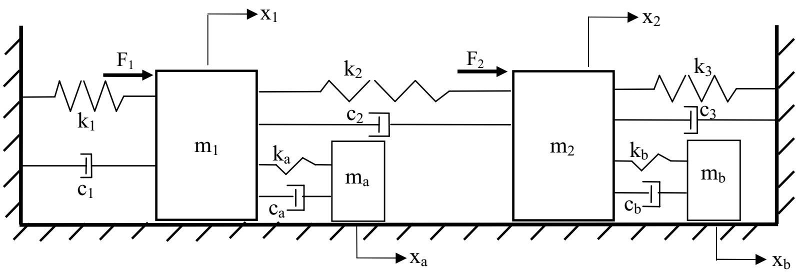

Fig. 5. System of two main masses with two absorbers.

In the present study, a new method based on using the Sherman–Morrison matrix inversion formula to calculate the optimal parameters of the absorbers is studied. The method also analyzes and provides the receptance of a system with more than one absorber.

2. Theory

2.1. Dynamic stiffness via linear graph methodology

The linear graph technique can be regarded as one of the most powerful graphical methods to represent the elements of a given physical system, as well as its structure, in a unified system. The study of linear graphs involves the branch of mathematics called topology, which deals with the properties of figures which are related to the way in which their various parts are interconnected. The procedure expressed by Mcphee [13] is to define a set of "through" and "across" variables that describe the state of the physical system and to write the constitutive equation for each element in terms of these variables. A through variable is a physical quantity that would be measured by an instrument placed in series with an element; current in an electrical system and force in a multibody system are two examples. An across variable is a quantity that can be measured by an instrument placed across the end of an element. This would be voltage for an electrical network, and displacement in a multibody system.

To obtain the dynamic stiffness, i.e., the inverse of the receptance, one can model the physical systems by a linear graph. Moreover, a methodology is derived to move between the dynamic stiffness and the linear graph representation. To calculate the dynamic stiffness of a system using linear graph representation, the series and parallel branches were considered.

2.1.1. Series branch

The inverse of the dynamic stiffnesses of each branch are added together. The dynamic stiffness of the total system (Z_{tot}) is equal to the inverse of the calculated value. $Z_1, Z_2, Z_3 \dots$ are the dynamic stiffnesses of each branch.

$$\frac{1}{Z_{tot}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \frac{1}{Z_4} + \frac{1}{Z_5} \cdots \tag{1}$$

2.1.2. Parallel branch

The dynamic stiffness of the total system is found by adding together the dynamic stiffness values of all the branches.

$$Z_{tot} = Z_1 + Z_2 + Z_3 + Z_4 + Z_5 \cdots \tag{2}$$

In order to obtain the dynamic stiffness, the problem is investigated on an example shown in Fig. 1

$$k_a(x - x_a) + c_a(\dot{x} - \dot{x}_a) + F = m_a\ddot{x}_a \tag{3}$$

$$-kx - c\dot{x} - k_a(x - x_a) - c_a(\dot{x} - \dot{x}_a) = m\ddot{x} \tag{4}$$

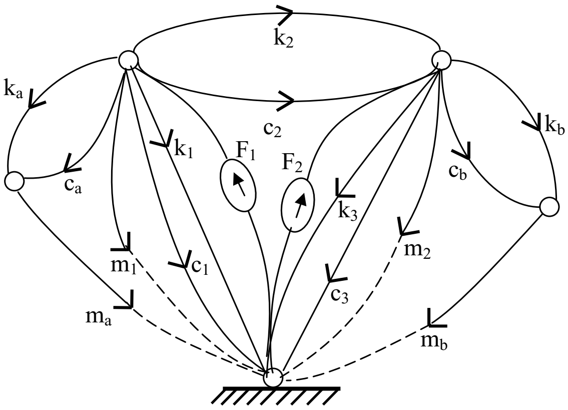

Fig. 6. Linear graph of system of two masses with two absorbers.

After some steps using Eqs (3) and (4), the dynamic stiffness between $X_a(s)$ and $F(s)$ (where $X_a(s)$ and $F(s)$ are the Laplace transform of the $x_a(t)$ and $f(t)$, respectively) is found to be

$$Z_{X_aF} = \frac{(m_a s^2 + c_a s + k_a)(ms^2 + cs + k + c_a s + k_a) - (c_a s + k_a)^2}{ms^2 + cs + k + c_a s + k_a} \tag{5}$$

The linear graph of the system is shown in Fig. 2. Z_1 is in series with Z_2, and the resulting system composed of Z_1 and Z_2, is in parallel with Z_3. Z_{X_aF} is the dynamic stiffness of the whole system.

$$Z_{X_aF} = \left[\frac{1}{Z_1} + \frac{1}{Z_2}\right]^{-1} + Z_3 \tag{6}$$

where

$$Z_1 = ms^2 + cs + k, \tag{7}$$

$$Z_2 = c_a s + k_a, \tag{8}$$

$$Z_3 = m_a s^2 \tag{9}$$

Substituting Eqs (7)–(9) into Eq. (6) and simplifying, we obtain

$$Z_{X_aF} = \frac{mc_a s^3 + \mathrm{mk}_a s^2 + cc_a s^2 + \mathrm{ck}_a\mathrm{s} + \mathrm{c}_a\mathrm{ks} + \mathrm{kk}_a + \mathrm{m}_a\mathrm{ms}^4 + \mathrm{m}_a\mathrm{cs}^3 + \mathrm{m}_a\mathrm{ks}^2 + \mathrm{m}_a\mathrm{c}_a\mathrm{s}^3 + \mathrm{m}_a\mathrm{k}_a\mathrm{s}^2}{ms^2 + cs + k + c_a s + k_a} \tag{10}$$

It is observed that, as expected, Eqs (5) and (10) are the same. In both the equation of motion and linear graph methodology, the same dynamic stiffness expression between $X_a(s)$ and $F(s)$ is found.

2.2. *System of N masses with M absorbers*

A new method to calculate the optimal absorber parameter values based on the Sherman–Morrison matrix inversion formula was studied. The general case for N masses with M absorbers is shown in Fig. 3. It should be noted that each main mass does not necessarily have an absorber(s) attached. As an example, the system of two masses with two absorbers is analyzed.

One can easily find $K_{\mathrm{add}}, C_{\mathrm{add}}$ ($\mathrm{K_{add}}$,$\mathrm{C_{add}}$ are the additive matrices that come upon the addition of the absorber/s to the main system) with both the equation of motion of the main system (Eq. (11)) and also the equation of motion of the absorbers (Eq. (16)). Displaying the equation of motion in matrix form we obtain

$$\begin{aligned}&\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 + c_a & -c_2 \\ -c_2 & c_2 + c_3 + c_b \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 + k_a & -k_2 \\ -k_2 & k_2 + k_3 + k_b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &+ \begin{bmatrix} -c_a & 0 \\ 0 & -c_b \end{bmatrix}\begin{bmatrix} \dot{x}_a \\ \dot{x}_b \end{bmatrix} + \begin{bmatrix} -k_a & 0 \\ 0 & -k_b \end{bmatrix}\begin{bmatrix} x_a \\ x_b \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}\end{aligned} \tag{11}$$

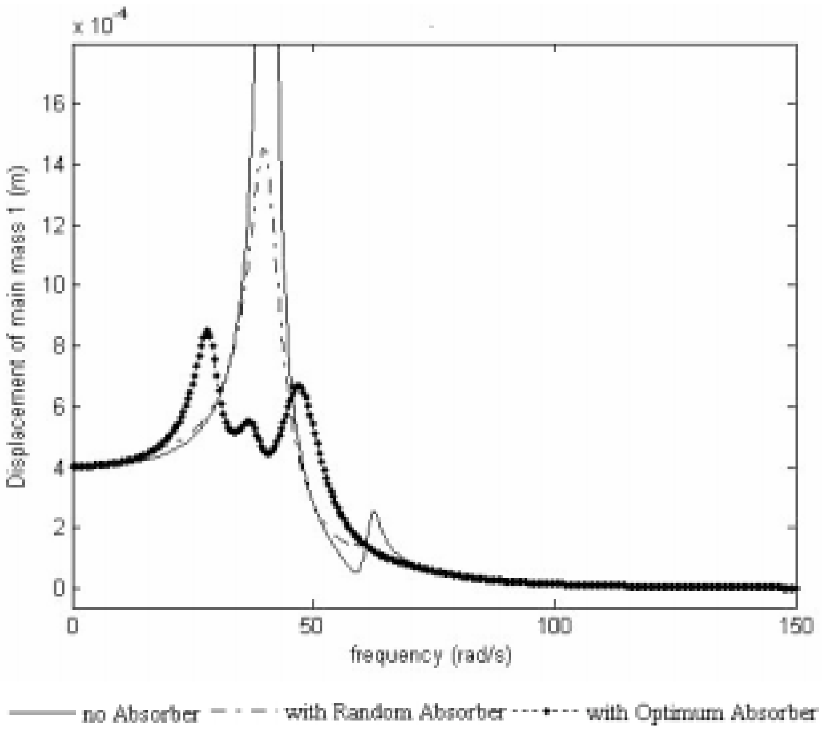

Fig. 7. Displacement of main mass 1.

where

$$C_{\text{add}} = \begin{bmatrix} -c_a & 0 \\ 0 & -c_b \end{bmatrix}, \tag{12}$$

$$K_{\text{add}} = \begin{bmatrix} -k_a & 0 \\ 0 & -k_b \end{bmatrix} \tag{13}$$

In general, the equations of motion of the main system and the absorbers are, respectively

$$M\ddot{x} + C\dot{x} + Kx + C_{\text{add}}\dot{\tilde{x}} + K_{\text{add}}\tilde{x} = F \tag{14}$$

$$\tilde{M}\ddot{\tilde{x}} + \tilde{C}\dot{\tilde{x}} + \tilde{K}\tilde{x} + C_{sec}\dot{x} + K_{sec}x = \bar{0} \tag{15}$$

K_{sec},C_{sec} are the additive matrices that exist in the equation of motion of the absorbers and they are the multipliers of the main system's generalized coordinates.

In this example, the equation of motion of the absorbers is

$$\begin{bmatrix} m_a & 0 \\ 0 & m_b \end{bmatrix} \begin{bmatrix} \ddot{x}_a \\ \ddot{x}_b \end{bmatrix} + \begin{bmatrix} c_a & 0 \\ 0 & c_b \end{bmatrix} \begin{bmatrix} \dot{x}_a \\ \dot{x}_b \end{bmatrix} + \begin{bmatrix} k_a & 0 \\ 0 & k_b \end{bmatrix} \begin{bmatrix} x_a \\ x_b \end{bmatrix} + \begin{bmatrix} -c_a & 0 \\ 0 & -c_b \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} -k_a & 0 \\ 0 & -k_b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{16}$$

2.2.1. Formulation for system of N masses with M absorbers

It is assumed that absorbers are attached to each main mass.

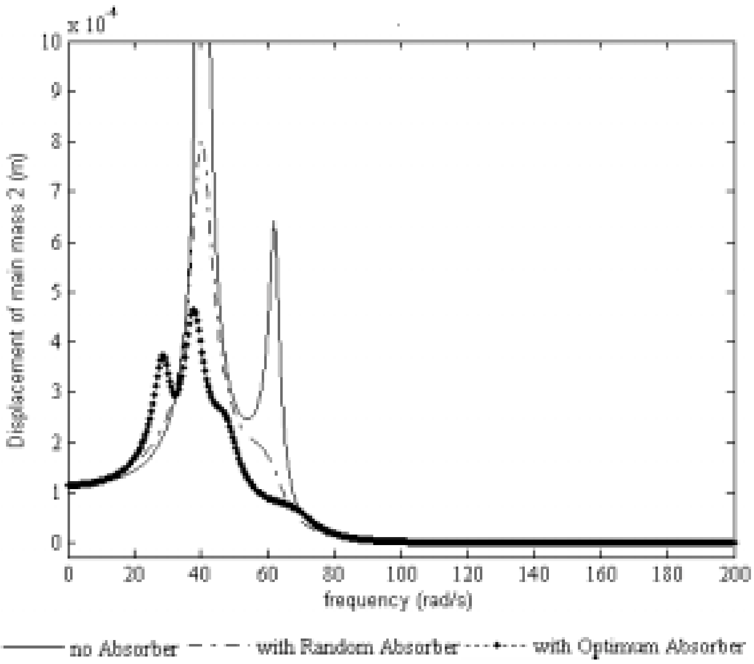

Fig. 8. Displacement of main mass 2.

$$M = \begin{bmatrix} m_1 & & & & & & \\ & m_2 & & & & & \\ & & \ddots & & 0 & & \\ & & & m_i & & & \\ & & & & m_{i+1} & & \\ & 0 & & & & \ddots & \\ & & & & & & m_n \end{bmatrix} \tag{17}$$

$$C = \begin{bmatrix} c_1+c_2+c_a & -c_2 & & & & & & \\ -c_2 & c_2+c_3+c_b & -c_3 & & & 0 & & \\ & -c_3 & \dots\dots\dots & -c_i & & & & \\ & & -c_i & c_i+c_{i+1}+c_z & -c_{i+1} & & & \\ & & & -c_{i+1} & c_{i+1}+c_{i+2}+c_q & & & \\ & 0 & & & & \dots\dots\dots & & \\ & & & & & & & -c_{n-1} \\ & & & & & & -c_{n-1} & c_{n-1}+c_n+c_m \end{bmatrix} \tag{18}$$

$$K = \begin{bmatrix} k_1+k_2+k_a & -k_2 & & & & & & \\ -k_2 & k_2+k_3+k_b & -k_3 & & & 0 & & \\ & -k_3 & \dots\dots\dots & -k_i & & & & \\ & & -k_i & k_i+k_{i+1}+k_z & -k_{i+1} & & & \\ & & & -k_{i+1} & k_{i+1}+k_{i+2}+k_q & & & \\ & 0 & & & & \dots\dots\dots & & \\ & & & & & & & -k_{n-1} \\ & & & & & & -k_{n-1} & k_{n-1}+k_n+k_m \end{bmatrix} \tag{19}$$

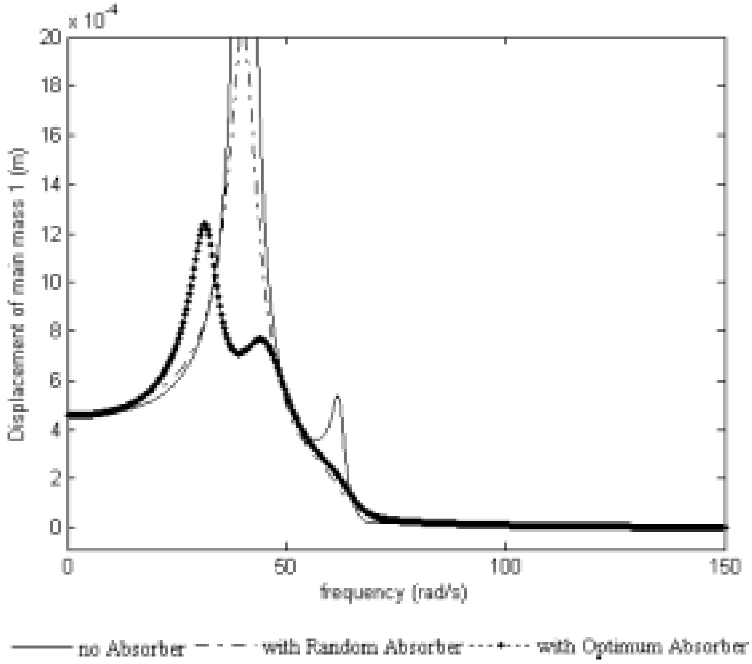

Fig. 9. Displacement of main mass 1, goal function is $(0.2x_1 + 0.8x_2)$.

By substituting; $x = e^{i\omega t}$ and its derivatives into Eqs (14) and (15), respectively, we obtain

$$\left[-\omega^2 M + i\omega C + K\right] x + \left[i\omega C_{\mathrm{add}} + K_{\mathrm{add}}\right] \tilde{x} = F \tag{20}$$

$$\left[-\omega^2 \tilde{M} + i\omega \tilde{C} + \tilde{K}\right] \tilde{x} + \left[i\omega C_{sec} + K_{sec}\right] x = \bar{0} \tag{21}$$

Substituting Eq. (21) into Eq. (20) and simplifying, the output equation is as follows:

$$\left(\left[-\omega^2 M + i\omega C + K\right] - \left[i\omega C_{\mathrm{add}} + K_{\mathrm{add}}\right] \left[-\omega^2 \tilde{M} + i\omega \tilde{C} + \tilde{K}\right]^{-1} \left[i\omega C_{sec} + K_{sec}\right]\right) x = F \tag{22}$$

The relation between x and F is known as the dynamic stiffness of the system which is

$$Zx = F \tag{23}$$

$$Z = Z_x + Z_{\mathrm{add}} = \left(\left[-\omega^2 M + i\omega C + K\right] - \left[i\omega C_{\mathrm{add}} + K_{\mathrm{add}}\right] \left[-\omega^2 \tilde{M} + i\omega \tilde{C} + \tilde{K}\right]^{-1} \left[i\omega C_{\mathrm{sec}} + K_{\mathrm{sec}}\right]\right) \tag{24}$$

where

$$Z_x = \left[-\omega^2 M + i\omega C + K\right] \tag{25}$$

$$Z_{\mathrm{add}} = -\left[i\omega C_{\mathrm{add}} + K_{\mathrm{add}}\right] \left[-\omega^2 \tilde{M} + i\omega \tilde{C} + \tilde{K}\right]^{-1} \left[i\omega C_{\mathrm{sec}} + K_{\mathrm{sec}}\right] \tag{26}$$

Z_x can be separated into the parameters of the main system and those of the absorber system.

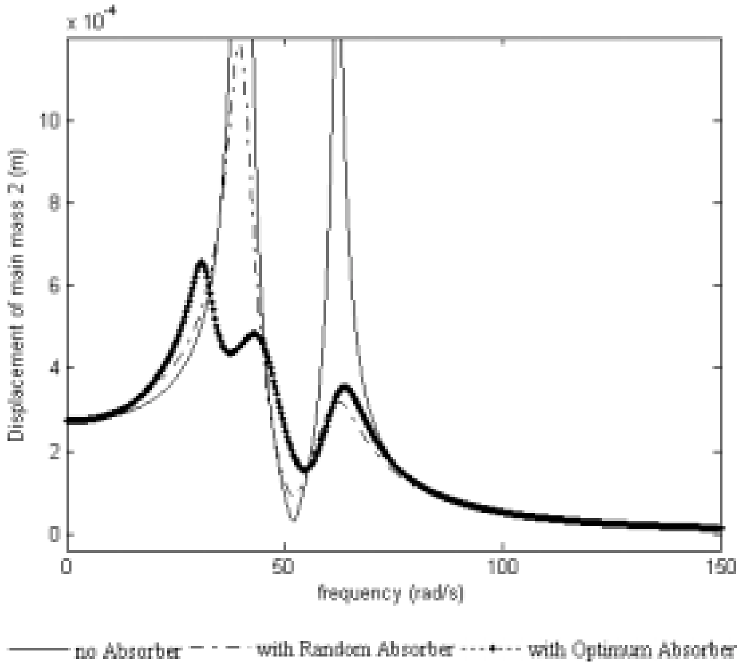

Fig. 10. Displacement of main mass 2, goal function is $(0.2x_1 + 0.8x_2)$.

$$Z_x = Z_{\text{main}} + \begin{bmatrix} i\omega c_a + k_a & & & & & & & 0 \\ 0 & i\omega c_b + k_b & & & & & & \\ \vdots & 0 & i\omega c_c + k_c & & & & & \\ \vdots & \vdots & 0 & \cdots\cdots\cdots & & & & \\ \vdots & \vdots & \vdots & 0 & \cdots\cdots\cdots & & & \\ \vdots & \vdots & \vdots & \vdots & 0 & \cdots\cdots\cdots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \cdots\cdots\cdots & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & i\omega c_m + k_m \end{bmatrix} \tag{27}$$

where

$$Z_{\text{main}} = -\omega^2 M_{\text{main}} + i\omega C_{\text{main}} + K_{\text{main}} \tag{28}$$

As an example of the case of two masses with two absorbers, we consider

$$Z_x = \left[-\omega^2 M + i\omega C + K\right] = -\omega^2 M_{\text{main}} + i\omega C_{\text{main}} + K_{\text{main}} + Z_1 Z_1^T + Z_2 Z_2^T \tag{29}$$

where

$$M_{\text{main}} = M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}; \quad C_{\text{main}} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}; \quad K_{\text{main}} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \tag{30}$$

$$Z_1 = \begin{bmatrix} \sqrt{i\omega c_a + k_a} \\ 0 \end{bmatrix}; \quad Z_2 = \begin{bmatrix} 0 \\ \sqrt{i\omega c_b + k_b} \end{bmatrix} \tag{31}$$

For the case of N masses with M absorbers, the following formulation is found:

$$Z = Z_{\text{main}} + Z_{\text{add}} + Z_1 Z_1^T + Z_2 Z_2^T + Z_3 Z_3^T + \ldots\ldots\ldots Z_m Z_m^T \tag{32}$$

The system receptance is displayed as follows:

$$\alpha = Z^{-1} = \left[Z_{\text{main}} + Z_{\text{add}} + Z_1 Z_1^T + Z_2 Z_2^T + Z_3 Z_3^T + \ldots\ldots\ldots Z_m Z_m^T\right]^{-1} \tag{33}$$

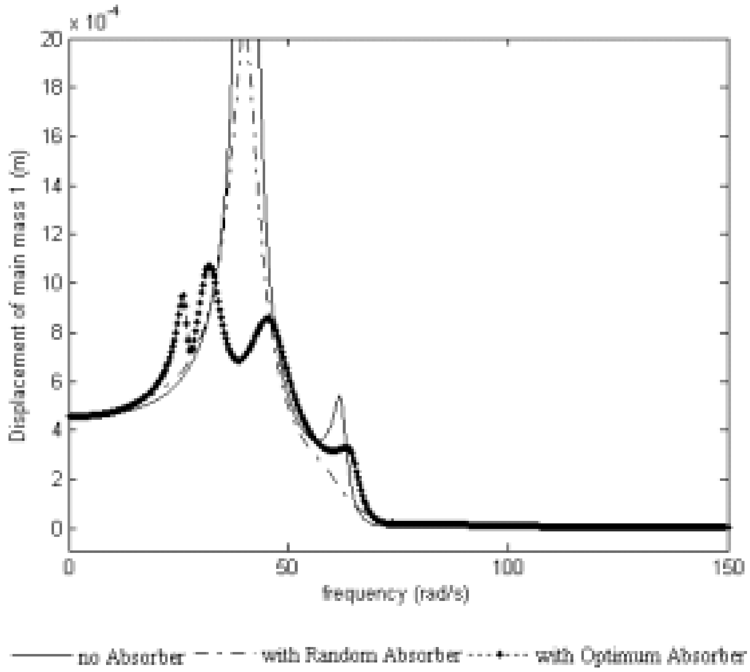

Fig. 11. Displacement of main mass 1, goal function is $(0.8x_1 + 0.2x_2)$.

2.2.2. *Sherman–Morrison matrix inversion formula*

$$\left[A + uv^T\right]^{-1} = \mathrm{A}^{-1} - \frac{A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u} \tag{34}$$

Applying the Sherman–Morrison formula to Eq. (33); the result is as follows:
The number of absorbers in the system is designated by i
$i = 0$ (no absorber)

$$\alpha_0 = [Z_{\text{main}} + Z_{\text{add}}]^{-1} \tag{35}$$

$i = 1$ (one absorber)

$$\alpha_1 = \left[Z_{\text{main}} + Z_{\text{add}} + Z_1Z_1^T\right]^{-1} = \alpha_0 - \frac{\alpha_0 Z_1 Z_1^T \alpha_0}{1 + Z_1^T \alpha_0 Z_1} \tag{36}$$

$i = 2$ (two absorbers)

$$\begin{aligned} \alpha_2 &= \left[Z_{\text{main}} + Z_{\text{add}} + Z_1Z_1^T + Z_2Z_2^T\right]^{-1} \\ &= \left[Z_{\text{main}} + Z_{\text{add}} + Z_1Z_1^T\right]^{-1} - \frac{\left[Z_{\text{main}} + Z_{\text{add}} + Z_1Z_1^T\right]^{-1} Z_2Z_2^T \left[Z_{\text{main}} + Z_{\text{add}} + Z_1Z_1^T\right]^{-1}}{1 + Z_2^T\left[Z_{\text{main}} + Z_{\text{add}} + Z_1Z_1^T\right]^{-1} Z_2} \\ &= \alpha_1 - \frac{\alpha_1 Z_2 Z_2^T \alpha_1}{1 + Z_2^T \alpha_1 Z_2} \end{aligned} \tag{37}$$

$i = m$ (m absorbers)

$$\begin{aligned} \alpha_m &= \left[Z_{\text{main}} + Z_{\text{add}} + Z_1Z_1^T + Z_2Z_2^T + \ldots\ldots\ldots + Z_{m-1}Z_{m-1}^T + Z_mZ_m^T\right]^{-1} \\ &= \left[Z_{\text{main}} + Z_{\text{add}} + Z_1Z_1^T + \ldots + Z_{m-1}Z_{m-1}^T\right]^{-1} \end{aligned}$$

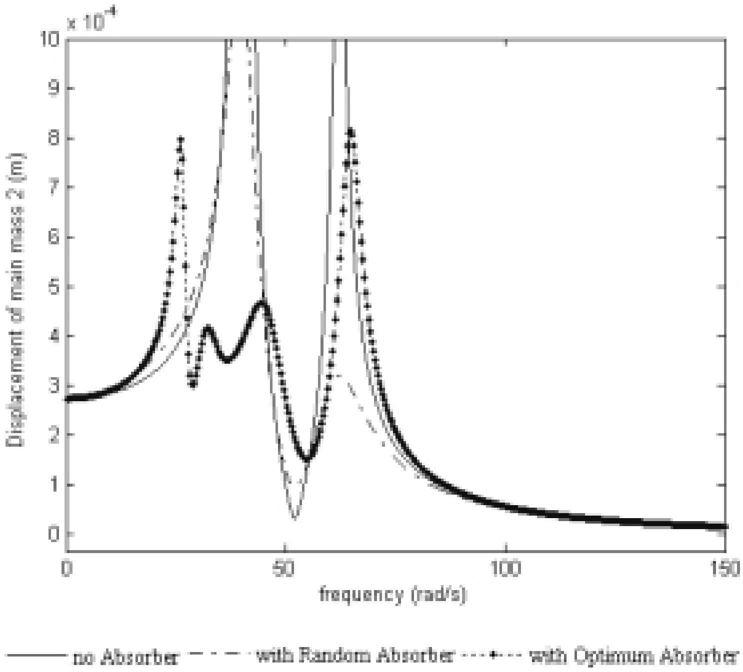

Fig. 12. Displacement of main mass 2, goal function is $(0.8x_1 + 0.2x_2)$.

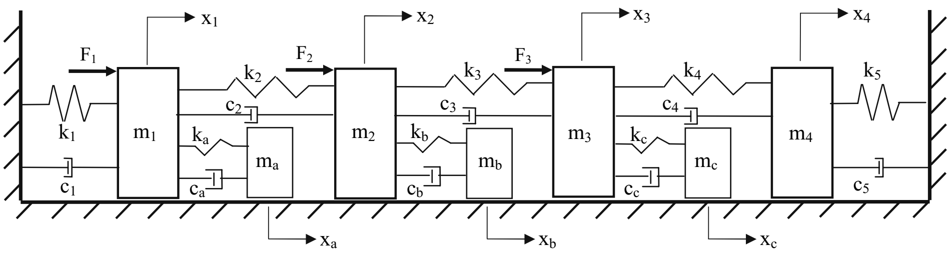

Fig. 13. System of four main masses with three absorbers.

$$-\frac{\left[Z_{\text{main}}+Z_{\text{add}}+Z_1Z_1^T+\ldots+Z_{m-1}Z_{m-1}^T\right]^{-1} Z_m Z_m^T \left[Z_{\text{main}}+Z_{\text{add}}+Z_1Z_1^T+\ldots+Z_{m-1}Z_{m-1}^T\right]^{-1}}{1+Z_m^T\left[Z_{\text{main}}+Z_{\text{add}}+Z_1Z_1^T+\ldots+Z_{m-1}Z_{m-1}^T\right]^{-1} Z_m}$$

$$= \alpha_{m-1} - \frac{\alpha_{m-1}Z_m Z_m^T \alpha_{m-1}}{1+Z_m^T\alpha_{m-1}Z_m} \qquad (38)$$

In general, by applying the Sherman–Morrison formula sequentially, the system receptance is found in terms of the receptance of the system with one less absorber.

$$\alpha_j = \alpha_{j-1} - \frac{\alpha_{j-1}Z_j Z_j^T \alpha_{j-1}}{1+Z_j^T\alpha_{j-1}Z_j} \qquad (39)$$

The main outcomes can be summarized as follows:

– In general, the mass, damping and stiffness matrices are symmetric, and the parameters of the absorbers have no effect on the off-diagonal terms.
– The parameters of the absorbers have no effect on the mass matrix.
– The damping and stiffness matrices have similar shapes due to the presence of both damper and spring between each of the masses.

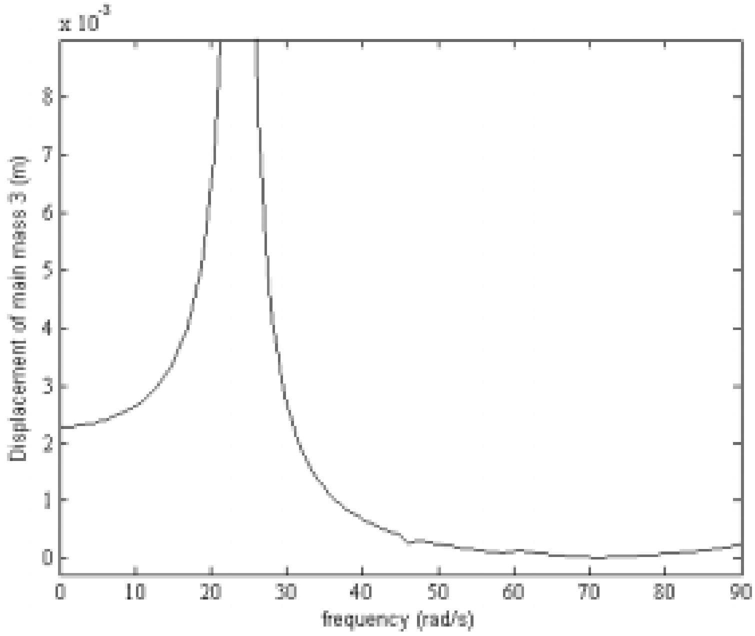

Fig. 14. Displacement of main mass 3 (no absorber).

- The parameters of the absorbers (k_a, c_a, k_b, c_b) appear on the nxnth elements if the nth mass has direct contact with the absorber.
- If the number of absorbers is equal to the number of main masses in the system, then

$$C_{\mathrm{sec}} = C_{\mathrm{add}}, \quad K_{\mathrm{sec}} = K_{\mathrm{add}} \tag{40}$$

otherwise

$$C_{\mathrm{sec}} = C_{\mathrm{add}}^T, \quad K_{\mathrm{sec}} = K_{\mathrm{add}}^T \tag{41}$$

- In C_{add}, K_{add} matrices, the $c_a, c_b, \ldots, c_m$ and $k_a, k_b, \ldots, k_m$ terms exist by x times (x is the number of contacts between absorber and main system masses).
- In the Z_x matrix, $i\omega c_a + k_a$, $i\omega c_b + k_b, \ldots, i\omega c_m + k_m$ terms exist by x times (x is the number of contact between absorber and main system masses).
- The absorber parameters can exist in off-diagonal terms if and only if the absorber spring and damper are in direct contact with both of the arbitrary main masses, but this situation is not physically realizable.
- The number of $Z_1 Z_1^T$, $Z_2 Z_2^T \ldots$ terms increases based on the number of contacts between the masses of the absorber and main system.
- One cannot use the Sherman–Morrison formula in one-step formulations for N masses with M absorbers.

3. Optimization of the parameters of the absorbers

An M-files package (consisting of 3 M files) was developed in Matlab in order to find the optimal absorber parameters. It can also analyze systems which have more than one absorber. The inputs of the M files are as follows:

- Mass, stiffness, damping parameters of the main system.
- Mass parameters of the absorber(s).

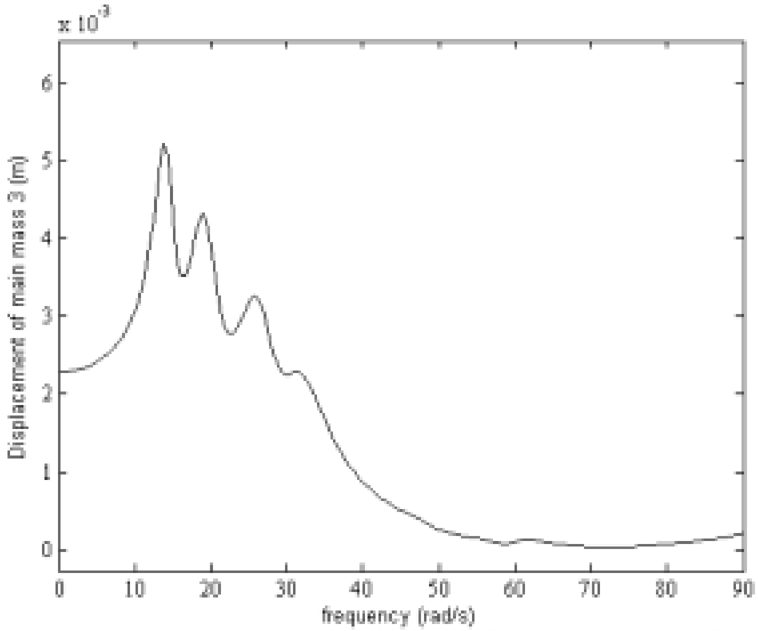

Fig. 15. Displacement of main mass 3 (with optimal absorber).

– Magnitudes of external force(s).

The outputs of the M files are as follows:

– Explicit expression of receptance.
– Explicit expression of displacement.
– Displacement vs. frequency plots (including all displacements).

The M files are able to fulfill the following tasks:

– Find the optimum absorber parameters according to the user's goal (working frequency range, resonance frequency, etc.)
– Analyze systems with N masses and M absorbers, where N and M are any positive number.
– Express the receptance explicitly (receptance of a main system with 1 absorber, 2 absorbers . . . . . . M absorbers are derived sequentially).
– Display plots of the receptance versus frequency and the displacement versus the frequency.

With the aid of the M-files, the user can easily optimize the parameters of the system absorbers, based on the minimization of the amplitude of the vibration at a specified frequency range or at resonance frequency, etc. Moreover, the user can define a mixed-goal with different weighing functions applied to the displacements of the main masses. The transfer function of the main system is defined as $G_{sys}(s)$. The weighing functions of the input force and output displacement, respectively, are $W_f(s)$ and $W_y(s)$. The user can easily define the desired output with weighing functions of the input and output (see Fig. 4).

The Gaussian quadrature numerical integration method, is utilized on the plot of the displacement versus the frequency. The area under this plot is calculated and the output function is optimized using the Matlab optimization toolbox. In the optimization pack, in order to converge the output values to the same optimum parameters (global minimum of the system); high working precision, low tolerance function and high number of maximum iterations were used. As a result, the optimal absorber parameters that minimize the amplitude of the vibration according to the user's requirements, are obtained.

4. Numerical applications and case studies

A numerical study was performed to verify the method on physical systems. Two masses with two absorbers were studied (see Fig. 5). The aim was to minimize the vibration amplitudes of the displacement of both main mass 1 (m_1) and main mass 2 (m_2) with equal weighing.

4.1. Case Study 1

The linear graph of the system is displayed in Fig. 6 with an external force applied on the first main mass (m_1). The dynamic stiffness between x_1 and F can be found by using the dynamic stiffness via the linear graph method.

$$\begin{aligned} Z_{x_1F} = &\left[\frac{1}{c_a s + k_a} + \frac{1}{m_a s^2}\right]^{-1} + m_1 s^2 + c_1 s + k_1 \\ &+ \left[\frac{1}{c_2 s + k_2} + \left[\left[\frac{1}{m_b s^2} + \frac{1}{c_b s + k_b}\right]^{-1} + m_2 s^2 + c_3 s + k_3\right]^{-1}\right]^{-1} \end{aligned} \tag{42}$$

The same result can be verified by solving the relation between x_1 and F_1 using the equations of motion of the main and absorber systems as follows:

$$\begin{aligned} &-k_1 x_1 - c_1 \dot{x}_1 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) + k_a(x_a - x_1) + c_a(\dot{x}_a - \dot{x}_1) + F_1 = m_1 \ddot{x}_1 \\ &-k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) - k_3 x_2 - c_3 \dot{x}_2 + k_b(x_b - x_2) + c_b(\dot{x}_b - \dot{x}_2) = m_2 \ddot{x}_2 \\ &\qquad -k_a(x_a - x_1) - c_a(\dot{x}_a - \dot{x}_1) = m_a \ddot{x}_a \\ &\qquad -k_b(x_b - x_2) - c_b(\dot{x}_b - \dot{x}_2) = m_b \ddot{x}_b \end{aligned} \tag{43}$$

The receptance is found by applying the formulation for N masses with M absorbers (derived in Section 2.2).

The working frequency range is set as 30–70 rad/s (the resonance frequency is in the same range). The inputs are taken as

$$F = [F_1; F_2] = \left[\left|4000/\left((s+40)^2 + 225\right)\right| \ ; 0\right] \tag{44}$$

$$\begin{aligned} &m_1 = 3\ \mathrm{kg};\ m_2 = 2\ \mathrm{kg};\ c_1 = 0.8\ \mathrm{Ns/m};\ c_2 = 2\ \mathrm{Ns/m};\ c_3 = 3\ \mathrm{Ns/m}; \\ &k_1 = 4000\ \mathrm{N/m}; k_2 = 2000\ \mathrm{N/m}; k_3 = 5000\ \mathrm{N/m}; m_a = 0.6\ \mathrm{kg}; m_b = 0.7\ \mathrm{kg} \end{aligned} \tag{45}$$

The displacements of main mass 1 (m_1) and main mass 2 (m_2) are plotted in Figs 7 and 8, respectively. The results are compared with the cases for no absorber and for an absorber with randomly selected parameters. The randomly selected parameters in the plots are

$$c_a = 10\ \mathrm{Ns/m}; c_b = 20\ \mathrm{Ns/m}; k_a = 200\ \mathrm{N/m}; k_b = 400\ \mathrm{N/m} \tag{46}$$

In the optimization package, the tolerances are very narrow so that, regardless of the value of the initial estimate, the output values converge to the same optimum parameters.

The optimal absorber parameters are as follows:

$$c_a = 5.5\ \mathrm{Ns/m}; c_b = 10.7\ \mathrm{Ns/m}; k_a = 585.3\ \mathrm{N/m}; k_b = 1469.1\ \mathrm{N/m} \tag{47}$$

4.2. *Case Study 2* (*with different weighing functions*)

The same system as is displayed in Fig. 5 is used as a multi-input–multi-output (MIMO) system. Two input forces, applied on the two main masses with their respective output main mass displacements, are analyzed.

$$F = [F_1; F_2] = \left[\left|4000/\left((s+40)^2+225\right)\right| ; |100/(s+100)|\right] \tag{48}$$

The values of the input parameter given in Eq. (45) are used.

One can define a mixed-goal with different weighing functions on the displacements of the main masses. Figures 9 and 10 show the displacements of main mass 1 and main mass 2, respectively, with a higher weighing on displacement x_2 than on displacement x_1. Figures 11 and 12 show the reverse situation. As a result, the optimization package is set to work on the user's aim of minimizing the amplitudes in the desired weights. The results for the different weighing assumptions are shown in Figs 9–12. The goal function, expressed as $(0.2x_1 + 0.8x_2)$, is displayed in Figs 9 and 10 for main mass 1 and main mass 2, respectively. In the reverse case, the goal function is expressed as $(0.8x_1 + 0.2x_2)$ and is shown in Figs 11 and 12 for main mass 1 and main mass 2, respectively.

4.3. *Case Study 3* (*with constrained minimization*)

Figure 13 shows an example with constrained minimization. A system of four main masses with three absorbers system is used to verify the constrained minimization approach. The aim is to minimize the vibration amplitudes of the displacement of main mass 3 with the constrained parameters of the damping lower than 10 N s/m and the stiffness lower than 1000 N/m.

The optimal values of the parameters of the absorber are shown in Figs 14 and 15 for the cases with no absorber and with an optimal absorber, respectively.

$$\begin{aligned} &c_a = 4.9Ns/m; c_b = 4.9Ns/m; c_c = 3.8Ns/m \\ &k_a = 574.9N/m; k_b = 336.5N/m; k_c = 213.5N/m \end{aligned} \tag{49}$$

5. Conclusions

A new and efficient method for calculating the system receptance (α) has been developed. The receptance matrix (also called the frequency response matrix) is an important matrix which interrelates the input and output of a damped mechanical system which is subject to external forcing as the input. Although there are many papers on this subject in the vibration literature, to the best of the author's knowledge, there has not been any investigation of the method studied in this article to provide receptance of a translational mass–spring–damper system having N masses with M absorbers, where N and M are any positive integer.

The combined system's receptance can be obtained separately in terms of the parameters of the main system and the absorber system by using the Sherman–Morrison formula. But one cannot use this formula in one-step formulations which have N masses with M absorbers. In general, by applying the Sherman–Morrison formula sequentially (in a series manner), the system receptance is found in terms of the receptance of the system with one less absorber.

Finally, using the receptance method and the newly developed optimal absorber parameter method, one can obtain the optimal parameters of the absorbers of a translational mass–spring–damper system based on specified design criteria and physical constraints.

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