Power delivered to mechanical systems by random vibrations

Timothy S. Edwards

Analytical Structural Dynamics, Org.1523, Sandia National Laboratories, MS0557, P.O. Box 5800, Albuquerque, NM 87185, USA*

E-mail: tsedwar@sandia.gov

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Abstract. This paper develops deformational response power descriptions of multiple degree-of-freedom systems due to stationary random vibration excitation. Two new concepts are developed. The deformational response power density (DRPD) can be computed when a structure's natural frequencies and modal masses are available. The DRPD shows the spectral content of the deformational power delivered to a specific structure by the stationary, random excitation. This function can be found through a weighted windowing of the power spectrum of the input acceleration excitation. Deformational response input power spectra (DRIPS), similar to the input energy spectrum and shock response spectrum, give the power delivered to single-degree-of-freedom systems as a function of natural frequency. It is shown that the DRIPS is simply a smoothed version of the power spectrum of the input acceleration excitation. The DRIPS gives rise to a useful power-based data smoothing operation.

Keywords: Power, energy, random vibration

Nomenclature

d : modal displacement

E_K^R : relative kinetic energy

E_I^R : relative input energy

E_D : dissipated energy

E_A : absorbed energy

f_d : dissipative forces

f_s : restorative forces.

$G(\omega)$: Fourier transform of modal coordinates

H_d : acceleration input, relative displacement output transmissibility

H_v : acceleration input, relative velocity output transmissibility

i : $\sqrt{-1}$

m : mass

M : mass matrix

v : modal velocity

ω : circular frequency (rad/sec)

ω_n : natural frequency (rad/sec)

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$\ddot{x}_g$: ground acceleration
 z : relative displacement
 η : modal coordinate
 Φ : mass-normalized eigenvectors
 Λ : modal participation factor
 Λ^2 : effective modal mass
 ζ : fraction of critical damping
 $\{u\}$: rigid body mode shape
 $[I]$: identity matrix
 $[\cdot]^T$: denotes matrix transpose
 $E[\cdot]$: denotes expected value
 $\Gamma_{\ddot{x}\ddot{x}}$: Auto spectrum of excitation
 P_I^R : Total deformational response power
 $P_I^R(\omega)$: Deformational response power density
 $[N] = \int [\Gamma_{\ddot{x}\ddot{x}}(\omega)] d\omega$

1. Introduction

The use of energy in the analysis of earthquake excitation and response is widely accepted. Hudson [7] and Housner [6] are credited with proposing the use of energy quantities to characterize transient base excitation. Since then, many authors have contributed to the study of building response to earthquake excitation using the framework of energy [8,11,13,14].

The use of these energy methods is much less widely practiced outside of the earthquake community. Because the majority of development on these methods has focused on seismic excitation little emphasis has previously been given to developing them for steady-state excitation. This paper develops deformational response power descriptions of uniaxial, base excited, multiple degree-of-freedom systems due to stationary random vibration excitation. Extending the work of other researchers, who focused on transient, deterministic excitation, to stationary, stochastic excitation is relatively straightforward. In fact, the derivation in this paper follows the form of Ordaz et al. [11] whose work proved the relation between the Fourier transform of a transient, deterministic waveform and the input energy spectrum.

Two new concepts are developed in this work. The deformational response power density (DRPD) can be computed when a structure's particular natural frequencies and modal masses are available. The DRPD shows the spectral content of the deformational power delivered to the structure by the stationary, random excitation. This function can be found through a weighted windowing of the autospectrum of the base acceleration excitation. Integration of the DRPD over all frequencies yields the total deformational power delivered to the structure, and so the DRPD is a true density function. The second new concept, the deformational response input power spectra (DRIPS) presents the power delivered to single-degree-of-freedom (SDOF) systems as a function of natural frequency. This is similar to shock response spectra and input energy spectra where the relevant quantities are also presented as a function of SDOF natural frequency. These functions are most useful when the structure's properties are largely unknown and its natural frequencies are unavailable. This paper shows that the DRIPS is simply a smoothed version of the autospectrum of the base acceleration excitation.

Note that although sometimes interchanged, the terms autospectrum and power spectrum are not strictly equivalent. The autospectrum can be computed for any signal, regardless of dimension. In this paper the term power spectrum is reserved for a quantity whose dimension is truly that of power. It is important to keep this distinction in mind to avoid confusion when discussing deformational response input power spectra and power densities that both depend on the autospectrum of the input acceleration.

2. Energy in base-excited, MDOF systems

A brief review of the energy equations for MDOF systems is presented for completeness. The base-excited equations of motion are then decoupled using modal analysis for use in the subsequent section. References 1, 4, 5, 6, and 11 contain more thorough development of these concepts.

The equations of free vibration are the starting point for analysis of base-driven structures. Thus, the equations of motion governing a multiple-degree-of-freedom structure excited at its base through ground motion are given by

$$[M]\{\ddot{x}\} + \{f_d\} + \{f_s\} = \{0\} \quad (1)$$

where $\{f_d\}$ are the viscous damper forces and $\{f_s\}$ are the restorative forces. These equations define the motion of the system and ground in absolute coordinates. The restorative forces, $\{f_s\}$, can contain a variety of force-displacement properties including elastic, viscoelastic, and hysteretic terms.

Define a vector of the ground motion, $\{x_g\} = \{u\} x_g$ where $\{u\}$ is a $N \times 1$ rigid body mode shape. If a relative coordinate, $\{z\}$, is defined as $\{z\} = \{x\} - \{x_g\}$, then the equation of motion written in terms of this relative coordinate is

$$[M]\{\ddot{z}\} + \{f_d\} + \{f_s\} = -[M]\{\ddot{x}_g\} \quad (2)$$

In terms of the relative coordinates, the work¹ done by the forces acting on the system is found by multiplying both sides of Eq. (2) by dz^T and integrating, yielding

$$\frac{1}{2} \{\dot{z}\}^T [M] \{\dot{z}\} + \int \{\dot{z}\}^T \{f_d\} dt + \int \{\dot{z}\}^T \{f_s\} dt = - \int \{\dot{z}\}^T [M] \{\ddot{x}_g\} dt \quad (3)$$

The following energy quantities are defined in Reference 1.

$$E_K^R = \frac{1}{2} \{\dot{z}\}^T [M] \{\dot{z}\} \quad (4)$$

$$E_D = \int \{\dot{z}\}^T \{f_d\} dt \quad (5)$$

$$E_A = \int \{\dot{z}\}^T \{f_s\} dt \quad (6)$$

$$E_I^R = - \int \{\dot{z}\}^T [M] \{\ddot{x}_g\} dt \quad (7)$$

where E_K^R is the relative kinetic energy, E_D is the dissipated energy, E_A is the absorbed energy and E_I^R is the relative input energy. If the spring forces contain hysteretic terms, then the absorbed energy can be separated into recoverable, elastic strain energy and non-recoverable energy lost to hysteresis. Using this notation the energy balance is

$$E_K^R + E_D + E_A = E_I^R. \quad (8)$$

It will be shown that deformational energy delivered to a MDOF system is a weighted sum of the uncoupled modal contributions where the weighting factors are the modal masses. It can also be shown that the internal storage and loss energies are uncoupled and combined in a weighted sum according to the modal masses.

To decouple the equations of motion through modal analysis [4,5,11] we begin by writing the equations of motion of a base-excited, linear, MDOF system. Linearity imposes a special case of the more general case described by Eq. (2).

$$[M] \{\ddot{z}\} + [C] \{\dot{z}\} + [K] \{z\} = -\ddot{x}_g [M] \{u\} \quad (9)$$

¹Writing these equations in terms of coordinates whose origin is in an accelerating frame results in quantities that are not strictly energy. They will nevertheless be referred to as such.

where $\{u\}$ is a $N \times 1$ rigid body mode shape and the coordinates, $\{z\}$, are defined as the relative displacement between the degrees of freedom and ground.

If $[\Phi]$ is an $N \times N$ matrix whose columns are the mass-normalized mode shapes such that $[\Phi]^T [M] [\Phi] = [I]$ where $[I]$ is the identity matrix, then the modal transformation is $\{z\} = [\Phi] \{\eta\}$ where $\{z\}$ is an $N \times 1$ column vector of physical coordinates and $\{\eta\}$ is an $N \times 1$ column vector of the uncoupled modal coordinates.

Rewriting Eq. (9) in terms of the uncoupled modal coordinates yields

$$[M] [\Phi] \{\ddot{\eta}\} + [C] [\Phi] \{\dot{\eta}\} + [K] [\Phi] \{\eta\} = -\ddot{x}_g [M] \{u\} \quad (10)$$

To further simplify the equations, premultiply both sides by $[\Phi]^T$.

$$[\Phi]^T [M] [\Phi] \{\ddot{\eta}\} + [\Phi]^T [C] [\Phi] \{\dot{\eta}\} + [\Phi]^T [K] [\Phi] \{\eta\} = -\ddot{x}_g [\Phi]^T [M] \{u\} \quad (11)$$

The term $[\Phi]^T [M] \{u\}$ is exactly the modal participation factor, $\{\Lambda\}$ [4]. Sometimes called the earthquake participation factor, the modal participation factor is the projection of the inertial forces on the mode shapes and determines the degree to which the mode will be excited by base excitation. These equations can then be simplified to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix} \{\ddot{\eta}\} + \begin{bmatrix} 2\zeta_1 \Omega_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 2\zeta_N \Omega_N \end{bmatrix} \{\dot{\eta}\} + \begin{bmatrix} \Omega_1^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \Omega_N^2 \end{bmatrix} \{\eta\} = -\ddot{x}_g \{\Lambda\} \quad (12)$$

where the modal participation factor is given by

$$\{\Lambda\} = [\Phi]^T [M] \{u\} \quad (13)$$

Note the assumption that the viscous damping matrix is of an appropriate form to be represented by modal damping. Then, take the Fourier transform of Eq. (12) to yield

$$\left[\begin{bmatrix} -\omega^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & -\omega^2 \end{bmatrix} + \begin{bmatrix} 2\zeta_1 \Omega_1 i\omega & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 2\zeta_N \Omega_N i\omega \end{bmatrix} + \begin{bmatrix} \Omega_1^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \Omega_N^2 \end{bmatrix} \right] \{G(\omega)\} = -\ddot{X}_g(\omega) \{\Lambda\}. \quad (14)$$

Rearranging Eq. (14) yields

$$\{G(\omega)\} = \ddot{X}_g(\omega) \begin{bmatrix} \frac{-1}{-\omega^2 + 2\zeta_1 i\omega \Omega_1 + \Omega_1^2} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{-1}{-\omega^2 + 2\zeta_N i\omega \Omega_N + \Omega_N^2} \end{bmatrix} \{\Lambda\}. \quad (15)$$

In shortened notation Eq. (15) may be written as

$$\{G(\omega)\} = \ddot{X}_g(\omega) [H_d(\omega)] \{\Lambda\} \quad (16)$$

where $[H_d(\omega)]$ is a diagonal matrix whose elements are the FRFs that relate input force excitation and relative displacement response for SDOF oscillators with parameters equal to the modal parameters of the MDOF system and $\{G(\omega)\}$ is a column vector whose terms are the Fourier transforms of the displacements of the modal degrees of freedom. This result will be used in the subsequent section.

3. Input power of random vibrations

The relative input energy for an integrable, deterministic input is given by Eq. (7) as

$$E_I^R = - \int \{\dot{z}\}^T [M] \{\ddot{x}_g\} dt.$$

For stationary random vibration, the input energy is unbounded and the integral in the above equation does not exist. So, we integrate over a finite time period, τ , as the integral will exist for finite time periods.

$$E_I^R(\tau) = - \int_0^\tau \{\dot{z}(t)\}^T [M] \{\ddot{x}_g(t)\} dt \quad (17)$$

The expected relative input energy per unit time is the relative input power, P_I^R , of the stationary random vibration excitation. Accordingly, P_I^R is defined by taking the expected value of E_I^R , dividing by τ , and taking the limit as τ goes to infinity.

$$P_I^R = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} E[E_I^R(\tau)] \quad (18)$$

where $E[\cdot]$ denotes expected value. The relative input power can now be written as

$$P_I^R = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} E \left[- \int_0^\tau \{\dot{z}(t)\}^T [M] \{\ddot{x}_g(t)\} dt \right]. \quad (19)$$

The modal coordinate transformation, $\{z\} = [\Phi] \{\eta\}$, can be written in the frequency domain as $\{Z\} = [\Phi] \{G\}$. Differentiating to relate the modal velocities to the input excitation, and making use of Eq. (16) yields

$$\{\dot{Z}(\omega, \tau)\} = i\omega \ddot{X}_g(\omega, \tau) [\Phi] [H_d(\omega)] \{\Lambda\} \quad (20)$$

or equivalently,

$$\{\dot{Z}(\omega, \tau)\} = \ddot{X}_g(\omega, \tau) [\Phi] [H_v(\omega)] \{\Lambda\} \quad (21)$$

where $\{\dot{Z}(\omega, \tau)\}$ is the finite Fourier transform of $\{\dot{z}(t)\}$ and $\ddot{X}_g(\omega, \tau)$ is the finite Fourier transform of $\ddot{x}_g(t)$, both taken over the interval τ .

Writing the inverse finite Fourier transform of Eq. (21) yields

$$\{\dot{z}(t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \ddot{X}_g(\omega, \tau) [\Phi] [H_v(\omega)] \{\Lambda\} e^{j\omega t} d\omega. \quad (22)$$

Substituting this result into Eq. (19) yields

$$P_I^R = - \lim_{\tau \rightarrow \infty} \frac{1}{\tau} E \left[\int_0^\tau \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \ddot{X}_g(\omega, \tau) [\Phi] [H_v(\omega)] \{\Lambda\} e^{j\omega t} d\omega \right]^T [M] \{\ddot{x}_g(t)\} dt \right]. \quad (23)$$

To begin simplifying the above equation, recognize that $[H_v]$ is diagonal and so that $[H_v] = [H_v]^T$. Also recall that $\{\ddot{x}_g(t)\} = \{u\} \ddot{x}_g(t)$. This leads to the expression

$$P_I^R = - \frac{1}{2\pi} \{\Lambda\}^T \lim_{\tau \rightarrow \infty} \frac{1}{\tau} E \left[\int_0^\tau \int_{-\infty}^{\infty} \ddot{X}_g(\omega, \tau) [H_v(\omega)] e^{j\omega t} d\omega [\Phi]^T [M] \{u\} \ddot{x}_g(t) dt \right]. \quad (24)$$

Making use of the modal participation factor to simplify gives

$$P_I^R = - \frac{1}{2\pi} \{\Lambda\}^T \lim_{\tau \rightarrow \infty} \frac{1}{\tau} E \left[\int_0^\tau \int_{-\infty}^{\infty} \ddot{X}_g(\omega, \tau) [H_v(\omega)] e^{j\omega t} d\omega \ddot{x}_g(t) dt \right] \{\Lambda\}. \quad (25)$$

Separating the integrals is permitted because none of the variables are dependant on both time and frequency. This separation results in

$$P_I^R = - \frac{1}{2\pi} \{\Lambda\}^T \lim_{\tau \rightarrow \infty} \frac{1}{\tau} E \left[\int_{-\infty}^{\infty} \ddot{X}_g(\omega, \tau) [H_v(\omega)] d\omega \int_0^\tau \ddot{x}_g(t) e^{j\omega t} dt \right] \{\Lambda\}. \quad (26)$$

Invoking the definition of the finite Fourier transform, recognize that $\ddot{X}^*(\omega, \tau) = \int_0^\tau \ddot{x}_g(t) e^{j\omega t} dt$. Substitution of this result yields

$$P_I^R = -\frac{1}{2\pi} \{\Lambda\}^T \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \mathbb{E} \left[\int \left| \ddot{X}_g(\omega, \tau) \right|^2 [H_v(\omega)] d\omega \right] \{\Lambda\}. \quad (27)$$

This result is the stationary, stochastic, multiple degree-of-freedom equivalent to the results that Ordaz, et al. [12] obtained for single degree-of-freedom response to transient, deterministic excitation.

The autospectrum of the base motion can be defined from finite Fourier transforms [12] as

$$\Gamma_{\ddot{x}\ddot{x}}(\omega) = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \mathbb{E} \left| \ddot{X}_g(\omega, \tau) \right|^2 \quad (28)$$

so that

$$P_I^R = -\frac{1}{2\pi} \{\Lambda\}^T \int \Gamma_{\ddot{x}\ddot{x}}(\omega) [H_v(\omega)] d\omega \{\Lambda\}. \quad (29)$$

The quantity inside the integral is real since $\Gamma_{\ddot{x}\ddot{x}}$ is real and the imaginary part of $[H_v]$ is odd while the real part is even. This result states that the energy delivered to a single mode is given by a windowed version of the autospectrum of the excitation, weighted by the appropriate modal mass. The windowing function is the input acceleration, velocity output FRF of a base-excited SDOF oscillator.

Recognizing that $\Gamma_{\ddot{x}\ddot{x}}(\omega) [H_v(\omega)] = \Gamma_{\ddot{x}\dot{\eta}}(\omega)$ Eq. (29) can be rewritten to give

$$P_I^R = -\frac{1}{2\pi} \{\Lambda\}^T \int [\Gamma_{\ddot{x}\dot{\eta}}(\omega)] d\omega \{\Lambda\}. \quad (30)$$

Notice that the quantity $[\Gamma_{\ddot{x}\dot{\eta}}]$ is diagonal and so the power delivered to a structure by random vibration is uncoupled between modes. The term $\int [\Gamma_{\ddot{x}\dot{\eta}}(\omega)] d\omega$ is also a diagonal matrix, whose terms contain the mean square value of the spectral density between the excitation acceleration and the modal velocities. Defining $[N] = \int [\Gamma_{\ddot{x}\dot{\eta}}(\omega)] d\omega$ allows writing Eq. (30) as

$$P_I^R = -\frac{1}{2\pi} \sum_{j=1}^N \Lambda_j^2 N_{j,j} \quad (31)$$

where $N_{j,j}$ is the mean square value of the spectral density of the input acceleration excitation and the j 'th modal velocity.

Equation (31) states that the input energy delivered to a MDOF system is equal to the weighted sum of the energy delivered to each uncoupled mode. The weighting factors are the participating effective modal masses, Λ_j^2 .

3.1. The deformational response power density

If a structure's modal properties are known, the deformational power delivered to it can be found using Eq. (31), which can be rewritten as shown below.

$$P_I^R = -\frac{1}{2\pi} \int \sum_{j=1}^N \Gamma_{\ddot{x}\ddot{x}}(\omega) \Lambda_j^2 H_{v,j}(\omega) d\omega \quad (32)$$

The quantity $-\frac{1}{2\pi} \sum_{j=1}^N \Gamma_{\ddot{x}\ddot{x}}(\omega) \Lambda_j^2 H_{v,j}(\omega)$ that appears in this equation can be interpreted as a power density as a function of frequency, $P_I^R(\omega)$, as shown below.

$$P_I^R(\omega) = -\frac{1}{2\pi} \sum_{j=1}^N \Gamma_{\ddot{x}\ddot{x}}(\omega) \Lambda_j^2 H_{v,j}(\omega) \quad (33)$$

$P_I^R(\omega)$ is the density of the total deformational power delivered to all modes of the structure as a function of frequency. The integral of this density is the total power delivered to the structure.

Table 1
Modal parameters

Mode	Natural freq. (Hz)	Modal mass (slug)	Power absorbed (ft-lb /sec)	Percent power absorbed
1	441	4.23×10^{-2}	5.45×10^{-2}	20.6%
2	445	4.23×10^{-2}	5.58×10^{-2}	21.1%
3	605	9.31×10^{-5}	0.79×10^{-3}	0.30%
4	1528	3.50×10^{-2}	6.08×10^{-1}	23.0%
5	1628	2.51×10^{-3}	1.06×10^{-2}	4.0%
6	1774	2.17×10^{-4}	2.39×10^{-2}	0.90%
7	1785	7.36×10^{-3}	7.86×10^{-1}	29.7%
8	1807	9.31×10^{-5}	8.90×10^{-3}	0.33%

Fig. 1. Graphical representation.

Some physical significance of the above equations can be offered by representing them graphically. As an example, consider the structure whose first eight fixed-base modes are described in Table 1. The modal parameters were found from a finite element solution and each mode is assumed to have five percent critical damping. It is desired to predict which modes will absorb the greatest deformational power in response to a random vibration excitation that was measured in the structure's operational environment. The excitation autospectrum and the modal mass-weighted FRFs are shown in Fig. 1.

Plotting the excitation autospectrum with the modal mass-weighted FRFs emphasizes the frequencies of excitation that will impart the greatest deformational power as well as the modes absorbing the greatest share. In this example, the first two, closely-spaced modes have the highest modal mass, but are aligned with a trough in the excitation autospectrum. The fourth mode is somewhat near a peak in the excitation and has a high modal mass. Therefore, it absorbs a high fraction of the total deformational power. The seventh mode is very closely aligned with the peak in the excitation autospectrum at 1763 Hz. Despite its relatively low modal mass, this mode absorbs nearly one third of the total deformational power.

The deformational response power density, $P_I^R(\omega)$, is plotted in Fig. 2. The resemblance to the input autospectrum is strong, but the effects of the structure's modal properties are also apparent.

Fig. 2. Deformational response power density.

3.2. The deformational response input power spectrum

When a structure's specific modal properties are unknown, computations can be made for single-degree-of-freedom oscillators of unit mass, such as is done for the shock response spectrum and the input energy spectrum. The deformational power per unit mass can easily be found for a range of SDOF oscillators with varying natural frequency. For a single degree of freedom oscillator of given natural frequency and damping, Eq. (29) reduces to

$$\frac{P_I^R}{m} = -\frac{1}{2\pi} \int \Gamma_{\ddot{x}\ddot{x}}(\omega) H_v(\omega; \omega_n, \zeta) d\omega \quad (34)$$

We wish to plot the scalar quantity represented by Eq. (34) as a function of natural frequency. The result is the deformational response input power spectrum (DRIPS). Equation (34) states that the input power for a single oscillator can be found from a windowed version of the excitation autospectrum, it is then possible to see that the DRIPS is a smoothed version of the excitation autospectrum. The smoothing function is dependant on both natural frequency and damping, as shown in the equation below.

$$\frac{P_I^R(\omega_n; \zeta)}{m} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma_{\ddot{x}\ddot{x}}(\Omega) H_v(\Omega - \omega_n; \omega_n, \zeta) d\Omega \quad (35)$$

Note that for zero damping, H_v becomes an impulse centered at ω_n . The value of the impulse is such that $\int_{-\infty}^{\infty} H_v(\omega; \omega_n, \zeta) d\omega = -\pi$, regardless of natural frequency or damping. Therefore,

$$\frac{P_I^R(\omega_n; 0)}{m} = \frac{1}{2} \Gamma_{\ddot{x}\ddot{x}}(\omega_n). \quad (36)$$

The DRIPS, $\frac{P_I^R}{m}(\omega_n; 0.05)$, was computed for the same excitation autospectrum as shown in Fig. 1. The results are shown in Fig. 3.

3.3. Power-based data smoothing

The DRIPS gives rise to a power-based smoothing algorithm for measured test data. Traditionally, random vibration data collected on aerospace structures is smoothed using a fractional-octave band analysis before being

Fig. 3. Deformational response input power spectrum.

converted to test specifications. It has been shown [2,12] that the optimum bandwidth for minimizing the sum of the squares of the random and bias errors in the spectral estimate is proportional to the center frequency, assuming constant damping ratio.² Piersol [12] suggests that 1/12th octave averaging is near optimum for minimum-error spectral estimation in most scenarios. However, in some cases the data will be used as the basis for test input specification for base-excited systems rather than for spectral estimation alone. In these cases it is desirable to generate smoothed versions of the measured data that both have the same damage potential as the measured data and minimize the dynamic range of the test.

Equation (35) also describes a proportional bandwidth smoothing operation, but the smoothing function is H_v instead of the moving average kernel as would traditionally be used. The smoothing kernel H_v conserves the deformational power, which represents the damage potential, of the excitation delivered to a base-excited structure. When two autospectra have equal DRIPS then their damage potential is also equal, independent of the actual natural frequencies of the structure under test. The asymptotic behavior of H_v yields an upper bound on the dynamic range (as a function of the spacing and relative magnitudes of the peaks in the autospectrum of the excitation and the assumed damping) required to produce a test of equal damage potential. This effect is shown in the example below.

Numerical procedures can be used to arrive at a smoothed autospectrum that has approximately equal damage potential and minimizes the dynamic range (under the assumed, constant damping) of the test specification. Any number of error terms can be minimized when iterating on the data. The following example minimized the squared difference between the DRIPS of the raw data and the DRIPS of the smoothed data at every natural frequency, as shown in Eq. (37).

$$\varepsilon = \left| \frac{P_I^R}{m} (\omega_n; 0.05)_{original} - \frac{P_I^R}{m} (\omega_n; 0.05)_{smoothed} \right|^2 \quad (37)$$

The MATLAB [9] function *fminbnd* was used to minimize this error term at every natural frequency for the example data. Of course, there is a trade-off between the smoothness of the results and the match to the DRIPS of the original data. The results of the smoothing operation are shown in Fig. 4. The DRIPS of the original and smoothed data are shown in Fig. 5.

²It should be noted that the optimum bandwidth changes at every frequency line. Therefore this discussion presumes a fractional-octave frequency smoothing operation whose center frequencies occur at every frequency and are not fixed, such as described in [1].

Fig. 4. Original and smoothed autospectra.

Fig. 5. DRIPS of original and smoothed data.

3.4. Summary

One of the most attractive properties of the energy quantities originally developed to describe shock excitation is the elegant mathematical framework in which they reside. This framework allows them to describe any type of excitation, transient or stationary. This paper has developed the measure of deformational input power due to stationary random vibration. Two descriptions of the available deformational power arise: the deformational response power density (DRPD) and the deformational response input power spectrum (DRIPS).

The DRPD can be found when a structure's modal parameters are known. It represents the spectral density of power that results in deformation absorbed by a specific structure through base excitation. The integral of the DRPD over all frequencies is the total deformational power imparted to a base-excited structure.

The DRIPS can be used when the structures modal properties are unknown. The DRIPS represents the deformational power absorbed by single degree-of-freedom oscillators of unit mass. The DRIPS is simply a smoothed version of the excitation autospectrum. Its integral is meaningless.

The DRIPS gives rise to a power-based data smoothing operation. This smoothing operation can be used on measured data to create input test specifications for base-excited systems. Under the assumption of constant damping, the resulting autospectra will have approximately equal damage potential to the measured data and will minimize the dynamic range of the test.

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