Free in-plane vibration of super-elliptical plates

Murat Altekin

Department of Civil Engineering, Yildiz Technical University, 34210 Esenler, Istanbul, Turkey
Tel.: +90 212 3835148; Fax: +90 212 3835133; E-mail: altekin@yildiz.edu.tr

Received 4 March 2008

Revised 17 April 2009

Abstract. Free in-plane vibration of super-elliptical plates of uniform thickness was investigated by the Ritz method. A large variety of plate shapes ranging from an ellipse to a rectangle were examined. Two cases were considered: (1) a completely free, and (2) a point-supported plate. The geometrical boundary conditions were satisfied by the Lagrange multipliers. The results were compared with those of rectangular plates. Basically good agreement was obtained. Matching results were reported, and the discrepancies were highlighted.

Keywords: Super-elliptical, plate, Ritz, in-plane, vibration

1. Introduction

In the vast literature devoted to the dynamic analysis of plates [1–38], transverse vibration has received significant attention and the solutions have been available for a wide range of shapes and boundary conditions [21]. This is probably partly because transverse motion is prone to excitation from external sources [16], and partly because the natural frequencies of plates in lateral vibration are generally much lower than those of plates in in-plane vibration [25]. However, in-plane modes can be practically important in aerospace structures [21], and in the design of ship hulls [24].

The studies on the in-plane vibration of plates have basically focused on isotropic rectangular plates solved numerically by various methods [1–8, 21–25]. As far as the author knows the unique paper examining the in-plane vibration of rectangular plates with intermediate point-supports was submitted by Kobayashi et al. [1]. They investigated the point-supported rectangular plates and presented the eigenvalues of a completely free plate as well. Bardell et al. [2] dealt with rectangular plates with either completely free or constrained edges. Higher modes of rectangular plates were presented by Hyde et al. [3]. An improved Fourier series method was utilized by Du et al. [4] to solve rectangular plates with elastically restrained edges. Gorman [5, 6] employed the superposition method to analyze rectangular plates. Wang and Wereley [21] presented the solutions of a rectangular plate with constrained edges by the Kantorovich-Krylov method. Larsson [23] examined a free isotropic plate experimentally.

The present study was motivated by the scarcity of contributions on the in-plane vibrations of super-elliptical plates [30–38]. The fundamental (lowest) frequency was obtained and the optimum support location¹ was searched. Two cases were handled: (1) a completely free plate, (2) a plate with symmetrically distributed four point-supports on the diagonals. The Ritz method – one of the most prevalent methods used in applied mechanics (e.g., Refs. [1–3]) – was employed, and the geometrical boundary conditions were satisfied by the Lagrange multipliers. The results obtained in the present study were checked with those of rectangular plates.² Almost identical values were found for the first case. Good agreement was obtained for SS modes for the second case. However, lower frequencies were reported for SA, and AA modes. The corresponding mode shapes were demonstrated.

¹Optimum location of the support is defined by the support position which yields the highest fundamental frequency.

²The results obtained from Refs. [1–4] were scaled – where necessary – by Murat Altekin.

Fig. 1. Geometry of the plate.

2. Mathematical formulation

The perimeter of a homogeneous, isotropic, flat, thin super-elliptical elastic plate of uniform thickness h is defined by the equation given by

$$\left(\frac{x}{a}\right)^{2k} + \left(\frac{y}{b}\right)^{2k} = 1, \quad k = 1, 2, \dots, \infty \quad (1)$$

where a , b , and k denote the semi-major axis, the semi-minor axis, and the super-elliptical power respectively [32, 38].

The positions of the point-supports are introduced by

$$d_x = \Delta_1 a, \quad d_y = \Delta_2 b \quad (2)$$

where d_x and d_y denote the distance [39] from the central axes of the plate (Fig. 1).

The strain energy (U_S), the kinetic energy (T), and the governing equations (L_1 , L_2) of the plate are written in terms of u and v -the in-plane displacements in x -direction and y -direction, respectively-, as [1,6]

$$U_S = \frac{D}{2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left[\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 - 2(1-\mu) \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{1-\mu}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] dx dy, \quad (3)$$

$$T = \frac{1}{2} \rho h \omega^2 \int_{x_1}^{x_2} \int_{y_1}^{y_2} (u^2 + v^2) dx dy, \quad (4)$$

$$L_1 = C_{11} \frac{\partial^2 u}{\partial x^2} + C_{12} \frac{\partial^2 v}{\partial x \partial y} + C_{66} \left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} \right) + \rho u \omega^2 = 0, \quad (5)$$

$$L_2 = C_{66} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} \right) + C_{12} \frac{\partial^2 u}{\partial x \partial y} + C_{22} \frac{\partial^2 v}{\partial y^2} + \rho v \omega^2 = 0. \quad (6)$$

Here; ρ denotes the mass density, ω indicates the frequency, and μ stands for the Poisson's ratio. The coefficients (C_{11} , C_{12} , C_{22} , C_{66}) are given by [6]

$$C_{11} = C_{22} = \frac{E}{(1 - \mu^2)}, \quad C_{12} = \frac{\mu E}{(1 - \mu^2)}, \quad C_{66} = \frac{E}{2(1 + \mu)}. \quad (7)$$

The extensional rigidity D , the aspect ratio c , and the non-dimensional counterparts of x , y , u , v , ω are introduced by

$$D = \frac{Eh}{(1 - \mu^2)}, \quad c = \frac{a}{b}, \quad X = \frac{x}{a}, \quad Y = \frac{y}{b}, \quad U = \frac{u}{a}, \quad V = \frac{v}{b}, \quad \lambda^2 = a^2 \frac{\rho h}{D} \omega^2 \quad (8)$$

where E is the Young's modulus. The plate domain can be characterized by

$$x_1 = -a, \quad x_2 = a, \quad y_1 = -\frac{1}{c} \sqrt{a^{2k} - x^{2k}}, \quad y_2 = \frac{1}{c} \sqrt{a^{2k} - x^{2k}}. \quad (9)$$

Due to the symmetry of the geometry of the plate, it is sufficient to consider only a quarter of the plate [1]. Therefore, U_S , T , L_1 and L_2 can be written as

$$U_S = \frac{1}{2} abD \int_0^{X_2} \int_0^{Y_2} \left[\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} \right)^2 - 2(1 - \mu) \frac{\partial U}{\partial X} \frac{\partial V}{\partial Y} + \frac{1 - \mu}{2} \left(\frac{1}{c} \frac{\partial V}{\partial X} + c \frac{\partial U}{\partial Y} \right)^2 \right] dX dY, \quad (10)$$

$$T = \frac{1}{2} abD \lambda^2 \int_0^{X_2} \int_0^{Y_2} \left(U^2 + \frac{1}{c^2} V^2 \right) dX dY, \quad (11)$$

$$L_1 = \frac{\partial^2 U}{\partial X^2} + \mu \frac{\partial^2 V}{\partial X \partial Y} + \frac{(1 - \mu)}{2} \left(\frac{\partial^2 V}{\partial X \partial Y} + c^2 \frac{\partial^2 U}{\partial Y^2} \right) + U \lambda^2 = 0, \quad (12)$$

$$L_2 = \frac{(1 - \mu)}{2} \left(\frac{1}{c} \frac{\partial^2 V}{\partial X^2} + c \frac{\partial^2 U}{\partial X \partial Y} \right) + \mu c \frac{\partial^2 U}{\partial X \partial Y} + c \frac{\partial^2 V}{\partial Y^2} + \frac{1}{c} V \lambda^2 = 0 \quad (13)$$

where

$$X_2 = 1, \quad Y_2 = \sqrt[2k]{1 - X^{2k}}. \quad (14)$$

To improve the accuracy of the results the total residual error is set to zero by

$$G_1 = \int_0^{X_2} \int_0^{Y_2} L_1 dX dY = 0, \quad G_2 = \int_0^{X_2} \int_0^{Y_2} L_2 dX dY = 0. \quad (15)$$

The shape functions U and V are complete two-dimensional polynomial functions of degree d defined by [1]

$$U = \sum_{m=0}^M \sum_{n=0}^N A_{mn} X^m Y^n, \quad V = \sum_{m=0}^M \sum_{n=0}^N B_{mn} X^m Y^n, \quad d = m + n. \quad (16)$$

Here; A_{mn} and B_{mn} are the unknown coefficients. Depending on the mode category, the shape functions are constructed (Table 1). Since the in-plane displacement is prevented at the point-supports, the subsidiary conditions [40] can be written as

$$G_3 = U(X = \Delta_1, Y = \Delta_2), \quad G_4 = V(X = \Delta_1, Y = \Delta_2). \quad (17)$$

Table 1
Construction of the shape functions [1]

Mode	U		V	
	m	n	m	n
SS	odd	even	even	odd
SA	even	even	odd	odd
AS	odd	odd	even	even
AA	even	odd	odd	even

Table 2
Convergence study for λ ($k = 200$)

c	Mode	d	Case 1			Case 2 ($\Delta = 0.5$)			Case 2 ($\Delta = 0.9$)		
			Mode-1	Mode-2	Mode-3	Mode-1	Mode-2	Mode-3	Mode-1	Mode-2	Mode-3
1	SS	11	1.3143	1.4938	1.7263	1.4890	2.0687	2.2726	1.3168	2.0956	2.4490
		13	1.3143	1.4938	1.7263	1.4888	2.0177	2.2424	1.3168	2.0858	2.4313
		15	1.3143	1.4938	1.7263	1.4886	1.9538	2.2151	1.3168	2.0763	2.4139
1	SA	10	1.2359	1.8616	2.4857	1.3253	1.7635	2.4463	0.7099	1.8296	2.2346
		12	1.2359	1.8616	2.4845	1.3052	1.5973	2.3834	0.6943	1.8188	2.2160
		14	1.2359	1.8616	2.4844	1.2996	1.5574	2.3616	0.6808	1.8079	2.2000
1	AA	11	1.1604	2.1516	2.6285	1.1713	2.0554	2.6285	1.0392	1.7270	2.5920
		13	1.1604	2.1516	2.6285	1.1395	2.0118	2.6285	1.0247	1.6941	2.5904
		15	1.1604	2.1516	2.6285	1.0918	1.9614	2.6285	1.0114	1.6762	2.5888
2	SS	11	1.4805	2.6023	3.0734	2.5228	2.8713	3.2863	1.8798	3.1506	3.4352
		13	1.4805	2.6023	3.0734	2.4909	2.8017	3.2857	1.8668	3.1475	3.4183
		15	1.4805	2.6023	3.0734	2.4454	2.7448	3.2850	1.8590	3.1448	3.4079
2	SA	10	2.6285	2.6827	3.3746	2.5030	2.8361	3.0762	1.0288	2.6333	3.4964
		12	2.6285	2.6826	3.3746	2.2686	2.7389	3.0387	1.0002	2.6332	3.4911
		14	2.6285	2.6826	3.3746	2.2083	2.7293	3.0302	0.9826	2.6331	3.4865
2	AS	10	0.9770	2.3922	3.2285	1.3042	1.7665	3.1676	0.8118	2.2264	3.2996
		12	0.9770	2.3921	3.2240	1.2558	1.6888	3.0556	0.7972	2.2012	3.2640
		14	0.9770	2.3921	3.2238	1.2440	1.6615	3.0047	0.7817	2.1591	3.2346
2	AA	11	1.6336	2.3633	3.4289	1.3474	2.9460	3.2347	1.4300	2.4166	3.1466
		13	1.6336	2.3633	3.4286	1.3195	2.8818	3.1876	1.4068	2.3987	3.0983
		15	1.6336	2.3633	3.4286	1.2781	2.8126	3.1189	1.3818	2.3831	3.0539

Consequently, the modified total potential energy functional of the plate becomes

$$F = U_S - T + \beta_1 G_1 + \beta_2 G_2 + \eta (\beta_3 G_3 + \beta_4 G_4) \quad (18)$$

where $\beta_1, \beta_2, \beta_3, \beta_4$ are the Lagrange multipliers. The scalar indicator [35] η is designated as follows:

- $\eta = 0$ for a completely free plate (Case 1),
- $\eta = 1$ for a point-supported plate (Case 2).

The Ritz method is applied by

$$\frac{\partial F}{\partial A_{mn}} = 0, \quad \frac{\partial F}{\partial B_{mn}} = 0, \quad \frac{\partial F}{\partial \beta_1} = 0, \quad \frac{\partial F}{\partial \beta_2} = 0, \quad \frac{\partial F}{\partial \beta_3} = 0, \quad \frac{\partial F}{\partial \beta_4} = 0. \quad (19)$$

The formulation becomes

$$[K_3] = [K_1] - \lambda^2 [K_2], \quad [K_3] \{s\} = \{0\} \quad (20)$$

where $\{s\}$ is the vector composed of the unknowns $A_{mn}, B_{mn}, \beta_1, \beta_2, \beta_3, \beta_4$. The square matrices $[K_1]$, and $[K_2]$ are obtained by using Eq. (19). For a non-trivial solution, the determinant of the coefficient matrix $[K_3]$ is set to zero to obtain λ .

3. Numerical results

The integration with respect to “ X ” was evaluated by the 12-point Gaussian quadrature technique. The free in-plane vibration modes were classified into SS, SA, AS, and AA groups. The convergence studies reveal that a

Table 3
Frequency parameter λ (Case 1, $c = 1$)

$c = 1$	$d = 15$			$d = 14$			$d = 15$		
k	SS-1	SS-2	SS-3	SA-1	SA-2	SA-3	AA-1	AA-2	AA-3
1	1.3892	2.0500	2.5125	1.6177	2.1303	3.3771	1.3894	2.5115	2.7749
2	1.3261	1.8522	1.8617	1.4078	1.9592	2.8991	1.2351	2.5062	2.6493
3	1.3178	1.6880	1.7932	1.3302	1.9161	2.7264	1.1981	2.3857	2.6543
4	1.3157	1.6151	1.7675	1.2954	1.8970	2.6458	1.1832	2.3141	2.6402
5	1.3149	1.5770	1.7541	1.2769	1.8865	2.6003	1.1756	2.2701	2.6338
6	1.3147	1.5545	1.7463	1.2657	1.8800	2.5718	1.1713	2.2417	2.6313
7	1.3145	1.5400	1.7413	1.2586	1.8758	2.5525	1.1685	2.2222	2.6300
8	1.3143	1.5302	1.7380	1.2538	1.8728	2.5390	1.1666	2.2084	2.6294
9	1.3143	1.5232	1.7357	1.2504	1.8707	2.5291	1.1653	2.1982	2.6291
10	1.3143	1.5179	1.7339	1.2478	1.8692	2.5217	1.1645	2.1904	2.6289
11	1.3143	1.5139	1.7326	1.2458	1.8679	2.5159	1.1638	2.1845	2.6288
12	1.3143	1.5110	1.7316	1.2444	1.8669	2.5114	1.1632	2.1798	2.6287
13	1.3143	1.5085	1.7308	1.2432	1.8663	2.5077	1.1628	2.1760	2.6286
14	1.3143	1.5065	1.7302	1.2422	1.8656	2.5047	1.1625	2.1729	2.6286
15	1.3143	1.5050	1.7296	1.2414	1.8652	2.5023	1.1621	2.1704	2.6286
16	1.3143	1.5037	1.7292	1.2408	1.8647	2.5002	1.1619	2.1682	2.6286
17	1.3143	1.5025	1.7289	1.2402	1.8644	2.4985	1.1619	2.1663	2.6285
18	1.3143	1.5015	1.7286	1.2398	1.8641	2.4970	1.1617	2.1648	2.6285
19	1.3143	1.5008	1.7283	1.2394	1.8639	2.4958	1.1615	2.1636	2.6285
20	1.3143	1.5002	1.7281	1.2390	1.8636	2.4947	1.1615	2.1624	2.6285
50	1.3143	1.4947	1.7264	1.2363	1.8619	2.4859	1.1606	2.1531	2.6285
200	1.3143	1.4938	1.7263	1.2359	1.8616	2.4844	1.1604	2.1516	2.6285
Ref.									
[1]	1.3140	1.4940	1.7260				1.1600		
[2]	1.3140	1.4935	1.7260	1.2360			1.1605		

Table 4
Frequency parameter λ (Case 1, $c = 2$)

$c = 2$	$d = 15$			$d = 14$			$d = 14$			$d = 15$		
k	SS-1	SS-2	SS-3	SA-1	SA-2	SA-3	AS-1	AS-2	AS-3	AA-1	AA-2	AA-3
1	1.7101	3.2295	3.6598	2.8418	3.4679	4.1233	1.2538	2.8283	3.7467	1.9836	2.6012	3.6831
2	1.5601	3.0004	3.2599	2.6700	3.3060	3.5942	1.0890	2.5978	3.4461	1.7779	2.4893	3.5781
3	1.5209	2.8705	3.1650	2.6415	3.0707	3.4845	1.0368	2.5095	3.3624	1.7133	2.4361	3.5249
4	1.5048	2.7879	3.1270	2.6338	2.9372	3.4475	1.0141	2.4678	3.3200	1.6842	2.4105	3.4954
5	1.4967	2.7367	3.1084	2.6311	2.8619	3.4270	1.0022	2.4448	3.2946	1.6685	2.3962	3.4775
6	1.4920	2.7036	3.0980	2.6299	2.8158	3.4139	0.9952	2.4308	3.2781	1.6591	2.3876	3.4658
7	1.4891	2.6812	3.0915	2.6294	2.7853	3.4051	0.9910	2.4217	3.2667	1.6530	2.3819	3.4578
8	1.4871	2.6654	3.0873	2.6290	2.7642	3.3989	0.9879	2.4154	3.2585	1.6489	2.3779	3.4520
9	1.4858	2.6539	3.0844	2.6289	2.7490	3.3944	0.9856	2.4109	3.2524	1.6459	2.3752	3.4478
10	1.4848	2.6452	3.0823	2.6287	2.7376	3.3909	0.9841	2.4076	3.2478	1.6438	2.3731	3.4446
11	1.4841	2.6385	3.0807	2.6287	2.7288	3.3884	0.9829	2.4050	3.2441	1.6421	2.3715	3.4421
12	1.4836	2.6332	3.0795	2.6286	2.7220	3.3863	0.9821	2.4030	3.2413	1.6407	2.3702	3.4401
13	1.4831	2.6290	3.0786	2.6286	2.7166	3.3847	0.9813	2.4015	3.2390	1.6398	2.3692	3.4385
14	1.4827	2.6255	3.0778	2.6286	2.7122	3.3833	0.9808	2.4002	3.2371	1.6389	2.3684	3.4372
15	1.4824	2.6227	3.0773	2.6286	2.7086	3.3822	0.9803	2.3992	3.2355	1.6383	2.3678	3.4361
16	1.4822	2.6203	3.0768	2.6285	2.7055	3.3813	0.9798	2.3983	3.2342	1.6377	2.3673	3.4353
17	1.4821	2.6183	3.0764	2.6285	2.7031	3.3806	0.9795	2.3976	3.2330	1.6372	2.3667	3.4345
18	1.4819	2.6167	3.0760	2.6285	2.7008	3.3799	0.9793	2.3971	3.2321	1.6369	2.3664	3.4339
19	1.4817	2.6152	3.0757	2.6285	2.6991	3.3794	0.9790	2.3966	3.2313	1.6365	2.3661	3.4334
20	1.4816	2.6140	3.0755	2.6285	2.6975	3.3790	0.9788	2.3961	3.2306	1.6362	2.3658	3.4329
50	1.4807	2.6039	3.0736	2.6285	2.6848	3.3752	0.9772	2.3927	3.2248	1.6340	2.3636	3.4291
200	1.4805	2.6023	3.0734	2.6285	2.6826	3.3746	0.9770	2.3921	3.2238	1.6336	2.3633	3.4286
Ref.												
[1]	1.4807	2.6022					0.9768	2.3914		1.6334	2.3632	
[2]	1.4805	2.6025					0.9770	2.3920		1.6335	2.3630	

Table 5
Comparison of λ for the first six modes (Case 1, $k = 200$)

c	Mode number					
	1	2	3	4	5	6
1	1.1604	1.2359	1.2359	1.3143	1.4938	1.7263
	1.1605 [2]	1.2360 [2]	1.2360 [2]	1.3140 [2]	1.4935 [2]	1.7260 [2]
	1.1505 [3]	1.2355 [3]	1.2355 [3]	1.2945 [3]	1.4855 [3]	1.7505 [3]
	1.1605 [4]	1.2360 [4]	1.2360 [4]	1.3145 [4]	1.4940 [4]	1.7260 [4]
	1.1600 [5]	1.2360 [5]	1.2360 [5]	1.3140 [5]	1.4940 [5]	1.7260 [5]
	1.1600 [6]	1.2360 [6]	1.2360 [6]	1.3140 [6]	1.4940 [6]	1.7260 [6]
2	0.9770	1.4805	1.6336	2.3633	2.3921	2.6023
	0.9768 [1]	1.4807 [1]	1.6334 [1]	2.3632 [1]	2.3914 [1]	2.6022 [1]
	0.9770 [2]	1.4805 [2]	1.6350 [2]	2.3630 [2]	2.3920 [2]	2.6025 [2]
	0.9690 [3]	1.4635 [3]	1.6190 [3]	2.3510 [3]	2.3760 [3]	2.5890 [3]
	0.9770 [4]	1.4805 [4]	1.6340 [4]	2.3625 [4]	2.3925 [4]	2.6025 [4]
3	0.7855	1.4916	1.6112	2.4740	2.8758	2.9149
	0.7855 [4]	1.4915 [4]	1.6120 [4]	2.4755 [4]	2.8770 [4]	2.9150 [4]

Table 6
Frequency parameter λ (Case 2, $c = 1$, $d = 15$, SS-1)

k	$\Delta = 0.1$	$\Delta = 0.2$	$\Delta = 0.3$	$\Delta = 0.4$	$\Delta = 0.5$	$\Delta = 0.6$	$\Delta = 0.7$	$\Delta = 0.8$	$\Delta = 0.9$
1	1.5246	1.7696	1.9707	2.0653	1.8130	1.5734	1.4274	–	–
2	1.4353	1.5870	1.7032	1.8538	1.7238	1.5269	1.4023	1.3429	–
3	1.4209	1.5346	1.6061	1.6790	1.6439	1.4983	1.3907	1.3372	–
4	1.4147	1.5053	1.5568	1.6065	1.5928	1.4804	1.3842	1.3351	1.3180
5	1.4112	1.4878	1.5294	1.5691	1.5624	1.4685	1.3799	1.3338	1.3174
6	1.4089	1.4767	1.5126	1.5469	1.5434	1.4605	1.3771	1.3330	1.3172
7	1.4073	1.4692	1.5018	1.5328	1.5307	1.4550	1.3750	1.3325	1.3172
8	1.4061	1.4639	1.4943	1.5232	1.5220	1.4509	1.3735	1.3321	1.3172
9	1.4052	1.4601	1.4890	1.5164	1.5158	1.4479	1.3724	1.3317	1.3170
10	1.4046	1.4574	1.4849	1.5113	1.5110	1.4457	1.3717	1.3315	1.3170
11	1.4041	1.4552	1.4821	1.5075	1.5073	1.4438	1.3709	1.3314	1.3170
12	1.4036	1.4534	1.4797	1.5045	1.5045	1.4424	1.3704	1.3312	1.3170
13	1.4034	1.4522	1.4778	1.5022	1.5023	1.4414	1.3700	1.3310	1.3170
14	1.4030	1.4510	1.4763	1.5003	1.5005	1.4405	1.3697	1.3310	1.3170
15	1.4029	1.4502	1.4751	1.4988	1.4990	1.4396	1.3693	1.3308	1.3170
16	1.4027	1.4495	1.4741	1.4975	1.4978	1.4391	1.3691	1.3308	1.3170
17	1.4025	1.4490	1.4733	1.4965	1.4968	1.4384	1.3689	1.3308	1.3170
18	1.4025	1.4484	1.4726	1.4955	1.4960	1.4381	1.3688	1.3308	1.3170
19	1.4023	1.4479	1.4719	1.4948	1.4952	1.4377	1.3686	1.3306	1.3170
20	1.4023	1.4476	1.4714	1.4942	1.4947	1.4374	1.3684	1.3306	1.3168
50	1.4018	1.4445	1.4673	1.4890	1.4895	1.4346	1.3675	1.3302	1.3168
200	1.4016	1.4441	1.4668	1.4881	1.4886	1.4342	1.3673	1.3302	1.3168
Ref. [1]					1.4880				1.3170

polynomial function of degree $d = 15$ (for SS and AA modes), and $d = 14$ (for SA and AS modes) is sufficient for admissible accuracy (Table 2).

The first three in-plane free vibration frequencies of either completely free (Tables 3–5) or point-supported super-elliptical plates were carried out numerically for four mode categories for the parameters shown in the following:

$$k = \{1, 2, \dots, 20, 50, 200\}, \quad c = \{1, 2\}, \quad \mu = 0.3, \quad \Delta = \Delta_1 = \Delta_2, \quad 0.1 \leq \Delta \leq 0.9 \quad (21)$$

To avoid ambiguity of excessive data, some of the results were presented for the second case (Tables 6–13).

4. Concluding remarks

The in-plane free vibration frequencies corresponding to the first three SS, SA, AS, and AA modes of isotropic super-elliptical plates of uniform thickness were investigated by the Ritz method. Various plate shapes for two

Table 7
Frequency parameter λ (Case 2, $c = 1$, $d = 14$, SA-1)

k	$\Delta = 0.1$	$\Delta = 0.2$	$\Delta = 0.3$	$\Delta = 0.4$	$\Delta = 0.5$	$\Delta = 0.6$	$\Delta = 0.7$	$\Delta = 0.8$	$\Delta = 0.9$
1	0.8698	1.0766	1.2839	1.5547	1.4753	1.1679	0.8755	–	–
2	0.7714	0.9341	1.0954	1.2917	1.5158	1.2311	0.9962	0.7864	–
3	0.7457	0.8950	1.0387	1.2122	1.4381	1.2367	1.0164	0.8204	–
4	0.7352	0.8789	1.0141	1.1783	1.3878	1.2381	1.0247	0.8355	0.6438
5	0.7301	0.8709	1.0015	1.1608	1.3605	1.2390	1.0291	0.8429	0.6584
6	0.7270	0.8663	0.9942	1.1507	1.3442	1.2396	1.0315	0.8468	0.6671
7	0.7253	0.8634	0.9897	1.1443	1.3338	1.2400	1.0327	0.8488	0.6719
8	0.7242	0.8614	0.9864	1.1400	1.3265	1.2404	1.0334	0.8500	0.6749
9	0.7232	0.8599	0.9841	1.1369	1.3214	1.2406	1.0339	0.8509	0.6768
10	0.7228	0.8591	0.9826	1.1347	1.3176	1.2408	1.0342	0.8512	0.6779
11	0.7225	0.8582	0.9813	1.1329	1.3145	1.2412	1.0344	0.8515	0.6786
12	0.7221	0.8579	0.9803	1.1316	1.3122	1.2414	1.0346	0.8515	0.6793
13	0.7218	0.8573	0.9795	1.1307	1.3105	1.2416	1.0346	0.8518	0.6797
14	0.7215	0.8570	0.9790	1.1298	1.3090	1.2416	1.0349	0.8518	0.6797
15	0.7215	0.8567	0.9785	1.1292	1.3079	1.2418	1.0349	0.8518	0.6801
16	0.7215	0.8564	0.9780	1.1285	1.3069	1.2420	1.0349	0.8518	0.6801
17	0.7211	0.8564	0.9778	1.1281	1.3061	1.2420	1.0351	0.8521	0.6804
18	0.7211	0.8562	0.9772	1.1278	1.3054	1.2422	1.0351	0.8521	0.6804
19	0.7211	0.8562	0.9770	1.1274	1.3048	1.2424	1.0351	0.8521	0.6804
20	0.7211	0.8559	0.9770	1.1272	1.3044	1.2424	1.0351	0.8521	0.6804
50	0.7208	0.8550	0.9752	1.1252	1.3004	1.2436	1.0354	0.8521	0.6804
200	0.7208	0.8550	0.9749	1.1249	1.2996	1.2440	1.0356	0.8521	0.6808
Ref. [1]					1.4690				1.7980

Table 8
Frequency parameter λ (Case 2, $c = 1$, $d = 15$, AA-1)

k	$\Delta = 0.1$	$\Delta = 0.2$	$\Delta = 0.3$	$\Delta = 0.4$	$\Delta = 0.5$	$\Delta = 0.6$	$\Delta = 0.7$	$\Delta = 0.8$	$\Delta = 0.9$
1	0.3225	0.5688	0.8772	1.2085	1.6206	1.7563	1.3387	–	–
2	0.2608	0.4588	0.6845	0.9703	1.2744	1.5435	1.5897	1.2871	–
3	0.2540	0.4387	0.6515	0.9203	1.1889	1.4607	1.5973	1.3396	–
4	0.2520	0.4307	0.6395	0.8989	1.1502	1.4271	1.5992	1.3605	0.9612
5	0.2510	0.4266	0.6340	0.8874	1.1300	1.4093	1.5969	1.3671	0.9907
6	0.2510	0.4243	0.6309	0.8803	1.1185	1.3984	1.5930	1.3697	1.0025
7	0.2510	0.4225	0.6289	0.8758	1.1113	1.3914	1.5889	1.3706	1.0072
8	0.2510	0.4219	0.6277	0.8726	1.1066	1.3867	1.5856	1.3711	1.0087
9	0.2510	0.4207	0.6269	0.8706	1.1034	1.3833	1.5829	1.3715	1.0092
10	0.2510	0.4207	0.6265	0.8689	1.1009	1.3809	1.5807	1.3717	1.0095
11	0.2510	0.4201	0.6261	0.8678	1.0991	1.3791	1.5788	1.3719	1.0097
12	0.2510	0.4201	0.6261	0.8669	1.0977	1.3779	1.5773	1.3720	1.0100
13	0.2510	0.4195	0.6257	0.8660	1.0968	1.3768	1.5762	1.3720	1.0102
14	0.2510	0.4195	0.6257	0.8654	1.0959	1.3759	1.5753	1.3722	1.0102
15	0.2510	0.4195	0.6257	0.8649	1.0952	1.3753	1.5743	1.3724	1.0104
16	0.2510	0.4195	0.6257	0.8646	1.0948	1.3748	1.5737	1.3724	1.0104
17	0.2510	0.4189	0.6257	0.8643	1.0943	1.3742	1.5731	1.3726	1.0104
18	0.2510	0.4189	0.6257	0.8640	1.0941	1.3739	1.5726	1.3726	1.0107
19	0.2510	0.4189	0.6257	0.8637	1.0936	1.3735	1.5721	1.3728	1.0107
20	0.2520	0.4189	0.6257	0.8634	1.0934	1.3733	1.5718	1.3728	1.0107
50	0.2520	0.4183	0.6257	0.8617	1.0920	1.3715	1.5686	1.3739	1.0112
200	0.2520	0.4183	0.6257	0.8614	1.0918	1.3713	1.5681	1.3740	1.0114
Ref. [1]					1.9220				1.6590

Table 9
Frequency parameter λ (Case 2, $c = 2$, $d = 15$, SS-1)

k	$\Delta = 0.1$	$\Delta = 0.2$	$\Delta = 0.3$	$\Delta = 0.4$	$\Delta = 0.5$	$\Delta = 0.6$	$\Delta = 0.7$	$\Delta = 0.8$	$\Delta = 0.9$
1	1.8319	2.0305	2.3103	2.6615	2.8497	2.5846	2.2139	–	–
2	1.6488	1.7909	1.9940	2.3170	2.6709	2.5617	2.3042	2.0277	–
3	1.6069	1.7381	1.9289	2.2345	2.5937	2.5498	2.2982	2.0408	–
4	1.5903	1.7168	1.9037	2.1980	2.5472	2.5465	2.2933	2.0457	1.8321
5	1.5821	1.7062	1.8913	2.1783	2.5187	2.5434	2.2888	2.0466	1.8442
6	1.5775	1.7001	1.8843	2.1661	2.5002	2.5404	2.2851	2.0462	1.8507
7	1.5746	1.6963	1.8800	2.1581	2.4876	2.5375	2.2821	2.0456	1.8539
8	1.5729	1.6937	1.8772	2.1526	2.4786	2.5351	2.2798	2.0450	1.8557
9	1.5716	1.6919	1.8752	2.1486	2.4720	2.5330	2.2779	2.0445	1.8567
10	1.5707	1.6907	1.8739	2.1456	2.4672	2.5312	2.2766	2.0440	1.8573
11	1.5700	1.6897	1.8730	2.1434	2.4634	2.5297	2.2754	2.0436	1.8577
12	1.5696	1.6889	1.8722	2.1416	2.4606	2.5285	2.2745	2.0433	1.8580
13	1.5692	1.6883	1.8716	2.1402	2.4583	2.5275	2.2738	2.0430	1.8581
14	1.5689	1.6879	1.8711	2.1390	2.4564	2.5267	2.2731	2.0428	1.8582
15	1.5686	1.6876	1.8707	2.1382	2.4550	2.5260	2.2727	2.0427	1.8584
16	1.5684	1.6873	1.8704	2.1374	2.4538	2.5254	2.2722	2.0425	1.8584
17	1.5683	1.6870	1.8702	2.1368	2.4528	2.5248	2.2718	2.0424	1.8585
18	1.5681	1.6867	1.8700	2.1364	2.4518	2.5244	2.2715	2.0423	1.8585
19	1.5681	1.6866	1.8698	2.1359	2.4511	2.5240	2.2712	2.0422	1.8585
20	1.5680	1.6864	1.8696	2.1354	2.4505	2.5237	2.2710	2.0422	1.8585
50	1.5675	1.6852	1.8690	2.1325	2.4460	2.5208	2.2690	2.0416	1.8589
200	1.5675	1.6851	1.8691	2.1320	2.4454	2.5204	2.2686	2.0416	1.8590

Table 10
Frequency parameter λ (Case 2, $c = 2$, $d = 14$, SA-1)

k	$\Delta = 0.1$	$\Delta = 0.2$	$\Delta = 0.3$	$\Delta = 0.4$	$\Delta = 0.5$	$\Delta = 0.6$	$\Delta = 0.7$	$\Delta = 0.8$	$\Delta = 0.9$
1	1.3010	1.6837	2.0515	2.3997	2.2367	1.8714	1.3330	–	–
2	1.1762	1.5010	1.8307	2.1476	2.2210	1.9609	1.6008	1.2060	–
3	1.1500	1.4637	1.7752	2.0845	2.2333	1.9612	1.6257	1.2596	–
4	1.1404	1.4502	1.7524	2.0596	2.2311	1.9618	1.6380	1.2804	0.9255
5	1.1358	1.4438	1.7408	2.0478	2.2270	1.9639	1.6451	1.2911	0.9503
6	1.1336	1.4403	1.7344	2.0416	2.2234	1.9665	1.6494	1.2963	0.9636
7	1.1320	1.4384	1.7302	2.0379	2.2207	1.9688	1.6520	1.2992	0.9711
8	1.1311	1.4372	1.7274	2.0353	2.2187	1.9708	1.6538	1.3008	0.9752
9	1.1307	1.4363	1.7254	2.0336	2.2171	1.9726	1.6550	1.3015	0.9778
10	1.1303	1.4358	1.7239	2.0324	2.2160	1.9741	1.6557	1.3021	0.9793
11	1.1300	1.4354	1.7228	2.0314	2.2149	1.9755	1.6564	1.3025	0.9803
12	1.1298	1.4353	1.7219	2.0306	2.2142	1.9766	1.6568	1.3027	0.9808
13	1.1298	1.4351	1.7212	2.0301	2.2135	1.9776	1.6573	1.3029	0.9813
14	1.1296	1.4349	1.7206	2.0298	2.2129	1.9785	1.6576	1.3029	0.9818
15	1.1296	1.4347	1.7202	2.0294	2.2125	1.9793	1.6577	1.3031	0.9818
16	1.1296	1.4347	1.7197	2.0293	2.2120	1.9799	1.6580	1.3031	0.9821
17	1.1296	1.4346	1.7194	2.0290	2.2117	1.9805	1.6582	1.3031	0.9823
18	1.1296	1.4346	1.7192	2.0289	2.2113	1.9810	1.6583	1.3031	0.9823
19	1.1296	1.4346	1.7189	2.0288	2.2111	1.9815	1.6585	1.3031	0.9823
20	1.1296	1.4344	1.7187	2.0288	2.2109	1.9819	1.6586	1.3031	0.9823
50	1.1296	1.4342	1.7167	2.0288	2.2087	1.9868	1.6595	1.3033	0.9826
200	1.1298	1.4342	1.7164	2.0290	2.2083	1.9881	1.6597	1.3033	0.9826

Table 11
Frequency parameter λ (Case 2, $c = 2$, $d = 14$, AS-1)

k	$\Delta = 0.1$	$\Delta = 0.2$	$\Delta = 0.3$	$\Delta = 0.4$	$\Delta = 0.5$	$\Delta = 0.6$	$\Delta = 0.7$	$\Delta = 0.8$	$\Delta = 0.9$
1	0.8334	0.9423	1.1156	1.3784	1.4891	1.2106	0.9905	–	–
2	0.7106	0.7944	0.9287	1.1338	1.3869	1.2544	1.0529	0.8741	–
3	0.6786	0.7550	0.8769	1.0677	1.3265	1.2568	1.0696	0.8983	–
4	0.6656	0.7389	0.8556	1.0409	1.2969	1.2564	1.0759	0.9097	0.7540
5	0.6592	0.7311	0.8447	1.0271	1.2808	1.2558	1.0789	0.9152	0.7642
6	0.6554	0.7263	0.8385	1.0196	1.2708	1.2556	1.0805	0.9179	0.7704
7	0.6531	0.7235	0.8343	1.0146	1.2645	1.2556	1.0814	0.9195	0.7743
8	0.6515	0.7218	0.8316	1.0114	1.2600	1.2554	1.0819	0.9203	0.7769
9	0.6504	0.7204	0.8298	1.0092	1.2570	1.2554	1.0824	0.9209	0.7785
10	0.6496	0.7194	0.8283	1.0075	1.2546	1.2554	1.0826	0.9211	0.7791
11	0.6492	0.7187	0.8273	1.0062	1.2528	1.2556	1.0826	0.9214	0.7797
12	0.6488	0.7180	0.8264	1.0055	1.2514	1.2556	1.0828	0.9214	0.7804
13	0.6485	0.7176	0.8258	1.0047	1.2504	1.2556	1.0828	0.9217	0.7807
14	0.6481	0.7173	0.8255	1.0042	1.2496	1.2556	1.0831	0.9217	0.7807
15	0.6481	0.7173	0.8249	1.0037	1.2488	1.2558	1.0831	0.9217	0.7810
16	0.6477	0.7169	0.8246	1.0032	1.2482	1.2558	1.0831	0.9217	0.7810
17	0.6477	0.7166	0.8243	1.0030	1.2478	1.2558	1.0831	0.9217	0.7813
18	0.6477	0.7166	0.8240	1.0027	1.2474	1.2558	1.0833	0.9217	0.7813
19	0.6473	0.7166	0.8240	1.0025	1.2470	1.2560	1.0833	0.9220	0.7813
20	0.6473	0.7162	0.8237	1.0022	1.2466	1.2560	1.0833	0.9220	0.7813
50	0.6469	0.7155	0.8225	1.0010	1.2444	1.2566	1.0833	0.9220	0.7817
200	0.6469	0.7155	0.8222	1.0007	1.2440	1.2568	1.0833	0.9220	0.7817

Table 12
Frequency parameter λ (Case 2, $c = 2$, $d = 15$, AA-1)

k	$\Delta = 0.1$	$\Delta = 0.2$	$\Delta = 0.3$	$\Delta = 0.4$	$\Delta = 0.5$	$\Delta = 0.6$	$\Delta = 0.7$	$\Delta = 0.8$	$\Delta = 0.9$
1	0.4207	0.7043	1.0102	1.3620	1.8727	2.3160	1.8108	–	–
2	0.3413	0.5736	0.8056	1.1057	1.4844	1.9927	1.9865	1.6213	–
3	0.3309	0.5477	0.7649	1.0452	1.3855	1.8585	2.0116	1.6630	–
4	0.3279	0.5371	0.7497	1.0196	1.3426	1.8040	2.0200	1.6794	1.3388
5	0.3263	0.5315	0.7423	1.0062	1.3206	1.7758	2.0224	1.6857	1.3594
6	0.3256	0.5287	0.7382	0.9985	1.3081	1.7594	2.0226	1.6883	1.3691
7	0.3256	0.5263	0.7355	0.9932	1.3002	1.7489	2.0223	1.6897	1.3740
8	0.3256	0.5254	0.7342	0.9899	1.2950	1.7420	2.0216	1.6903	1.3766
9	0.3256	0.5244	0.7328	0.9874	1.2913	1.7371	2.0210	1.6907	1.3780
10	0.3256	0.5239	0.7321	0.9856	1.2888	1.7335	2.0205	1.6912	1.3789
11	0.3256	0.5235	0.7318	0.9844	1.2869	1.7309	2.0200	1.6913	1.3795
12	0.3256	0.5230	0.7314	0.9834	1.2853	1.7290	2.0197	1.6914	1.3800
13	0.3256	0.5225	0.7311	0.9823	1.2841	1.7274	2.0193	1.6916	1.3804
14	0.3256	0.5225	0.7308	0.9818	1.2834	1.7261	2.0190	1.6917	1.3808
15	0.3256	0.5220	0.7308	0.9813	1.2826	1.7251	2.0188	1.6917	1.3809
16	0.3256	0.5220	0.7308	0.9808	1.2820	1.7244	2.0187	1.6919	1.3811
17	0.3256	0.5220	0.7304	0.9806	1.2814	1.7237	2.0184	1.6919	1.3813
18	0.3256	0.5220	0.7304	0.9801	1.2810	1.7231	2.0183	1.6920	1.3815
19	0.3256	0.5215	0.7304	0.9798	1.2806	1.7226	2.0182	1.6920	1.3815
20	0.3256	0.5215	0.7304	0.9795	1.2804	1.7222	2.0180	1.6920	1.3817
50	0.3263	0.5211	0.7304	0.9778	1.2785	1.7194	2.0171	1.6925	1.3826
200	0.3263	0.5211	0.7304	0.9775	1.2781	1.7192	2.0169	1.6926	1.3828

Table 13
Frequency parameter λ (Case 2, $c = 1$)

$c = 1$	$\Delta = 0.5$						$\Delta = 0.9$					
	$d = 15$		$d = 14$		$d = 15$		$d = 15$		$d = 14$		$d = 15$	
	SS-2	SS-3	SA-2	SA-3	AA-2	AA-3	SS-2	SS-3	SA-2	SA-3	AA-2	AA-3
1	2.3173	3.0868	1.8829	2.9579	2.2051	2.9977	–	–	–	–	–	–
2	2.1617	2.2521	1.5511	2.6300	2.0822	2.7048	–	–	–	–	–	–
3	2.0214	2.2396	1.5427	2.5089	2.0276	2.6525	–	–	–	–	–	–
4	1.9834	2.2319	1.5505	2.4552	2.0019	2.6382	2.0491	2.3495	1.7710	2.1677	1.6216	2.5963
5	1.9690	2.2270	1.5535	2.4267	1.9881	2.6330	2.0580	2.3865	1.7909	2.1798	1.6397	2.5941
6	1.9621	2.2238	1.5550	2.4096	1.9800	2.6309	2.0645	2.4025	1.8000	2.1875	1.6523	2.5930
7	1.9584	2.2217	1.5560	2.3982	1.9751	2.6298	2.0685	2.4086	1.8040	2.1919	1.6603	2.5922
8	1.9563	2.2202	1.5564	2.3905	1.9718	2.6294	2.0709	2.4107	1.8060	2.1947	1.6651	2.5916
9	1.9550	2.2191	1.5569	2.3848	1.9695	2.6291	2.0724	2.4114	1.8068	2.1962	1.6681	2.5911
10	1.9542	2.2183	1.5571	2.3807	1.9679	2.6289	2.0734	2.4117	1.8072	2.1974	1.6700	2.5908
11	1.9537	2.2177	1.5572	2.3775	1.9667	2.6288	2.0740	2.4119	1.8073	2.1982	1.6712	2.5906
12	1.9533	2.2172	1.5574	2.3751	1.9658	2.6287	2.0745	2.4122	1.8075	2.1986	1.6721	2.5904
13	1.9532	2.2169	1.5574	2.3732	1.9651	2.6286	2.0747	2.4124	1.8076	2.1990	1.6727	2.5902
14	1.9531	2.2165	1.5574	2.3716	1.9644	2.6286	2.0750	2.4125	1.8078	2.1993	1.6732	2.5900
15	1.9529	2.2163	1.5574	2.3703	1.9641	2.6286	2.0751	2.4127	1.8078	2.1995	1.6735	2.5899
16	1.9529	2.2161	1.5574	2.3693	1.9637	2.6286	2.0752	2.4128	1.8078	2.1997	1.6738	2.5898
17	1.9528	2.2160	1.5574	2.3683	1.9634	2.6285	2.0753	2.4129	1.8079	2.1998	1.6739	2.5897
18	1.9528	2.2159	1.5574	2.3676	1.9632	2.6285	2.0755	2.4130	1.8079	2.1998	1.6741	2.5896
19	1.9528	2.2157	1.5574	2.3670	1.9629	2.6285	2.0755	2.4131	1.8079	2.1999	1.6742	2.5895
20	1.9529	2.2156	1.5574	2.3664	1.9627	2.6285	2.0755	2.4131	1.8079	2.1999	1.6744	2.5895
50	1.9536	2.2152	1.5574	2.3622	1.9615	2.6285	2.0761	2.4137	1.8079	2.2000	1.6757	2.5889
200	1.9538	2.2151	1.5574	2.3616	1.9614	2.6285	2.0763	2.4139	1.8079	2.2000	1.6762	2.5888
Ref.												
[1]	1.9190	2.1900	2.3280				2.0710	2.4050	2.1880			

Table 14
Fundamental mode

Case no	c	Interval of k	Support location	Fundamental mode
1	1	$k = 1$	–	SS-1~AA-1
1	1	$2 \leq k \leq 200$	–	AA-1
1	2	$1 \leq k \leq 200$	–	AS-1
2	1	$1 \leq k \leq 200$	$0.1 \leq \Delta \leq 0.4$	AA-1
2	1	$k = 1$	$\Delta = 0.5$	SA-1
2	1	$2 \leq k \leq 200$	$\Delta = 0.5$	AA-1
2	1	$1 \leq k \leq 200$	$0.6 \leq \Delta \leq 0.9$	AS-1
2	2	$1 \leq k \leq 200$	$0.1 \leq \Delta \leq 0.4$	AA-1
2	2	$1 \leq k \leq 200$	$0.5 \leq \Delta \leq 0.9$	AS-1

Table 15
Optimum support location (Case 2)

c	Interval of k	Support location
1	$1 \leq k \leq 2$	$\Delta = 0.5$
1	$3 \leq k \leq 200$	$\Delta = 0.6$
2	$1 \leq k \leq 9$	$\Delta = 0.5$
2	$10 \leq k \leq 200$	$\Delta = 0.6$

distinct aspect ratios were considered in the numerical computations which were carried out for two cases: (1) a completely free plate, (2) a plate with four intermediate point-supports symmetrically distributed about the central axes of the plate.

The results obtained in the current study were compared with those of rectangular plates by setting $k = 200$. For a completely free plate, almost identical results were obtained (Tables 3–5). For the point-supported case good

Fig. 2. Mode shapes (1^{st} mode, Case 1, $c = 1$).

Fig. 3. Mode shapes (2^{nd} mode, Case 1, $c = 1$).

Fig. 4. Mode shapes (3^{rd} mode, Case 1, $c = 1$).

Fig. 5. Mode shapes (1^{st} mode, Case 2, $c = 1$, $\Delta = 0.5$). Fig. 6. Mode shapes (2^{nd} mode, Case 2, $c = 1$, $\Delta = 0.5$). Fig. 7. Mode shapes (3^{rd} mode, Case 2, $c = 1$, $\Delta = 0.5$).

agreement was observed for SS modes (Tables 6, 13), but lower eigenvalues were computed for SA, and AA modes when $\Delta = 0.5$ and $\Delta = 0.9$. Interestingly, the frequencies corresponding to the second and the third SA and AA modes obtained in this paper seem to be in harmony with those corresponding to the first and the second modes reported in Ref. [1] (Tables 7–8, 13). The convergence study was presented (Table 2), and the mode shapes were depicted as an example (Figs 2–7).

The mode corresponding to the fundamental frequency for each case was determined. It was shown that AA-1 and AS-1 modes generally yield the fundamental frequency (Table 14). The optimum support location for a point-supported super-elliptical plate was presented. It can be concluded that for $k > 9$ the optimum support location is $\Delta = 0.6$ (Table 15).

References

- [1] Y. Kobayashi, G. Yamada and S. Honma, In-plane vibration of point-supported rectangular plates, *Journal of Sound and Vibration* **126**(3) (1988), 545–549.
- [2] N.S. Bardell, R.S. Langley and J.M. Dunsdon, On the free in-plane vibration of isotropic rectangular plates, *Journal of Sound and Vibration* **191**(3) (1996), 459–467.
- [3] K. Hyde, J.Y. Chang, C. Bacca and J.A. Wickert, Parameter studies for plane stress in-plane vibration of rectangular plates, *Journal of Sound and Vibration* **247**(3) (2001), 471–487.
- [4] J. Du, W.L. Li, G. Jin, T. Yang and Z. Liu, An analytical method for the in-plane vibration analysis of rectangular plates with elastically restrained edges, *Journal of Sound and Vibration* **306** (2007), 908–927.
- [5] D.J. Gorman, Free in-plane vibration analysis of rectangular plates by the method of superposition, *Journal of Sound and Vibration* **272** (2004), 831–851.
- [6] D.J. Gorman, Free in-plane vibration analysis of rectangular plates with elastic support normal to the boundaries, *Journal of Sound and Vibration* **285** (2005), 941–966.
- [7] N.H. Farag and J. Pan, Modal characteristics of in-plane vibration of rectangular plates, *Journal of the Acoustical Society of America* **105**(6) (1999), 3295–3310.
- [8] N.H. Farag and J. Pan, Free and forced in-plane vibration of rectangular plates, *Journal of the Acoustical Society of America* **103**(1) (1998), 408–413.
- [9] B.A. Ovunc, In-plane vibration of plates under external disturbances applied at singular points, *Thin-Walled Structures* **18**(1994), 83–95.
- [10] T. Irie, G. Yamada and Y. Muramoto, Natural frequencies of in-plane vibration of annular plates, *Journal of Sound and Vibration* **97**(1) (1984), 171–175.
- [11] A.N. Bercin, An assessment of the effects of in-plane vibrations on the energy flow between coupled plates, *Journal of Sound and Vibration* **191**(5) (1996), 661–680.
- [12] N.H. Farag and J. Pan, Modal characteristics of in-plane vibration of circular plates clamped at the outer edge, *Journal of the Acoustical Society of America* **113**(4) (2003), 1935–1946.
- [13] A.N. Bercin and R.S. Langley, Application of the dynamic stiffness technique to the in-plane vibrations of plate structures, *Computers and Structures* **59**(5) (1996), 869–875.
- [14] B. Ovunc, In-plane vibration of plates by continuous mass matrix method, *Computers and Structures* **8** (1978), 723–731.
- [15] A.V. Singh and T. Muhammad, Free in-plane vibration of isotropic non-rectangular plates, *Journal of Sound and Vibration* **273** (2004), 219–231.
- [16] C.I. Park, Frequency equation for the in-plane vibration of a clamped circular plate, *Journal of Sound and Vibration* **313**(1–2) (2008), 325–333.
- [17] I.V. Andrianov, J. Awrejcewicz and V. Chernetskyy, Analysis of natural in-plane vibration of rectangular plates using homotopy perturbation approach, *Mathematical Problems in Engineering* (2006), DOI 10.1155/MPE/2006/20598 1–8.
- [18] D.C. Chang, G. Wang and N.M. Wereley, A generalized Kantorovich method and its application to free in-plane plate vibration problem, *Applicable Analysis* **80**(3) (2001), 493–523.
- [19] A. Houmat, In-plane vibration of plates with curvilinear plan-forms by a trigonometrically enriched curved triangular p-element, *Thin-Walled Structures* **46**(2) (2008), 103–111.
- [20] A.R.H. Henni, C. Bacon and Liu, In-plane vibration of thin elliptic plates submitted to uniform pulsed microwave irradiations, *Journal of Sound and Vibration* **299** (2007), 298–313.
- [21] G. Wang and N.M. Wereley, Free in-plane vibration of rectangular plates, *AIAA Journal* **40**(5) (2002), 953–959.
- [22] R.H. Gutierrez and P.A.A. Laura, In-plane vibrations of thin, elastic rectangular plates elastically restrained against translation along the edges, *Journal of Sound and Vibration* **132**(3) (1989), 512–515.
- [23] D. Larsson, In-plane modal testing of a free isotropic rectangular plate, *Experimental Mechanics* **37**(3) (1997), 339–343.
- [24] D.J. Gorman, Accurate analytical type solutions for the free in-plane vibration of clamped and simply supported rectangular plates, *Journal of Sound and Vibration* **276** (2004), 311–333.
- [25] D.J. Gorman, Exact solutions for the free in-plane vibration of rectangular plates with two opposite edges simply supported, *Journal of Sound and Vibration* **294** (2006), 131–161.
- [26] D.C. Gazis and R.D. Mindlin, Extensional vibrations and waves in a circular disk and a semi-infinite plate, *Journal of Applied Mechanics* (1960), 541–547.

- [27] S.S.H. Chen and T.M. Liu, Extensional vibration of thin plates of various shapes, *Journal of the Acoustical Society of America* **58**(4) (1975), 828–831.
- [28] R. Kirisik and S. Yuksel, Free vibration analysis of a rectangular plate with Kelvin type boundary conditions, *Shock and Vibration* **14** (2007), 90–95.
- [29] W.J. Si, K.Y. Lam and S.W. Gong, Vibration analysis of rectangular plates with one or more guided edges via bicubic B-spline method, *Shock and Vibration* **12**(5) (2005), 363–376.
- [30] M. Altekin, Free vibration of orthotropic super-elliptical plates on intermediate supports, *Nuclear Engineering and Design* **239**(6) (2009), 981–999.
- [31] L. Wu and J. Liu, Free vibration analysis of arbitrary shaped thick plates by differential cubature method, *International Journal of Mechanical Sciences* **47** (2005), 63–81.
- [32] C.C. Chen, C.W. Lim, S. Kitipornchai and K.M. Liew, Vibration of symmetrically laminated thick super elliptical plates, *Journal of Sound and Vibration* **220**(4) (1999), 659–682.
- [33] K.M. Liew and Z.C. Feng, Three-dimensional free vibration analysis of perforated superelliptical plates via the p-Ritz method, *International Journal of Mechanical Sciences* **43** (2001), 2613–2630.
- [34] C.W. Lim and K.M. Liew, Vibrations of perforated plates with rounded corners, *Journal of Engineering Mechanics* **121**(2) (1995), 203–213.
- [35] C.M. Wang, L. Wang and K.M. Liew, Vibration and buckling of super elliptical plates, *Journal of Sound and Vibration* **171**(3) (1994), 301–314.
- [36] D. Zhou, S.H. Lo, Y.K. Cheung and F.T.K. Au, 3-D vibration analysis of generalized super elliptical plates using Chebyshev-Ritz method, *International Journal of Solids and Structures* **41** (2004), 4697–4712.
- [37] C.W. Lim, S. Kitipornchai and K.M. Liew, A free-vibration analysis of doubly connected super-elliptical laminated composite plates, *Composites Science and Technology* **58** (1998), 435–445.
- [38] K.M. Liew, S. Kitipornchai and C.W. Lim, Free vibration analysis of thick superelliptical plates, *Journal of Engineering Mechanics* **124**(2) (1998), 137–145.
- [39] D.J. Gorman, Free-vibration analysis of point-supported orthotropic plates, *Journal of Engineering Mechanics* **120**(1) (1994), 58–74.
- [40] C.A. Brebbia, *The Boundary Element Method for Engineers*, Pentech Press, London, 1984.