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Analysis of shearing viscoelastic beam under moving load

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Abstract. In this paper the dynamic behavior of a viscoelastic beam subjected to a moving distributed load has been studied analytically. The viscoelastic properties of the beam have been considered as the linear standard model in shear and incompressible in bulk. The stress components have been separated to the shear and dilatation components then, the governing equation in viscoelastic form has been obtained with direct method and it has been solved with the eigenfunction expansion method. Using the obtained dimensionless coefficients from the governing equation, an analytical procedure has been presented and by parametric studies the effects of the load properties and viscoelastic materials on the amplitude and frequency of the response have been investigated. Such results can present an idea for selecting some parameters in engineering design.

Keywords: Viscoelastic beam, Moving load, Analytical solution, linear standard model

1. Introduction

The moving load on a structure is one of the common problems in engineering. Travelling of a crane on a beam, moving cars on a bridge, transmitted fluid in a polymeric tube, air flow on a composite aircraft wing are some of the practical cases. Also, the material properties are important on the responses of structures due to this excitation. In the most analysis of structures under moving load, the material has been assumed elastic while the main of materials are in the viscoelastic field or they have time depended properties. Huang [1–3] investigated a infinite viscoelastic cylindrical shell response to moving pressure for a material that obeys the linear standard model. Fung et al. [4] investigated the dynamic stability of a viscoelastic beam subjected to the harmonic and parametric excitations simultaneously. They extracted the governing equation in both linear and nonlinear forms. Also they used the dimensionless parameters for simplifying the governing equations and results. Gonza'lez et al. [5] analyzed a steady state moving load along the boundaries of a linear viscoelastic two-dimensional solid by considering a linear standard model and correspondence principle without the inertia effects. Karnaukhov et al. [6] investigated the effects of the load velocity and dimensions of a viscoelastic Timoshenko beam on dissipated heat. Kocatürk et al. [7,8] and Şimşek et al. [9] analyzed Timoshenko and Euler-Bernoulli beams under harmonic moving load. They assumed the Kelvin-Voigt model for material. The governing equations have been solved numerically. Mofid et al. [10] analyzed a beam that obeys a Kelvin-Voigt model under a moving mass. They used the discrete element technique to solve the governing equation and they compared the results with an analytical – numerical solution. As a base reference of moving load Fryba [11] solved many problems in the elastic domain but there is a lack of a complete viscoelastic analysis for the moving load field.

In this paper a three-element linear standard model has been used to describe the viscoelastic material proprieties under moving load. In most published articles, the viscoelastic behavior presented by the Kelvin-Voigt model and

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the solution method was numerical but in this article, the results have been calculated analytically in terms of some dimensionless parameters. The presented form of solution is suitable for the parametric studies; the results are general and can be referred to many materials that obey the linear standard model. The results show the effects of different parameters on the response which can present an idea for selecting the parameters in engineering design.

2. Governing equation

2.1. Constitutive model

The differential constitutive law for a linear viscoelastic material can be written as [12]:

$$P(D)\sigma_{ij}(t) = Q(D)\gamma_{ij}(t) \tag{1}$$

where σ is stress, γ is strain and P, Q are differential operators which are defined as

$$Q(D) = \sum_{r=o}^{N} Q_r \frac{\partial^r}{\partial t^r} \tag{2}$$

$$P(D) = \sum_{r=o}^{N} P_r \frac{\partial^r}{\partial t^r} \tag{3}$$

Q_r and P_r are independent of time in linear form. The dilatational and deviatoric forms of the stress components are [12]:

$$P_1\sigma_{ij}^{de}(t) = Q_1\gamma_{ij}^{de}(t) \tag{4}$$

$$P_2\sigma_{KK}^{dil}(t) = Q_2\gamma_{KK}^{dil}(t) \tag{5}$$

where subscript *de* and *dil* designate as deviatoric and dilatational. P_1, P_2, Q_1, Q_2 are differential operators similar to Eqs (2) and (3). The tensile modulus of elasticity and the Poisson's ratio derived using the relationship between elastic Lame's coefficients and Eqs (4) and (5) as [12]

$$E(D) = \frac{3Q_1Q_2}{P_2Q_1 + 2P_1Q_2} = \frac{Q^E}{P^E} \tag{6}$$

$$\nu(D) = \frac{P_1Q_2 - P_2Q_1}{P_2Q_1 + 2P_1Q_2} = \frac{Q^\nu}{P^\nu} \tag{7}$$

For an incompressible material P_2 in Eq. (5) equals to zero and as a result, when the response in shear is viscoelastic and the dilatational response is incompressible in bulk, Eq. (6) reforms as

$$E\,(D) = \frac{3Q_1}{2P_1} = \frac{Q^E}{P^E} \tag{8}$$

2.2. Motion equation

The equation of transverse motion of an Euler-Bernoulli beam with constant properties is [4]

$$\rho A\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 M(x,t)}{\partial x^2} = f(x,t) \tag{9}$$

where $w(x,t)$, M, x and ρ represent the lateral deflection, bending moment, longitudinal coordinate, density and $f(x,t)$ is force per unit length, respectively. By assuming that each fiber of the beam obeys the same stress-strain-time relation of the viscoelastic model, and by employing the simple beam theory for small curvature, one obtains the strain-curvature-time relation as following [4]

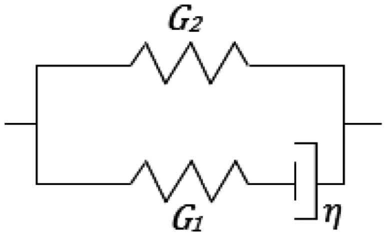


Fig. 1. Linear standard model.

$$\gamma(x,y,t) = \gamma_0 + \kappa(x,t)y \approx \gamma_0 - \frac{\partial^2 w(x,t)}{\partial x^2}y \tag{10}$$

γ_0 is the strain of the middle line, κ is the curvature of the beam and y is the distance from the middle line to the given fiber in the plane of the curvature. By defining the bending moment as [4]

$$M\,(x,t) = \int_A \sigma(x,y,t) y dA \tag{11}$$

For a simple viscoelastic beam the relation between the axial stress and strain is given as [12]

$$P^E\sigma(t) = Q^E\gamma(t) \tag{12}$$

By combining Eqs (10), (11) and (12), we have [4]:

$$P^E M\,(x,t) = IQ^E\kappa \;= -IQ^E\frac{\partial^2 w(x,t)}{\partial x^2} \tag{13}$$

Where I is the second moment of the beam cross-section. By using Eqs (13) and (9) the governing equation for a viscoelastic beam is obtained as

$$\rho A\frac{\partial^2 P^E w(x,t)}{\partial t^2} + I\frac{\partial^4 Q^E w(x,t)}{\partial x^4} = P^E f(x,t) \tag{14}$$

For a simply supported beam, the boundary conditions can be written as:

$$w(0,t) = w(L,t) \;= \frac{\partial^2 w(0,t)}{\partial x^2} = \frac{\partial^2 w(L,t)}{\partial x^2} = 0 \tag{15}$$

where L is the length of the beam. By using the eigenfunction expansion method, the lateral deflection for a simply supported beam is assumed as:

$$w(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\left(\frac{n\pi x}{L}\right) \tag{16}$$

And by substituting in the governing Eq. (14), the general form of the differential equation is obtained as

$$P^E(\ddot{a}_n) + \frac{I}{\rho A}\left(\frac{n\pi}{L}\right)^4 Q^E(a_n) = P^E\left[\frac{2}{\rho AL}\int_0^L f(x,t)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x\right] \tag{17}$$

Where $\ddot{a}_n = \frac{\mathrm{d}^2 a_n}{\mathrm{d}t^2}$.

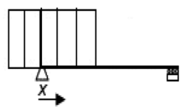
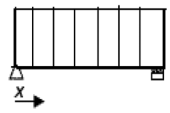
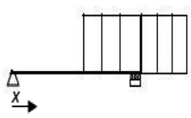
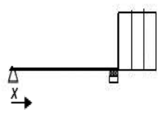
A three-parameter viscoelastic model (Fig. 1) by the following constitutive equation has been considered for the material behavior [12]:

$$\dot{\sigma} + \frac{G_1}{\eta}\sigma = 2\left((G_1+G_2)\,\dot{\gamma} + \frac{G_1G_2}{\eta}\gamma\right) \tag{18}$$

From Eqs (18) and (4), Q_1 and P_1 have been extracted as

$$Q_1 = 2\left((G_1+G_2)\frac{\mathrm{d}}{\mathrm{d}t} + \frac{G_1G_2}{\eta}\right) \quad P_1 = \frac{\mathrm{d}}{\mathrm{d}t} + \frac{G_1}{\eta} \tag{19}$$

Table 1
Different forms of distributed loads on the beam

Right hand side of Eq. (20)	f(x,t)	Initial conditions	Figure of load	Load status on a beam
$\frac{2P_0}{\rho A n\pi}\left[\frac{G_1}{\eta}-\frac{G_1}{\eta}\cos(n\omega t)+n\omega\sin(n\omega t)\right]$	$P_0H[Vt-x]$	$a(t)\big\vert_{t=0}=0$ $\frac{\mathrm{d}a(t)}{\mathrm{d}t}\Big\vert_{t=0}=0$ $\frac{\mathrm{d}^2a(t)}{\mathrm{d}t^2}\Big\vert_{t=0}=0$		arrival
$\frac{2P_0}{\rho A n\pi}\left[\frac{G_1}{\eta}-\frac{G_1}{\eta}\cos(n\pi)\right]$	P_0	$a(t)\big\vert_{t=L/V}=b(t)\big\vert_{t=L/V}$ $\frac{\mathrm{d}a(t)}{\mathrm{d}t}\Big\vert_{t=L/V}=\frac{\mathrm{d}b(t)}{\mathrm{d}t}\Big\vert_{t=L/V}$ $\frac{\mathrm{d}^2a(t)}{\mathrm{d}t^2}\Big\vert_{t=L/V}=\frac{\mathrm{d}^2b(t)}{\mathrm{d}t^2}\Big\vert_{t=L/V}$		stand
$\frac{2P_0}{\rho A n\pi}\left[-(-1)^n\frac{G_1}{\eta}+\frac{G_1}{\eta}\cos(n\omega t)-n\omega\sin(n\omega t)\right]$	$P_0H[x-Vt]$	$b(t)\big\vert_{t=T}=c(t)\big\vert_{t=T}$ $\frac{\mathrm{d}b(t)}{\mathrm{d}t}\Big\vert_{t=T}=\frac{\mathrm{d}c(t)}{\mathrm{d}t}\Big\vert_{t=T}$ $\frac{\mathrm{d}^2b(t)}{\mathrm{d}t^2}\Big\vert_{t=T}=\frac{\mathrm{d}^2c(t)}{\mathrm{d}t^2}\Big\vert_{t=T}$		departure
0	0	$c(t)\big\vert_{t=T+L/V}=d(t)\big\vert_{t=T+L/V}$ $\frac{\mathrm{d}c(t)}{\mathrm{d}t}\Big\vert_{t=T+L/V}=\frac{\mathrm{d}d(t)}{\mathrm{d}t}\Big\vert_{t=T+L/V}$ $\frac{\mathrm{d}^2c(t)}{\mathrm{d}t^2}\Big\vert_{t=T+L/V}=\frac{\mathrm{d}^2d(t)}{\mathrm{d}t^2}\Big\vert_{t=T+L/V}$		free vibration

and by substituting into Eq. (8), Eq. (17) reforms as:

$$\dddot{a}_n+\frac{G_1}{\eta}\ddot{a}_n+3\left(G_1+G_2\right)\frac{I}{\rho A}\left(\frac{n\pi}{L}\right)^4\dot{a}_n+\frac{3G_1G_2}{\eta}\frac{I}{\rho A}\left(\frac{n\pi}{L}\right)^4a_n=P^E\left[\frac{2}{\rho AL}\int_0^L f(x,t)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x\right] \quad (20)$$

The right hand side of Eq. (20) depends on the load function. For a unit step load $f(x,t)=P_0(1-H[x-Vt])$, the initial conditions and the different forms of the load have been given in Table 1 where H is the unit step function, V is the load velocity and P_0 is the magnitude of the load per unit length.

In Table 1, a, b, c and d are modal coordinates of each load case on the beam, $\omega=\frac{\pi V}{L}$ is the circular frequency of excitation and T is the standing time of the load on the beam.

By introducing the following dimensionless parameters

$$\tau=\Omega_1 t \quad \alpha=\frac{\omega}{\Omega_1} \quad K=\frac{G_1}{\eta\Omega_1} \quad E_1=\frac{G_2}{G_1+G_2} \quad (21)$$

Where $\Omega_1^2=3(\frac{\pi}{L})^4\frac{(G_1+G_2)I}{\rho A}$, $T'=\frac{2\pi}{\Omega_1}$ is the period of the first natural frequency, $\tau'=\frac{\eta}{G_1}$ is the relaxation time. By inserting into Eq. (20), this equation for the arrival of load simplifies as:

$$\frac{\mathrm{d}^3a_n}{\mathrm{d}\tau^3}+K\frac{\mathrm{d}^2a_n}{\mathrm{d}\tau^2}+n^4\frac{\mathrm{d}a_n}{\mathrm{d}\tau}+K\cdot E_1\cdot n^4a_n=\frac{2P_0}{\rho A\Omega_1^2 n\pi}\left[K-K\cos(n\alpha\tau)+n\alpha\sin(n\alpha\tau)\right] \quad (22)$$

Where α is the dimensionless speed parameter and τ is the dimensionless time. For a viscoelastic beam subjected to a full load P_0 along the length of the beam, the static deflection of the mid-span where time approaches to infinity is [Appendix A].

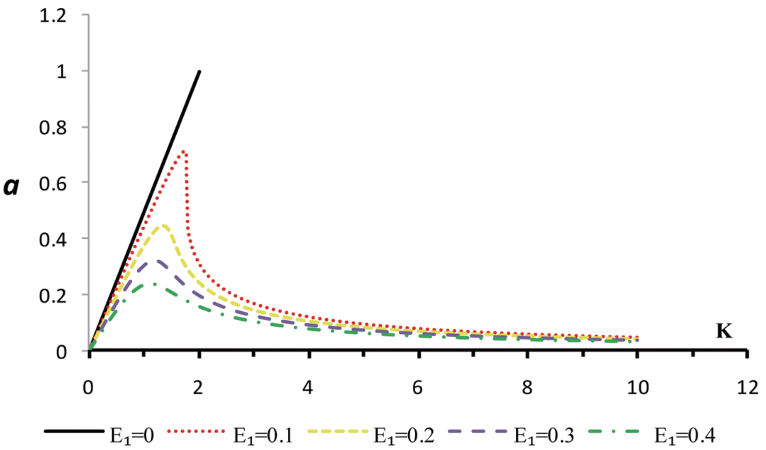


Fig. 2. Variations of a in terms of K for different values E_1.

$$y_0 = \frac{4P_0}{\rho A\pi\Omega_1^2 E_1} \tag{23}$$

With normalizing Eq. (16) with respect to y_0, one obtains a dimensionless form that its solution is the ratio of the dynamic deflection to static deflection. Eq. (22) is a linear equation of order- three and it has a solution for any n. Because all of the coefficients in left hand side of Eq. (22) are real positive, according to Descartes' Rule of Sign, its characteristic equation has three negative real roots or one negative real root and two complex conjugate roots with negative real parts. In the last case, the characteristic equation may be written as:

$$(\lambda + c)((\lambda + a)^2 + b^2) = 0 \tag{24}$$

where a, b and c are real positive numbers and $\lambda = -a \mp ib$, $\lambda = -c$ are the roots of the characteristics equation. For example, where $n = 1$, the solution of Eq. (22) is

$$\begin{aligned} a_1(\tau) &= \frac{E_1 y_0}{2}\left[\frac{K}{c(a^2+b^2)} + \frac{\alpha^2(c-K)e^{-c\tau}}{(c^2+\alpha^2)(c^2+b^2+a^2-2ca)c}\right. \\ &\quad \left. + A1\sin(\alpha\tau) + B1\cos(\alpha\tau) + e^{-a\tau}(A2\sin(b\tau) + B2\cos(b\tau))\right] \end{aligned} \tag{25}$$

where

$$A1 = \frac{\alpha^2(K-2a-c) - 2Kca + (a^2+b^2)(c-K)}{(\alpha^4 + 2(a^2-b^2)\alpha^2 + (b^2+a^2)^2)(\alpha^2+c^2)}$$

$$B1 = \frac{\alpha^2(K(c+2a) - (a+2c)a - b^2) - Kc(a^2+b^2) + \alpha^4}{((\alpha^2+b^2+a^2)^2 - 4\alpha^2 b^2)(\alpha^2+c^2)}$$

$$A2 = \frac{q_0}{b(a^2+(b+\alpha)^2)(a^2+(b-\alpha)^2)(b^2+c^2-2ca+a^2)(a^2+b^2)}$$

$$\begin{aligned} q_0 = &-\alpha^2[a^5 - (K+c)a^4 + (Kc+\alpha^2-2b^2)a^3 + (6b^2K - \alpha^2(K+c))a^2 \\ &-(b^2+Kc)(3b^2-\alpha^2)a + b^2(b^2-\alpha^2)(c-K)] \end{aligned}$$

$$B2 = \frac{q_1}{((b^2+a^2+\alpha^2)^2 - 4\alpha^2 b^2)(b^2+c^2-2ca+a^2)(a^2+b^2)}$$

$$\begin{aligned} q_1 = &\ \alpha^2[-3a^4 + (4K+2c)a^3 - (3Kc+\alpha^2+2b^2)a^2 + ((2c-4K)b^2 + 2\alpha^2 K)a \\ &+(b^2-\alpha^2)(b^2+Kc)] \end{aligned}$$

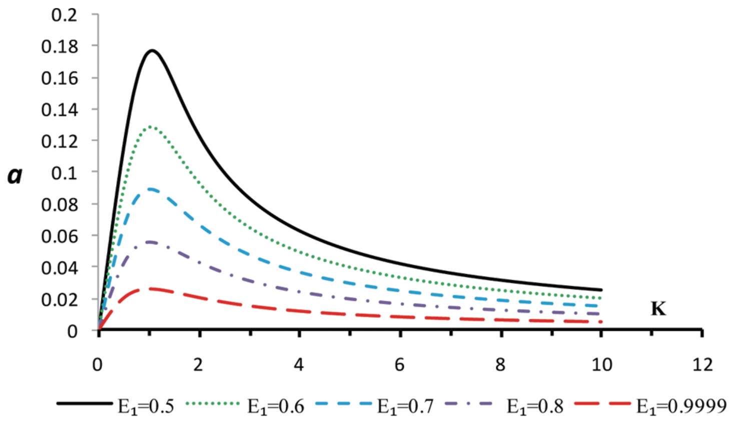


Fig. 3. Variations of a in terms of K for different values E_1.

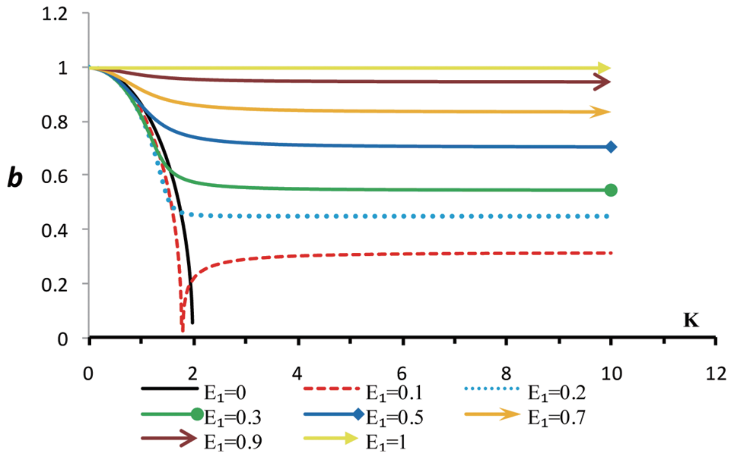


Fig. 4. Variations of b in terms of K for different values E_1.

The parameters a, b, c are different from Table 1. The last four terms of Eq. (25) show the dynamic behavior of a viscoelastic beam. The last term is important than the other terms because it contains the damping and oscillations properties of viscoelastic beam by a coefficient $e^{(-a\tau)}$ and $A2\sin(b\tau) + B2\cos(b\tau)$ respectively.

In Figs 2 and 3 the magnitude of a in terms of K has been shown for different values of E_1. It is seen that with increasing the parameter E_1 the magnitude of a decreases and approaches to zero and as a result, the values of damping decreases too. In these figures, there is a maximum for a in the range of $0 < K < 2$ for each value of E_1 and for $E_1 \geqslant 0.5$ this maximum is nearly at $K = 1$. When $E_1 < 0.1$ there is a discontinuity in the graph or it is not possible to find any value for a or the response does not conform Eq. (25).

In Fig. 4, the variations of b in terms of K has been plotted for different values of E_1. For $E_1 \leqslant 0.1$ there is a discontinuity in the graph and there is no value for b and at this range, there are three negative real roots. This range is the same for values of a, b and c. From Fig. 4, b approaches to a constant value for $K > 3$ and with increasing E_1 the magnitude of b increases or there is a decrease in the period of oscillations.

The second term in Eq. (25) includes a coefficient of $e^{(-c\tau)}$. This term damps the amplitude of oscillations. From Figs 5 and 6, when $K > 3$, the value of c is equals to the value of K and for large values of K, one can ignore this term.

A ramp with slope 45°, is a good approximation for graph with $E_1 < 0.5$. It is true for $E_1 > 0.5$ with $K > 3$. When $E_1 \to 0$ a same graph with slope 63° can be drawn and as a result the value of c can be approximated with straight lines with slopes between 45° to 63°.

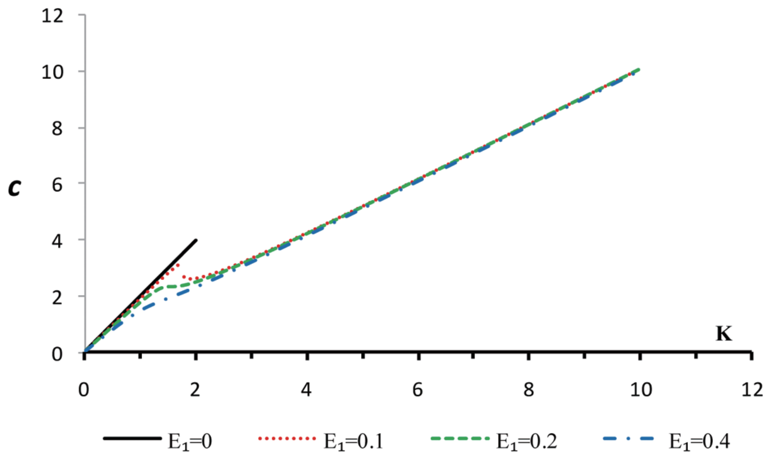


Fig. 5. Variations of c in terms of K for different values E_1.

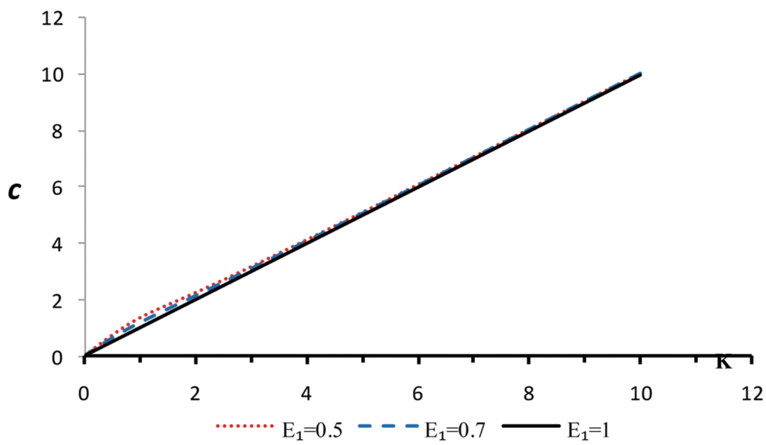


Fig. 6. Variations of c in terms of K for different values E_1.

3. Investigation of dimensionless parameters K and E_1

The parameter $K(K = \frac{G_1}{\eta\Omega_1} = \frac{T'}{2\pi\tau'}$, $\tau' = \eta/G_1$, $T' = 2\pi/\Omega_1)$ is a criterion for the damping of the beam and it equals to the ratio of the first period to 2π times of the relaxation time. The parameter E_1 equals to the ratio of the shearing modulus at infinity to the shearing modulus in the initial time [Appendix B]. Figures 7 and 8 show the Dynamic Load Factor (DLF) which is defined as the dynamic deflection to the absolute static deflection at the midspan of the beam in terms of the normalized time $\tau \cdot \alpha/\pi$. When the distributed load starts to depart from the beam the normalized time is $\tau \cdot \alpha/\pi = 15$ and $\tau \cdot \alpha/\pi = 1$ is the time that the load arrives to the end of the beam.

The parameter K has been varied from 0.01 to 50 in Figs 7 and 8. The maximum damping for DLF occurs when the first period is 2π times of the relaxation time i.e. $K = 1$. So, $K = 1$ is very important in these figures. Before departing of a load from the beam all oscillations are about DLF $= -1$ and by increasing the time of steady load on the beam, all the oscillations decay to this value or the dynamic and static deflections are the same. It is true for all values of E_1. This case has been demonstrated at $K = 50$ and $K = 0.01$. After the load exits from the beam, the oscillations approach to zerobut damping rate of oscillations depends on the parameter K. For values K>>1, the system presented a quasi-elastic behavior before departing the load from the beam. It corresponds to smallness of η or largeness of G_1. Also for $K \approx 0$ i.e. small damping, the system behaves similar to free vibrations of an elastic beam without damping.

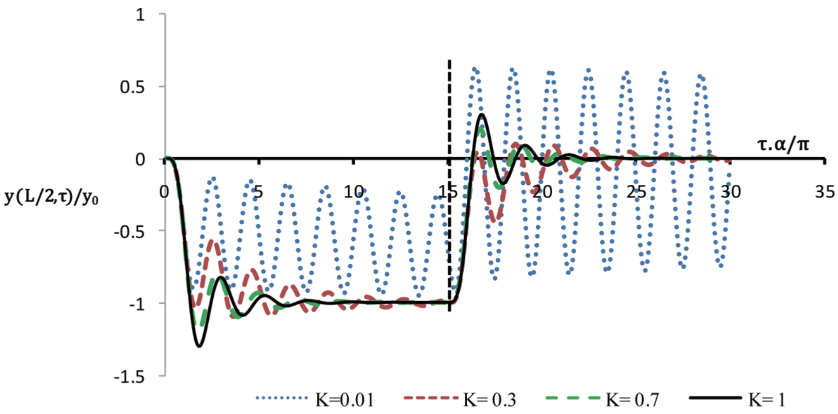


Fig. 7. DLF in terms of $\tau \cdot \alpha/\pi$ for $\alpha = 1$, $E_1 = 0.5$ and $K \leqslant 1$.

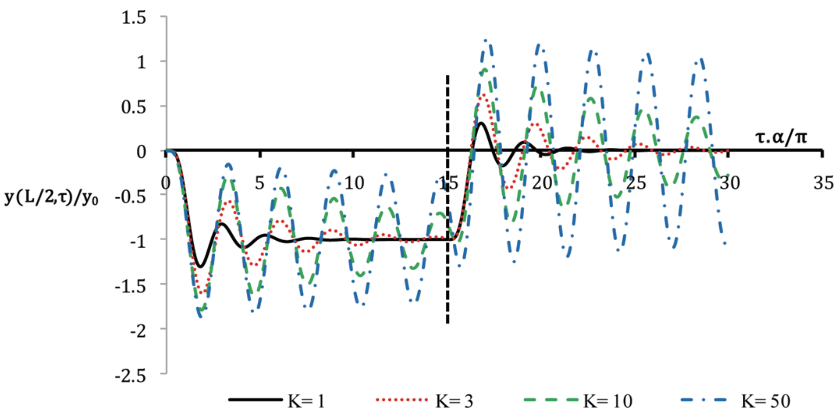


Fig. 8. DLF in terms of $\tau \cdot \alpha/\pi$ for $\alpha = 1$, $E_1 = 0.5$ and $K \geqslant 1$.

The effects of E_1 on DLF, has been presented in Figs 9 and 10 for $K = 1$ which corresponds to the maximum damping. When E_1 approaches to unity, the response is similar to free vibrations of an elastic beam without damping. In this case, the response does not depend to K. The perfect elastic mode is obtained when $\eta = 0$ and so $K \to \infty$, $E_1 = 1$ in Eq. (21). A comparison between elastic and viscoelastic state has been shown in Figs 9 and 10. Approximately for $E_1 < 0.25$ the beam deflection increases until its static value at infinity without oscillation. So, the small values of E_1 are suitable for designing of structures with low vibrations.

4. Investigation of dimensionless speed parameter

The effect of parameter α on DLF has been presented in Figs 11–13. In Fig. 11, DLF for the arrival of the moving load to the end of the beam has been shown. In this figure, the maximum deflection belongs to $\alpha = 0.5$ approximately and therefore, it is considered as a critical value. But this value depend on the parameters K and E_1 and according to Fig. 11 the exact value is $\alpha = 0.53$. For the values $\alpha \leqslant 0.25$ [quasi-static], one can expect that DLF approaches to -1 without large amplitude oscillations because the beam is excited slowly and it has enough time to respond to excitation. Figure 11 approves this behavior too.

From Fig. 12, by arriving the load to the end of the beam, the oscillations damped out very fast to DLF $= -1$ and by increasing α, the damping rate increases. For large values α the beam can not respond to excitation quickly and so, there is a delay in response.

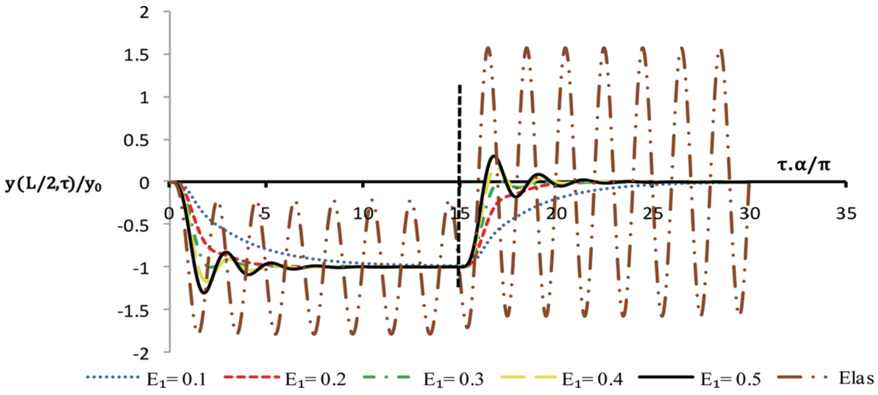


Fig. 9. DLF in terms of $\tau \cdot \alpha/\pi$ for $\alpha = 1$, $K = 1$ for Elastic(Elas) case K is infinity and $E_1 = 1$.

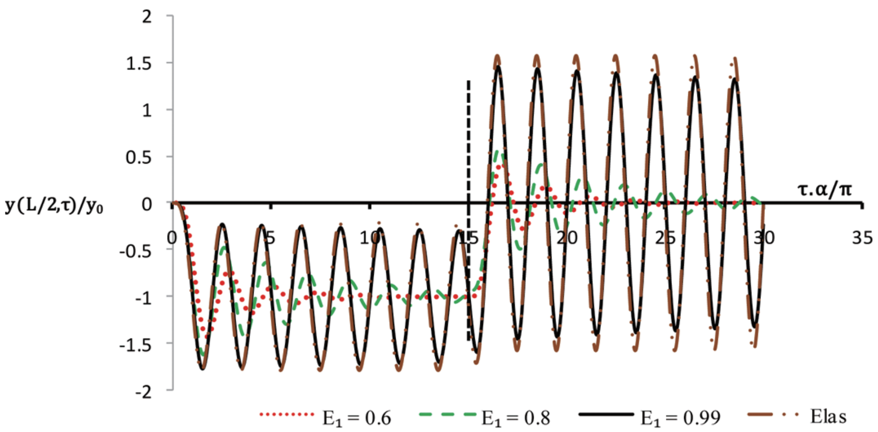


Fig. 10. DLF in terms of $\tau \cdot \alpha/\pi$ for $\alpha = 1$, $K = 1$ for Elastic(Elas) case K is infinity and E1 = 1.

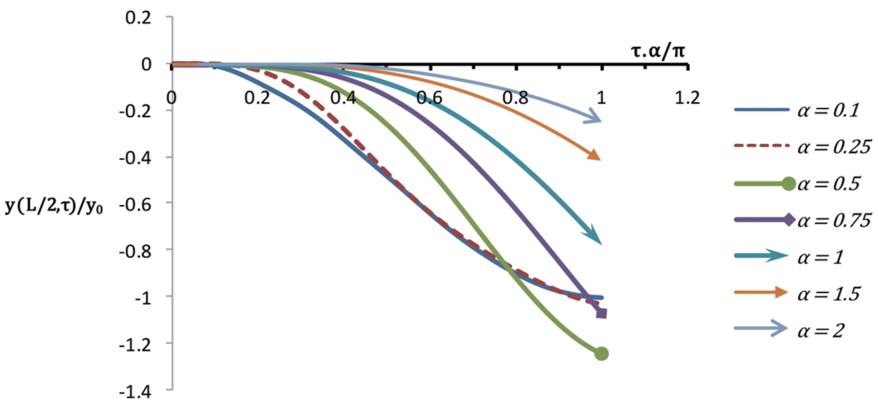


Fig. 11. DLF in terms of $\tau \cdot \alpha/\pi$ for different values of α at arrival and $E_1 = 0.75$, $K = 1$.

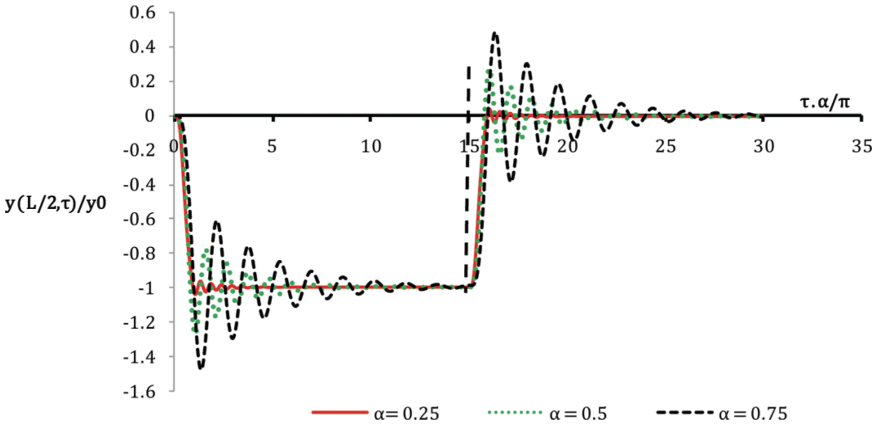


Fig. 12. DLF in terms of $\tau \cdot \alpha/\pi$ for $K = 1$, $E_1 = 0.75$ and $\alpha < 1$.

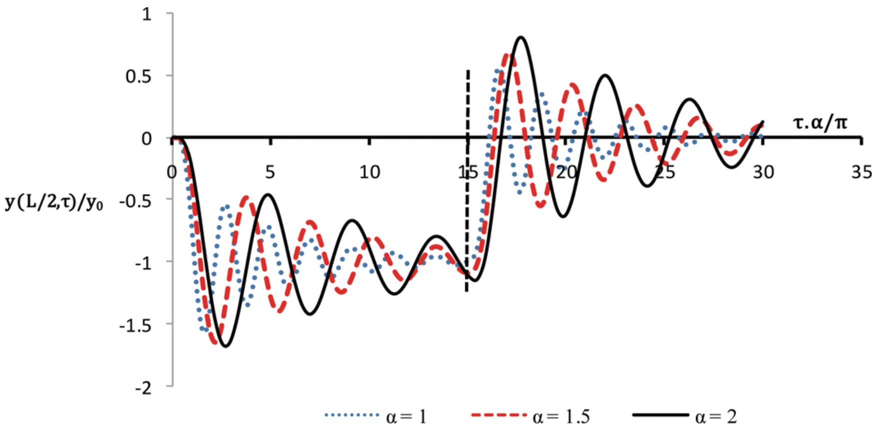


Fig. 13. DLF in terms of $\tau \cdot \alpha/\pi$ for $K = 1$, $E_1 = 0.75$ and $\alpha \geqslant 1$.

5. Investigation of maximum response

The maximum deflection is one of the most important quantities in analyzing the moving load problems. Figure 14 presents the maximum absolute DLF in terms of the dimensionless speed parameter. For $\alpha > 2.5$, the maximum deflections do not have noticeable change, and for different values E_1, it approaches to a constant value. With increasing the speed, for $E_1 \leqslant 0.25$, the absolute maximum deflection does not any change and it is near to one. For $E_1 = 0.99$, the absolute maximum DLF is 2 and one can be found a bound $1 <$ DLF < 2 for $K = 1$.

6. Conclusion

In this paper the effects of the shearing viscoelastic proprieties on a beam response under moving load has been investigated analytically. The discussions of results have been performed using defined dimensionless parameters and the effects of these parameters have been investigated on the response by calculating DLF of the mid-span of the beam. Some results are

- The maximum damping corresponds to $K = 1$ for a constant E_1 i.e. when the first natural period equals to 2π times of the relaxation time.
- After distributing the load on the beam, the oscillations decay to DLF $= -1$. The required time for this decay depends on the material parameters.

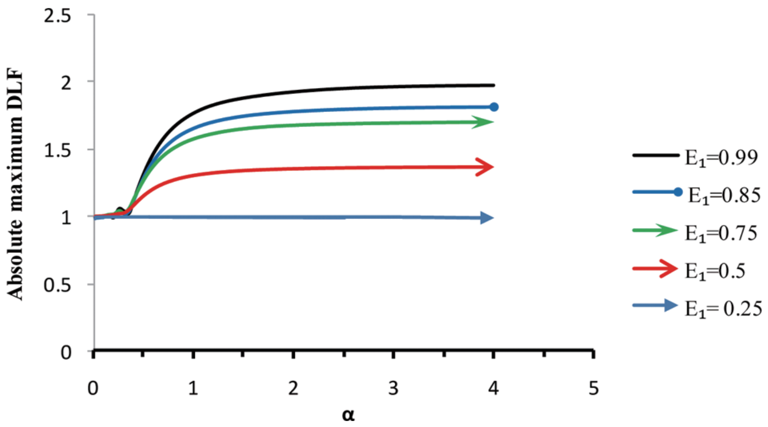


Fig. 14. The absolute maximum DLF graphs for $K = 1$.

- After the distributed load departs from a beam, the oscillations decay to zero.
- When E_1 approaches to zero, the deflection increases until its static value without oscillation. Also when E_1 approaches to one, the material represents a behavior similar to the free vibrations.
- For a constant damping ($K =$ constant), by decreasing E_1, the amplitude of vibrations will decrease.
- For $E_1 = 1$, the response is oscillatory for all values of K.
- When the load arrives on a beam, the maximum deflection belongs to $\alpha = 0.5$ and therefore in this part of moving load, $\alpha = 0.5$ can be found as the critical value.
- For $\alpha > 2.5$ the maximum deflection values do not have any change and they approach to a constant value for each E_1.

Appendix A

For calculating of the static deflection in a viscoelastic beam under perfect continuous load, it is assumed that it applies slowly, so, the first term in Eq. (17) which stands for the effect of the inertia in the system is removed.

$$\frac{I}{\rho A}\left(\frac{n\pi}{L}\right)^4 Q^E(a_n) = P^E\left[\frac{2}{\rho AL}\int_0^L f(x,t)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x\right] \tag{A.1}$$

Where $f(x,t) = P_0 H(t)$.

With applying the operators Q^E and P^E in Eq. (A.1), this equation reforms as

$$3\left(G_1+G_2\right)\frac{I}{\rho A}\left(\frac{n\pi}{L}\right)^4 \dot{a}_n + \frac{3G_1G_2}{\eta}\frac{I}{\rho A}\left(\frac{n\pi}{L}\right)^4 a = \frac{2P_0}{\rho A\Omega_1^2 n\pi}\left[\frac{G_1}{\eta} - \frac{G_1}{\eta}\cos(n\pi)\right] \tag{A.2}$$

By assuming zero initial deflection, $a(0) = 0$, the solution of Eq. (A.2) is

$$a_n(t) = \frac{2}{3}\frac{P_0L^4(1-(-1)^n)(1-e^{(-\frac{G_1G_2t}{\eta G_0})})}{n^5\pi^5 IG_2}H(t) \tag{A.3}$$

Where $G_0 = G_1 + G_2$ in Eq. (A.3). Therefore the static deflection at any time and any position of the beam is

$$y(x,t) = \sum_{n=1}^{\infty}\frac{2}{3}\frac{P_0L^4(1-(-1)^n)(1-e^{(-\frac{G_1G_2t}{\eta G_0})})}{n^5\pi^5 IG_2}H(t)\cdot\sin\left(\frac{n\pi x}{L}\right) \tag{A.4}$$

By using the Eq. (A.4), the static defection at mid-span of a viscoelastic beam subjected to perfect continuous load in infinity for $n = 1$ is:

$$y_0 = \frac{4}{3}\frac{P_0 L^4}{\pi^5 I G_2} \tag{A.5}$$

Also, using the Eq. (20), the Eq. (A.5) can be written as:

$$y_0 = \frac{4P_0}{\rho A \pi \Omega_1^2 E} \tag{A.6}$$

Appendix B

The response of the interest model in Fig. 1 to the input strain $\varepsilon(t) = \varepsilon_0 H(t)$ obtained from Eq. (18) as

$$\sigma(t) = \varepsilon_0 [G_2 + G_1 e^{(-t/\tau')}] \tag{B.1}$$

Form Eq. (A.7) the relaxation modulus in the time domain is given by

$$G(t) = G_2 + G_1 e^{(-t/\tau')} \tag{B.2}$$

As result from

$$_t\underline{\text{Limit}}_0 G(t) = G_2 + G_1 \text{ and } _t\underline{\text{Limit}}_\infty G(t) = G_2 \tag{B.3}$$

Then

$$E = \frac{_t\underline{\text{Limit}}_\infty G(t)}{_t\underline{\text{Limit}}_0 G(t)} = \frac{G_2}{G_2 + G_1} \tag{B.4}$$

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