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Nonlinear vibration of oscillation systems using frequency-amplitude formulation

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Abstract. In this paper we study the periodic solutions of free vibration of mechanical systems with third and fifth-order nonlinearity for two examples using He's Frequency-Amplitude Formulation (HFAF).The effectiveness and convenience of the method is illustrated in these examples. It will be shown that the solutions obtained with current method have a fabulous conformity with those achieved from time marching solution. HFAF is easy with powerful concepts and the high accuracy, so it can be found widely applicable in vibrations, especially strong nonlinearity oscillatory problems.

Keywords: Nonlinear vibration, He's frequency-amplitude formulation, periodic solution, approximate frequency

1. Introduction

Oscillation systems have been widely used in many areas of physics and engineering. These systems have significant importance in engineering particularly in mechanical and structural dynamics. Many practical engineering components consist of vibrating systems that can be modeled using oscillator systems such as elastic beams supported by two springs or mass-on-moving belt or nonlinear pendulum and vibration of a milling machine [1–3].

In recent years, much attention has been devoted to the new developed methods to construct an analytic solution of nonlinear vibration such as Variational Iteration Method [4–7], Homotopy Perturbation Method (HPM) [8,9], Energy Balance Method (EBM) [12–14], Max-Min Method [15,16], Differential Transform Method [17,18], He's Frequency-Amplitude Formulation [19,20], Parameter Expansion Method [21] and etc. Through the continuous development of these methods, many research works have been conducted as follows. Boumediene et al. [22] investigated nonlinear forced vibration of thin elastic rectangular plates subjected to harmonic excitation by asymptotic numerical method. Bayat et al. [23] employed Energy Balance Method to obtain analytical expressions for the non-linear fundamental frequency and deflection of Euler-Bernoulli beams defining the bending behavior of long isotropic beams. Only a first-order approximation leads them to accurate solutions compared to the work presented by Qaisi [24] using harmonic balance approach. Ganji et al. [25] studied static stability of a column by determining the nature of the singular point at $u = 0$ of the dynamic equations. Then they considered a two-mass system with three-springs while two equal masses are linked with the linear and nonlinear stiffness namely k_1, k_2 and k_3, respectively. Eventually, Max-Min approach was utilized to obtain the first and second-order approximate frequencies and periods for these single and two-degrees-of-freedom (SDOF and TDOF) systems [25]. Moreover, Parameter-Expansion Method was employed to develop a closed form solution to the governing equation of a system of nonlinear autonomous

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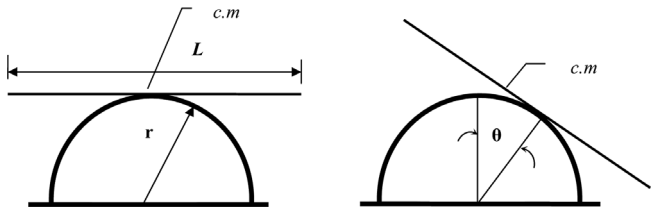

Fig. 1. Rigid rod rocks on a circular surface.

conservative oscillators, describing the large amplitude free vibrations of a restrained uniform beam carrying an intermediate lumped mass along its span [26].

Some other literature which used approximate methods for beam vibration problems are summarized in the following. Jacques et al. [27] analyzed nonlinear vibration of pre-stressed beams. Baghani et al. [28] represented large amplitude vibration and post-buckling analysis of composite beams on elastic foundation and used variational iteration method to solve the cubic nonlinearity equation. An improved He's Energy Balance Method for solving nonlinear oscillatory differential equation using a new trial function was presented by Sfahani et al. [29]. The problem considered represents the governing equations of the non-linear large amplitude free vibrations of a slender cantilever beam with a rotationally flexible root. The fourth-order parabolic differential equations presenting the transverse vibrations of a homogeneous beam are also approximated by Noor et al. [30].

In previous works, He's Frequency-Amplitude Formulation was introduced in simple cases [31–33]. However, through current research authors develop this method for two highly nonlinear vibration problems. In first case the motion equation of the rigid rod rocks on the circular surface is investigated [34]. It is also considered large amplitude vibration of a slender inextensible cantilever beam with intermediate lumped mass with fifth order nonlinearity followed by discussion of the results. Ultimately, the results are verified against time marching and other analytical solutions, and as will be depicted, approximate solutions obtained by current method are in excellent agreement with those obtained by the former one.

2. Application of He's frequency-amplitude formulation

In order to demonstrate the application and the accuracy of the frequency-amplitude formulation in oscillator systems, we will consider the following examples:

2.1. Case 1

Figure 1 shows the schematic of the rigid rod rocks on a circular surface without slipping [34]. The potential energy of the system can be written as:

$$U = \mathrm{mgr}(\theta \sin\theta + \cos\theta - 1).$$

The kinetic energy can be expressed as:

$$T = \frac{1}{2} m V_G^2 + \frac{1}{2} I_G \omega^2$$

Where, V_G, I_G and ω are as below:

$$V_G = r\theta\dot{\theta}, \qquad I_G = \frac{1}{12} m l^2, \qquad \omega = \dot{\theta}$$

The motion equation of the system by applying Lagrangian [34] can be found as:

$$\left(\frac{1}{12} l^2 + r^2\theta^2\right)\ddot{\theta} + r^2\theta\dot{\theta}^2 + gr\theta\cos\theta = 0. \tag{1}$$

Here dot denotes differentiation with respect to time.

By using the approximation $\cos\theta \approx 1 - \frac{\theta^2}{2} + \ldots$, and substituting into Eq. (1), we have:

$$\left(\frac{1}{12}l^2 + r^2\theta^2\right)\ddot{\theta} + r^2\theta\dot{\theta}^2 + gr\theta - \frac{gr}{2}\theta^3 = 0. \tag{2}$$

And initial conditions:

$$\theta(0) = A \text{ and } \frac{d\theta}{dt}(0) = 0, \tag{3}$$

According to He's Frequency-Amplitude Formulation [19,20], we choose two trial functions $\theta_1(t) = A\cos t$ and $\theta_2(t) = A\cos\omega t$, which are, respectively, the solutions of the following linear equations:

$$\ddot{\theta} + \omega_1^2\theta = 0, \quad \omega_1^2 = 1 \tag{4}$$

$$\ddot{\theta} + \omega_2^2\theta = 0, \ \omega_2^2 = \omega^2. \tag{5}$$

The residuals are:

$$R_1(t) = (-Al^2/12 + r^2A^3 + grA)\cos t - (2r^2A^3 + grA^3/2)\cos^3 t \tag{6}$$

$$R_2(t) = (-l^2\omega^2A/12 + r^2\omega^2A^3 + grA)\cos\omega t - (2r^2\omega^2A^3 + grA^3/2)\cos^3\omega t. \tag{7}$$

We introduce two new residual variables $\tilde{R}_1$ and $\tilde{R}_2$ as

$$\tilde{R}_1 = \frac{4}{T_1}\int_0^{T_1/4} R_1(t)\cos\left(\frac{2\pi}{T_1}t\right)dt \tag{8}$$

And

$$\tilde{R}_2 = \frac{4}{T_2}\int_0^{T_2/4} R_2(t)\cos\left(\frac{2\pi}{T_2}t\right)dt. \tag{9}$$

We can approximately determine ω^2 in the form

$$\omega^2 = \frac{\omega_1^2\tilde{R}_2 - \omega_2^2\tilde{R}_1}{\tilde{R}_2 - \tilde{R}_1}. \tag{10}$$

By a simple calculation, Eq. (2) can be expressed as:

$$\tilde{R}_1 = \frac{4grA - Al^2/3 - 2r^2A^3 - 1.5grA^3}{8} \tag{11}$$

And

$$\tilde{R}_2 = \frac{-(2r^2A^3 + l^2A/3)\omega^2 - 1.5grA^3 + 4grA}{8} \tag{12}$$

Substituting Eqs (4), (5), (11) and (12) into Eq. (10) leads to:

$$\omega^2 = \frac{(1.5grA^3 - 4grA)\omega^2 + 4grA - 1.5grA^3}{-(2r^2A^3 + l^2A/3)\omega^2 + 2r^2A^3 + l^2A/3}, \tag{13}$$

Its approximate frequency reads:

$$\omega = \sqrt{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}, \tag{14}$$

Where a, b and c are as follows:

$$a = 2r^2A^3 + l^2A/3, \qquad b = 1.5grA^3 - 2r^2A^3 - 4grA - l^2A/3, \qquad c = 4grA - 1.5grA^3. \tag{15}$$

The periodic solution is as follows:

$$\theta(t) = A\cos\omega t, \tag{16}$$

Table 1
Values of variables in Eq. (1), for two modes

Mode	L	r	g
1	12	3	9.81
2	12	1	9.81

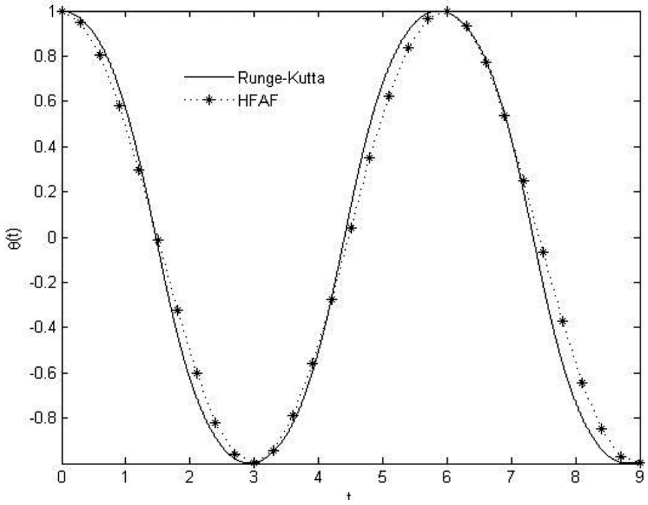

Fig. 2. The comparison between frequency-amplitude formulation with Runge-Kutta 4th order for first mode, when A = 1.

Where ω is evaluated from Eq. (14).

According to Eq. (14), two various frequencies are obtained that one of them is the main frequency apparent through Eqs (4) and (5). The main frequency is distinguished via the coefficients of the proposed equation. We illustrate these statements for two modes.

The values of parameters l, r and g associated with the calculation modes are shown in Table 1. We should use alternatively positive and negative symbols in Eq. (14), for the first and second modes. To show the remarkable accuracy of the obtained result, we compare the approximate periodic solutions with Runge-kutta 4th order in Figs 2 and 3.

Figures 4 and 5 also present the phase-plan diagrams of the analytical approach (HFAF) in comparison with Runge-Kutta 4th order.

2.2. Case 2

Free vibrations of an autonomous conservative oscillator with inertia and static type fifth-order non-linearity (Fig. 6), is expressed by [35,36];

$$\ddot{u} + \lambda u + \varepsilon_1 u^2 \ddot{u} + \varepsilon_1 u \dot{u}^2 + \varepsilon_2 u^4 \ddot{u} + 2\varepsilon_2 u^3 \dot{u}^2 + \varepsilon_3 u^3 + \varepsilon_4 u^5 = 0. \tag{17}$$

With initial conditions:

$$u(0) = A \text{ and } \dot{u}(0) = 0 \tag{18}$$

λ is an integer which may take values of $\lambda = 1$, 0 or -1, and ε_1, ε_2, ε_3 and ε_4 are positive parameters which are shown in Table 2.

This oscillation system is modeled as a restrained uniform beam carrying an intermediate lumped mass along its span. The effect of rotary inertia and shearing deformation is neglected, because the beam thickness is assumed to be small compared to the length.

According to He's Frequency-Amplitude Formulation [19,20], we choose two trial functions $u_1(t) = A\cos t$ and $u_2(t) = A\cos\omega t$.

Table 2
Values of dimensionless parameters ε_i in Eq. (17), for two modes

Mode	ε_1	ε_2	ε_3	ε_4
1	0.326845	0.129579	0.232598	0.087584
2	1.642033	0.913055	0.313561	0.204297

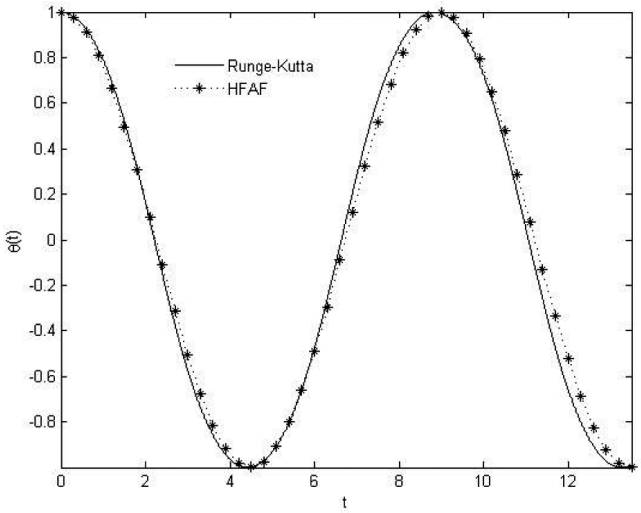

Fig. 3. The comparison between frequency-amplitude formulation with Runge-Kutta 4th order for second mode, when A = 1.

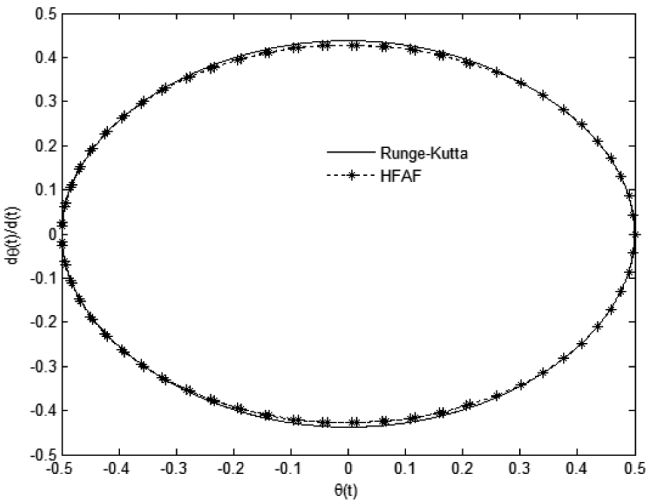

Fig. 4. Phase-plan diagram for first mode, when A = 0.5.

The residuals are:

$$R_1(t) = (-A + \lambda A + \varepsilon_1 A^3)\cos t - (2\varepsilon_1 A^3 - \varepsilon_3 A^3 - 2\varepsilon_2 A^5)\cos^3 t - (3\varepsilon_2 A^5 - \varepsilon_4 A^5)\cos^5 t \tag{19}$$

$$R_2(t) = (-A\omega^2 + \lambda A + \varepsilon_1 A^3\omega^2)\cos\omega t - (2\varepsilon_1 A^3\omega^2 - 2\varepsilon_2 A^5\omega^2 - \varepsilon_3 A^3)\cos^3\omega t - (3\varepsilon_2 A^5\omega^2 - \varepsilon_4 A^5)\cos^5\omega t. \tag{20}$$

Thus, for Eq. (17), we obtain:

$$\tilde{R}_1 = \frac{-8A + 8\lambda A - 4\varepsilon_1 A^3 - 3\varepsilon_2 A^5 + 6\varepsilon_3 A^3 + 5\varepsilon_4 A^5}{16} \tag{21}$$

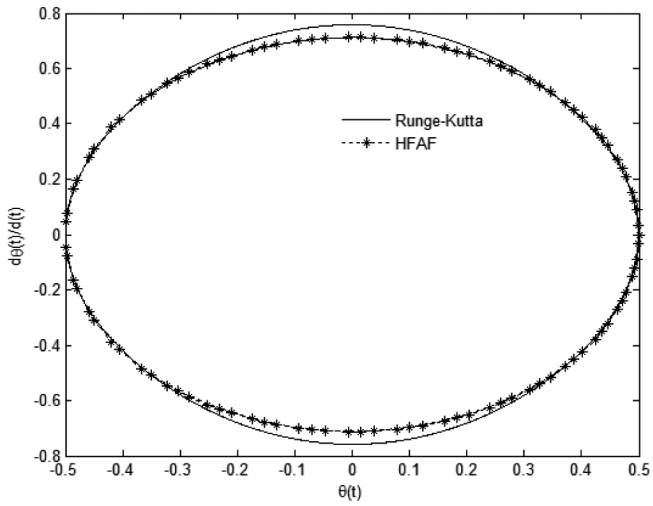

Fig. 5. Phase-plan diagram for second mode, when A = 0.5.

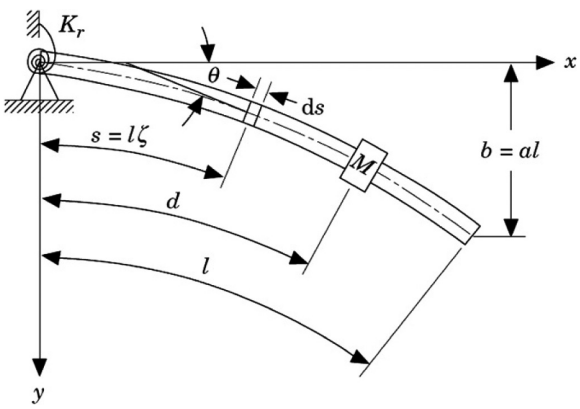

Fig. 6. A sketh of the beam system under study.

And,

$$\tilde{R}_2 = \frac{-8A\omega^2 + 8\lambda A - 4\varepsilon_1 A^3\omega^2 - 3\varepsilon_2 A^5\omega^2 + 6\varepsilon_3 A^3 + 5\varepsilon_4 A^5}{16} \tag{22}$$

According to Eq. (10), ω^2 can be written as:

$$\omega^2 = \frac{-(8\lambda A + 6\varepsilon_3 A^3 + 5\varepsilon_4 A^5)\omega^2 + 8\lambda A + 6\varepsilon_3 A^3 + 5\varepsilon_4 A^5}{-(8A + 4\varepsilon_1 A^3 + 3\varepsilon_2 A^5)\omega^2 + 8A + 4\varepsilon_1 A^3 + 3\varepsilon_2 A^5}, \tag{23}$$

Its approximate frequency reads:

$$\omega = \sqrt{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}, \tag{24}$$

Where a, b and c are as follows:

$$\begin{aligned} &a = 8A + 4\varepsilon_1 A^3 + 3\varepsilon_2 A^5, \qquad b = -(8A + 4\varepsilon_1 A^3 + 3\varepsilon_2 A^5 + 8\lambda A + 6\varepsilon_3 A^3 + 5\varepsilon_4 A^5), \\ &c = 8\lambda A + 6\varepsilon_3 A^5 + 5\varepsilon_4 A^5. \end{aligned} \tag{25}$$

Therefore the periodic solution is as follows:

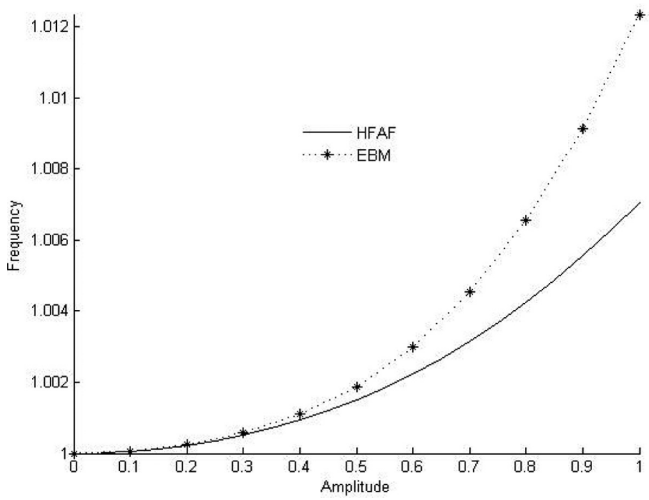

Fig. 7. The comparison between frequency-amplitude formulation and energy balance solution, when $\lambda = 1$, for the first mode.

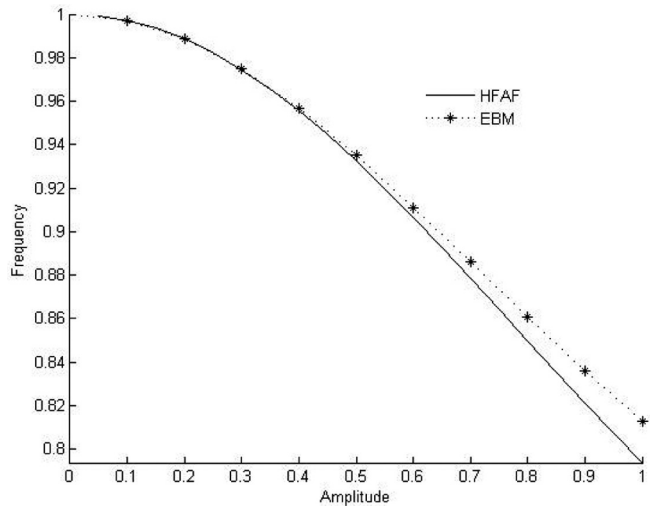

Fig. 8. The comparison between frequency-amplitude formulation with energy balance solution, when $\lambda = 1$, for the second mode.

$$u(t) = A\cos\omega t. \tag{26}$$

Where ω is evaluated from Eq. (24).

As explained in former example, Eq. (24) leads to two separate frequencies for the vibration of Eq. (17). In this example, the main frequency is alternatively obtained by using positive and negative symbols in Eq. (24) for the first and second modes.

The energy balance frequency for the periodic solution to Eq. (17) is [37]:

$$\omega_{ebm} = \frac{\sqrt{3}}{3}\sqrt{\frac{12\lambda + 9\varepsilon_3 A^2 + 7\varepsilon_4 A^4}{4 + 2\varepsilon_1 A^2 + \varepsilon_2 A^4}}, \tag{27}$$

The comparison between frequencies obtained by He's Frequency Amplitude Formulation and Energy Balance Method is shown in Figs 7, 8.

To show the remarkable accuracy of the obtained result, we compare the approximate periodic solutions with Runge-kutta 4th order in Figs 9 and 10.

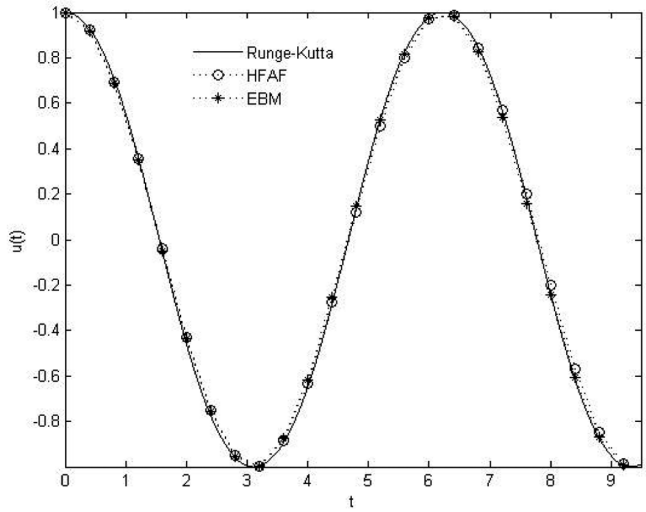

Fig. 9. The comparison between frequency-amplitude formulation with Runge-Kutta 4th order and energy balance solutions when $\lambda = 1$ and A $= 1$, for the first mode.

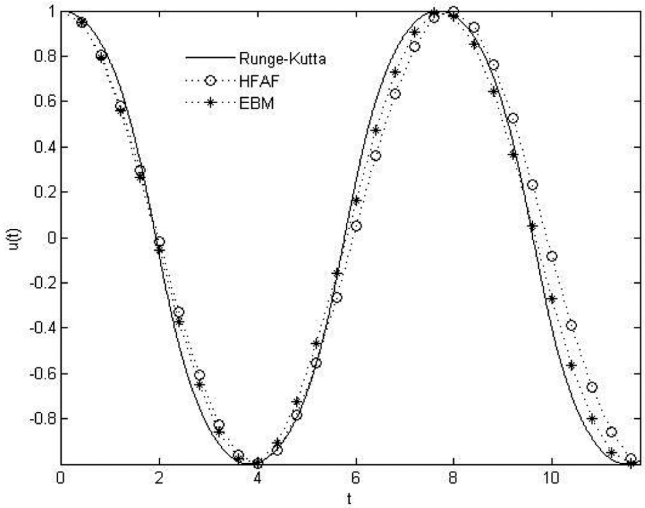

Fig. 10. The comparison between Frequency-amplitude formulation and Runge-Kutta 4th order and energy balance solutions, when $\lambda = 1$ and A $= 1$, for the second mode.

Figures 11 and 12 present the phase-plan diagrams of the analytical methods HFAF and EBM in comparison with Runge-Kutta 4th order.

3. Conclusions

In this paper, we applied a new method, called He's frequency-amplitude formulation to two examples. In first example we considered a rigid rod rock on a circular surface. As it was shown earlier, the obtained equation has third order nonlinearity. For second example large amplitude vibration of a slender inextensible cantilever beam with an intermediate lumped mass is analyzed while it presents a strongly nonlinear conservative oscillator with both inertia and static type nonlinearity.

Both cases showed the effectiveness and precision of the presented approach. We also compared the results with Runge-Kutta 4th order to visualize their drastic approximate solutions. This method is simple and doesn't need to

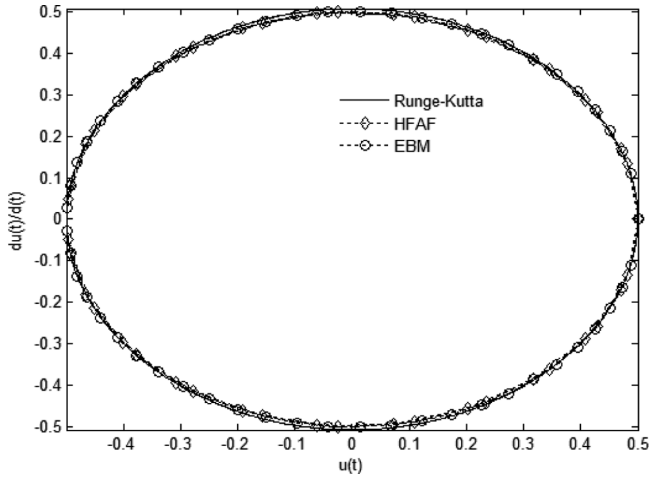

Fig. 11. Phase-plan diagram for first mode, when A = 0.5.

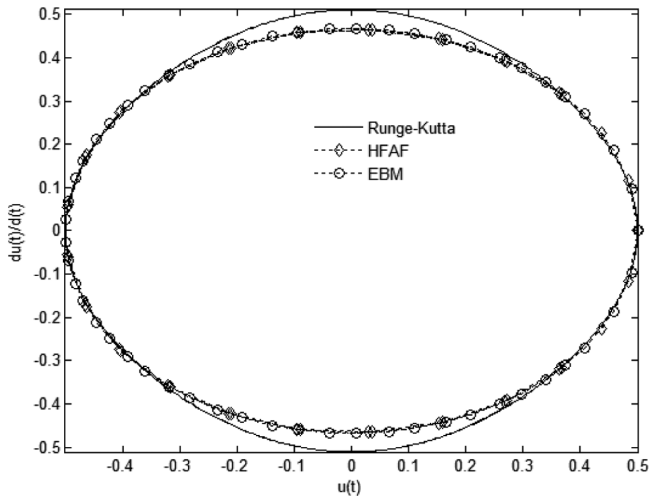

Fig. 12. Phase-plan diagram for second mode, when A = 0.5.

programming but it is important to choose the correct frequency for solving some complicated problems. It can be approved that HFAF is powerful and efficient technique in finding analytical solutions for a wide classes of nonlinear oscillator.

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