Sloshing of liquid in rigid cylindrical container with a rigid annular baffle. Part I: Free vibration

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Abstract. An analytical approach is presented to obtain the sloshing natural frequencies and modes of ideal liquid in a rigid cylindrical container with a rigid annular baffle. The free surface waves of the liquid are considered in the analysis. The artificial interfaces are introduced to divide the complicated liquid domain into several simple sub-domains. The exact analytical solutions of velocity potential of liquid corresponding to every sub-domain are obtained by using the method of separation of variables and the superposition principle. The Eigen-frequency equation is precisely derived by using the Fourier-Bessel expansion on the free surface and the artificial interfaces of the liquid. The convergence study shows high accuracy and fast convergence of the present approach. The comparative studies with those available from literature are made, excellent agreements have been achieved. Numerical results showing the variations of natural frequencies and modes versus position and inner diameter of the annular baffle are provided.

Keywords: Cylindrical container, baffle, sub-domain, sloshing, analytical solution

1. Introduction

Sloshing is one kind of liquid motion in a container with a free surface. Liquid sloshing constitutes a broad class of problems which have great practical importance with regard to the dynamic safety of liquid-structure coupling systems, such as space vehicles, road vehicle tanks and oil ships, storage tanks and elevated water towers under ground motion.

Numerous investigations, both theoretical and experimental, can be found on sloshing problem of liquid in rigid containers from Ibrahim's book [1]. The governing equations of liquid sloshing in rigid rectangular and circular containers with uniform cross-section under linearized boundary conditions were given by Rayleigh [2], Steklov [3], and Lamb [4]. Abramson [5] presented the analytical solutions of liquid sloshing in some simple containers such as rectangular containers and vertical circular (or annular) cylindrical containers by the method of separation of variables. However, this method is somewhat less powerful for cases where the cross-section of the container is non-uniform and other methods, such as the Ritz method, should be used. Budiansky [6] studied the natural frequencies and modes of small-amplitude sloshing of liquid in partially filled circular canals and spherical containers. He explicitly constructed the Green's function for the Laplace equation on these regions and formulated the sloshing problem as an integral equation which was solved. Davis [7] solved the sloshing problems in semi-circle canals by

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using the integral equation method. Chu [8] studied the liquid sloshing in spherical containers by using the similar method. Fox and Kuttler [9] investigated the two-dimensional problems on regions representing the cross-section of a cylindrical tank or canal. They transformed the sloshing problem on the given region to a weighted eigenvalue problem on a simple region such as a rectangle through the conformal mapping. The weighted problem could be treated very effectively by using the powerful methods of intermediate problems. Mciver [10] considered the two-dimensional sloshing of a fluid in a horizontal circular cylindrical container and the three-dimensional sloshing of a fluid in a spherical container. The special coordinate systems are used to formulate the sloshing problem as an integral equation which was solved numerically for eigenvalues. Evans and Linton [11] calculated the natural frequencies of fluid sloshing in certain containers by expressing the velocity potential of liquid as a linear combination of harmonic functions, each of which satisfies the free-surface boundary condition. The analysis was made for three kinds of containers: a two-dimensional cylinder with semicircular cross-section, a cylinder of finite length with semicircular cross-section and a hemisphere.

It is of great significance to understand the free vibration characteristics of liquid-container interaction and has been intensively studied in the last a few decades. Bhuta and Koval [12,13] studied the coupled vibration of liquid in a rigid circular cylindrical container with a flexible bottom which is treated as a flexible membrane or an elastic plate. The same problem was investigated by Tong [14]. Tsui and Small [15] investigated the coupled oscillation of liquid in an annular circular cylindrical container with a flexible bottom. Amabili [16] and Amabili and Dalpiaz [17] studied the bulging modes of an elastic bottom in circular and annular cylindrical containers partially filled with liquid by using the Ritz method. Cheung and Zhou [18] and Zhou and Cheung [19] analyzed the hydro-elastic vibration of a rectangular container bottom plate and a vertical rectangular plate in contact with water on one side. Moreover, Zhou and Liu [20] studied the free vibration of elastic rectangular tanks partially filled with liquid by using a combination of analytical method and Ritz method.

Most of the published works mentioned above are concerned with the natural frequencies and modes of liquid-filled containers without any slosh damping devices. However, baffles are generally used to provide the passive slosh damping in the liquid storage containers. A few of studies recently reported the effects of baffles on the dynamic characteristics of the liquid in containers. Bermudez et al. [21] investigated the vibration modes of the elastic baffle in a rigid liquid-filled container by the finite element method. Biswal et al. [22] studied the natural frequencies and modes of liquid in a liquid-filled cylindrical rigid container with/without baffles by the finite element method. Cho et al. [23] numerically calculated the resonance characteristics of liquid sloshing in a 2D baffled container subjected to lateral excitation. Gedikli and Ergüven [24] analyzed the seismic response of a rigid liquid-filled container with a rigid baffle by using the boundary element method. Malelci and Ziyaeifar [25] investigated the damping effect of baffles in reducing the seismic response of isolated cylindrical liquid storage containers. Gavriluk et al. [26] used the Green function method to develop a semi-analytical approach. They provided the accurate approximations to natural frequencies and modes of liquid sloshing in a rigid vertical circular cylindrical container with thin rigid-ring horizontal baffles. Based on the zoning method, a boundary element method was developed to determine the natural frequencies and mode shapes of liquid sloshing in 3D baffled tanks with arbitrary geometries by Firouz-Abadi et al. [27].

Up to now, most studies on the liquid sloshing of liquid-filled containers with baffles are focused on numerical simulations such as finite element method, boundary element method and finite difference method. These methods have provided many useful and satisfactory results. However, the success of these methods depends primarily on the refinement of the meshes. It is well known that the construction of such precise meshes is usually a difficult and time-consuming task. In addition, elements frequently become over-distorted during the simulation of liquid sloshing. In this paper, an analytical method is developed to solve the natural frequencies and modes of the sloshing liquid in a rigid cylindrical container with a rigid annular baffle. The effect of the liquid surface waves is taken into account in the analysis. The complicated liquid domain is divided into several simple sub-domains and the exact solutions of velocity potential corresponding to every fluid sub-domain are obtained by using the method of separation of variables and the superposition principle. The eigen-frequency equation is derived precisely by the boundary conditions and continuous conditions of the liquid. Results with high accuracy have been obtained by using the numerically searching root approach.

Fig. 1. Container and baffle.

Fig. 2. Sub-domains and interfaces.

2. Model and equations

Consider a thin rigid annular baffle placed in a rigid circular cylindrical container partially filled with inviscid, incompressible and irrotational liquid, as shown in Fig. 1. The liquid has a free surface which is orthogonal to axis of the container. Free-surface waves are considered in the present analysis, and a linear theory of liquid movement is adopted from the small-amplitude sloshing of the liquid. The baffle is very thin compared to its diameter and the effect of the thickness can be neglected. The fluid height h is divided into h_1 and h_2 where h_1 is the length between the free surface and the baffle's neutral plane and h_2 is the length between the baffle's neutral plane and the bottom. The radius of the container is R_2 and the inside radius of the baffle is R_1 . The cylindrical coordinates are used to describe the movement of the liquid and its origin is located at the center of the bottom of the container. According to the above assumptions and definitions, the velocity potential of liquid movement should satisfy the Laplace equation:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \text{ in } \Omega \quad (1)$$

where Ω is the liquid domain. The relations between ϕ and component velocities of the liquid are

$$v_r = \frac{\partial \phi}{\partial r}, \quad v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \quad v_z = \frac{\partial \phi}{\partial z} \quad (2)$$

The impermeability condition at rigid boundaries is

$$\frac{\partial \phi}{\partial \vec{n}} = 0 \text{ on } \Gamma \quad (3)$$

in which, Γ is the wetted solid-liquid interface. $\vec{n}$ is the normal vector outward from the liquid domain Ω . The velocity potential should be finite at $r = 0$, i.e.

$$\phi = \text{finite value at } r = 0 \quad (4)$$

It is obvious that ϕ should be 2π -periodic along the circumference, i.e.

$$\phi(\theta + 2\pi) = \phi(\theta) \quad (5)$$

The sloshing of the liquid can be distinctly classified into symmetric modes and anti-symmetric modes about the $\theta = 0$ axis. Without loss of generality, in the present analysis we shall only study the symmetric modes (the anti-symmetric modes can also be studied by the same procedure), i.e.

$$\frac{\partial \phi}{\partial \theta} = 0 \text{ at } \theta = 0 \quad (6)$$

Taking the effect the surface wave into account, the equation of the free surface is

$$\frac{\partial \phi}{\partial \vec{n}} + \frac{1}{g} \frac{\partial^2 \phi}{\partial t^2} = 0 \text{ on } S \quad (7)$$

where S is the free surface of the liquid.

3. Division of liquid domain

As mentioned above, the analytical solutions of liquid sloshing in some simple containers such as rectangular containers and vertical circular cylindrical containers can be directly got by the method of separation of variables. But this method is somewhat less powerful for the case of liquid in a baffled container. In order to develop an analytical method to solve the problem, the complex liquid domain is divided into several simple sub-domains in the following analysis. As shown in Fig. 2, the liquid domain in the cylindrical container with an annular baffle can be divided into four sub-domains, namely

$$\phi(r, \theta, z, t) = \phi_i(r, \theta, z, t), \quad (r, \theta, z) \in \Omega_i, \quad i = 1, 2, 3, 4 \quad (8)$$

Assume that Ω_i and Ω_l ($i < l$) are in contact each other and their interface is Γ_k . According to Fig. 2, the ordered triple (i, l, k) should belong to the set $\{(1, 2, 1), (2, 4, 2), (3, 4, 3)\}$. Obviously, Ω_i and Ω_l should satisfy the continuous conditions of pressure and velocity on Γ_k , i.e.

$$\phi_i = \phi_l \text{ on } \Gamma_k \quad (9)$$

$$\frac{\partial \phi_i}{\partial \vec{\tau}_k} = \frac{\partial \phi_l}{\partial \vec{\tau}_k} \text{ on } \Gamma_k \quad (10)$$

$$\frac{\partial \phi_i}{\partial \vec{n}_k} = \frac{\partial \phi_l}{\partial \vec{n}_k} \text{ on } \Gamma_k \quad (11)$$

in which, $\vec{n}_k$ is the normal vector of Γ_k . $\vec{\tau}_k$ is the tangential vector of Γ_k . Equation (10) can be easily deduced from Eq. (9). According to Eqs (1) and (8), the velocity potential of each liquid sub-domain should satisfy the following Laplace equation:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi_i}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi_i}{\partial \theta^2} + \frac{\partial^2 \phi_i}{\partial z^2} = 0 \text{ in } \Omega_i \quad (12)$$

The equivalent impermeability conditions at rigid boundaries can be given as follows:

$$\left. \frac{\partial \phi_1}{\partial z} \right|_{z=h_2} = 0, \quad \left. \frac{\partial \phi_1}{\partial r} \right|_{r=R_2} = 0 \quad (13)$$

$$\left. \frac{\partial \phi_3}{\partial z} \right|_{z=0} = 0, \quad \left. \frac{\partial \phi_3}{\partial z} \right|_{z=h_2} = 0, \quad \left. \frac{\partial \phi_3}{\partial r} \right|_{r=R_2} = 0 \quad (14)$$

$$\left. \frac{\partial \phi_4}{\partial z} \right|_{z=0} = 0 \quad (15)$$

Both Ω_1 and Ω_2 have the free surfaces. In such case, the equivalent conditions of Eq. (7) are

$$\left. \frac{\partial \phi_1}{\partial z} \right|_{z=h} + \frac{1}{g} \left. \frac{\partial^2 \phi_1}{\partial t^2} \right|_{z=h} = 0 \quad (16)$$

$$\left. \frac{\partial \phi_2}{\partial z} \right|_{z=h} + \frac{1}{g} \left. \frac{\partial^2 \phi_2}{\partial t^2} \right|_{z=h} = 0 \quad (17)$$

4. Solution of velocity potential

When liquid is undergoing free sloshing in container, ϕ and ϕ_i are the harmonic functions of time, therefore

$$\phi(r, \theta, z, t) = e^{j\omega t} \Phi(r, \theta, z), \quad \phi_i(r, \theta, z, t) = e^{j\omega t} \Phi_i(r, \theta, z) \quad (18)$$

in which, ω is the natural frequency of the liquid sloshing. j is the imaginary unit. Φ is the mode shape of the liquid domain Ω . Φ_i ($i = 1, 2, 3, 4$) is the mode shape of the sub-domain Ω_i . Substituting Eq. (18) into Eqs (13)–(15) gives

$$\left. \frac{\partial \Phi_1}{\partial z} \right|_{z=h_2} = 0, \quad \left. \frac{\partial \Phi_1}{\partial r} \right|_{r=R_2} = 0 \quad (19)$$

$$\left. \frac{\partial \Phi_3}{\partial z} \right|_{z=0} = 0, \quad \left. \frac{\partial \Phi_3}{\partial z} \right|_{z=h_2} = 0, \quad \left. \frac{\partial \Phi_3}{\partial r} \right|_{r=R_2} = 0 \quad (20)$$

$$\left. \frac{\partial \Phi_4}{\partial z} \right|_{z=0} = 0 \quad (21)$$

Here, the superposition principle can be used to solve the mode shapes of liquid sub-domains. The boundaries of each sub-domain can be separated into two classes: rigid boundaries and non-rigid boundaries. Assuming the number of non-rigid boundaries of sub-domain Ω_i is K_i , then, one has

$$\Phi_i = \sum_{q=1}^{K_i} \Phi_i^q \quad (22)$$

where Φ_i^q is the q th component of the mode shape of Φ_i , which can be directly solved by the method of separation of variables. According to Fig. 2 and Eqs (19)–(21), the mode shapes of liquid sub-domains in the baffled cylindrical container can be written in the form of

$$\Phi_1 = \Phi_1^1 + \Phi_1^2 \quad (23)$$

$$\Phi_2 = \Phi_2^1 + \Phi_2^2 + \Phi_2^3 \quad (24)$$

$$\Phi_3 = \Phi_3^2 \quad (25)$$

$$\Phi_4 = \Phi_4^2 + \Phi_4^3 \quad (26)$$

in which, Φ_i^1 ($i = 1, 2, 3, 4$) describes the mode shape of a liquid sub-domain with a free surface in a rigid container. Φ_i^2 ($i = 1, 2$) satisfies the non-rigid boundary condition on the side surface, however, rigid boundary condition on the bottom surface and zero pressure condition on the top surface. Φ_i^3 ($i = 3, 4$) satisfies the non-rigid boundary condition on the side surface, however, rigid boundary conditions on bottom surface and top surface. Φ_2^3 satisfies the non-rigid boundary condition on the bottom surface, however, rigid boundary condition on the side surface and zero pressure condition on the top surface. Φ_4^3 satisfies the non-rigid boundary condition on the top surface, however, rigid boundary condition on the side surface and the bottom surface. The boundary conditions of Φ_i^q ($i = 1, 2, 3, 4$, $q = 1, 2, 3$) can be obtained from Eqs (23)–(26). For sub-domain Ω_1 , one has

$$\begin{aligned} \left. \frac{\partial \Phi_1^1}{\partial r} \right|_{r=R_2} = 0, \quad \left. \frac{\partial \Phi_1^1}{\partial r} \right|_{r=R_1} = 0, \quad \left. \frac{\partial \Phi_1^1}{\partial z} \right|_{z=h_2} = 0 \\ \Phi_1^2|_{z=h} = 0, \quad \left. \frac{\partial \Phi_1^2}{\partial z} \right|_{z=h_2} = 0, \quad \left. \frac{\partial \Phi_1^2}{\partial r} \right|_{r=R_2} = 0 \end{aligned} \quad (27)$$

For sub-domain Ω_2 , one has

$$\left. \frac{\partial \Phi_2^1}{\partial r} \right|_{r=R_1} = 0, \quad \Phi_2^1|_{r=0} = \text{finite value}, \quad \left. \frac{\partial \Phi_2^1}{\partial z} \right|_{z=h_2} = 0$$

$$\begin{aligned} \Phi_2^2|_{z=h} = 0, \quad \frac{\partial \Phi_2^2}{\partial z}\bigg|_{z=h_2} = 0, \quad \Phi_2^2|_{r=0} = \text{finite value}, \\ \frac{\partial \Phi_2^3}{\partial r}\bigg|_{r=R_1} = 0, \quad \Phi_2^3|_{r=0} = \text{finite value}, \quad \Phi_2^3|_{z=h} = 0 \end{aligned} \quad (28)$$

For sub-domain Ω_3 , one has

$$\frac{\partial \Phi_3^2}{\partial z}\bigg|_{z=0} = 0, \quad \frac{\partial \Phi_3^2}{\partial z}\bigg|_{z=h_2} = 0, \quad \frac{\partial \Phi_3^2}{\partial r}\bigg|_{r=R_2} = 0 \quad (29)$$

For sub-domain Ω_4 , one has

$$\begin{aligned} \frac{\partial \Phi_4^2}{\partial z}\bigg|_{z=0} = 0, \quad \frac{\partial \Phi_4^2}{\partial z}\bigg|_{z=h_2} = 0, \quad \Phi_4^2|_{r=0} = \text{finite value} \\ \frac{\partial \Phi_4^3}{\partial r}\bigg|_{r=R_1} = 0, \quad \Phi_4^3|_{r=0} = \text{finite value}, \quad \frac{\partial \Phi_4^3}{\partial z}\bigg|_{z=0} = 0 \end{aligned} \quad (30)$$

For the simplification in expression and study, the following dimensionless co-ordinates and parameters are introduced:

$$\begin{aligned} \xi = \frac{r}{R_2}, \quad \zeta = \frac{z}{R_2}, \quad \Lambda = \omega \sqrt{\frac{R_2}{g}}, \quad \vec{\eta}_k = \frac{\vec{n}_k}{R_2}, \\ \alpha = \frac{R_1}{R_2}, \quad \beta_1 = \frac{h_1}{R_2}, \quad \beta_2 = \frac{h_2}{R_2}, \quad \beta = \frac{h}{R_2} \end{aligned} \quad (31)$$

According to Eqs (5) and (17), Φ , Φ_i and Φ_i^q should be a periodic function of θ and the period is 2π . Considering Eq. (6), one can conclude that

$$\begin{aligned} \Phi = \sum_{m=0}^{\infty} \Phi_m \cos m\theta, \quad \Phi_i = \sum_{m=0}^{\infty} \Phi_{im} \cos m\theta, \quad \Phi_i^q = \sum_{m=0}^{\infty} \Phi_{im}^q \cos m\theta, \\ (i = 1, 2, 3, 4, q = 1, 2, 3) \end{aligned} \quad (32)$$

where m denotes the circumferential wave number. The general solution of Φ_{im}^q corresponding to every liquid sub-domain can be analytically given in an infinite series form by using the method of separation of variables. For sub-domain Ω_1 , one has

$$\begin{aligned} \Phi_{1m}^1 = \sum_{n=1}^{\infty} A_{1mn}^1 \left[e^{\lambda_n^m \zeta} + e^{\lambda_n^m (2\beta_2 - \zeta)} \right] \left[N_m'(\lambda_n^m \alpha) J_m(\lambda_n^m \xi) - \right. \\ \left. J_m'(\lambda_n^m \alpha) N_m(\lambda_n^m \xi) \right] + A_{100}^1 \delta_m^1 \end{aligned} \quad (33)$$

$$\begin{aligned} \Phi_{1m}^2 = \sum_{n=1}^{\infty} A_{1mn}^2 \cos \left(\frac{(2n-1)\pi}{2\beta_1} (\zeta - \beta_2) \right) \times \\ \left[I_m \left(\frac{(2n-1)\pi \xi}{2\beta_1} \right) + k_1^b K_m \left(\frac{(2n-1)\pi \xi}{2\beta_1} \right) \right] \end{aligned} \quad (34)$$

in which, $(\cdot)$ denotes the derivative to ξ . δ_m^1 , λ_n^m and k_1^b satisfy the following equations:

$$\delta_m^1 = \begin{cases} 1 & (m = 0) \\ 0 & (m \neq 0) \end{cases} \quad (35)$$

$$N'_m(\lambda_n^m \alpha) J'_m(\lambda_n^m) - J'_m(\lambda_n^m \alpha) N'_m(\lambda_n^m) = 0 \quad (36)$$

$$I'_m\left(\frac{(2n-1)\pi}{2\beta_1}\right) + k_1^b K'_m\left(\frac{(2n-1)\pi}{2\beta_1}\right) = 0 \quad (37)$$

For sub-domain Ω_2 , one has

$$\Phi_{2m}^1 = \sum_{n=1}^{\infty} A_{2mn}^1 \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha} \zeta} + e^{\frac{\tilde{x}_n^{(m)}}{\alpha} (2\beta_2 - \zeta)} \right] J_m\left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi\right) + A_{200}^1 \delta_m^1 \quad (38)$$

$$\Phi_{2m}^2 = \sum_{n=1}^{\infty} A_{2mn}^2 \cos\left(\frac{(2n-1)\pi}{2\beta_1} (\zeta - \beta_2)\right) I_m\left(\frac{(2n-1)\pi \xi}{2\beta_1}\right) \quad (39)$$

$$\Phi_{2m}^3 = \sum_{n=1}^{\infty} A_{2mn}^3 \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha} \zeta} - e^{\frac{\tilde{x}_n^{(m)}}{\alpha} (2\beta - \zeta)} \right] J_m\left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi\right) + A_{200}^3 \delta_m^1 (\zeta - \beta) \quad (40)$$

in which, $\tilde{x}_n^{(m)}$ satisfies the following equation:

$$J'_m(\tilde{x}_n^{(m)}) = 0 \quad (41)$$

For sub-domain Ω_3 , one has

$$\begin{aligned} \Phi_{3m}^2 = \sum_{n=1}^{\infty} A_{3mn}^2 \cos\left(\frac{n\pi}{\beta_2} \zeta\right) \left[I_m\left(\frac{n\pi}{\beta_2} \xi\right) + k_3^b K_m\left(\frac{n\pi}{\beta_2} \xi\right) \right] + \\ A_{3m0}^2 (\xi^m + \xi^{-m}) \delta_m^2 + A_{300}^2 \delta_m^1 \end{aligned} \quad (42)$$

in which, k_3^b and δ_m^2 satisfy the following equation:

$$I'_m\left(\frac{n\pi}{\beta_2}\right) + k_3^b K'_m\left(\frac{n\pi}{\beta_2}\right) = 0 \quad (43)$$

$$\delta_m^2 = \begin{cases} 1 & (m \neq 0) \\ 0 & (m = 0) \end{cases} \quad (44)$$

For sub-domain Ω_4 , one has

$$\Phi_{4m}^2 = \sum_{n=1}^{\infty} A_{4mn}^2 \cos\left(\frac{n\pi}{\beta_2} \zeta\right) I_m\left(\frac{n\pi}{\beta_2} \xi\right) + A_{4m0}^2 \xi^m \delta_m^2 + A_{400}^2 \delta_m^1 \quad (45)$$

$$\Phi_{4m}^3 = \sum_{n=1}^{\infty} A_{4mn}^3 \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha} \zeta} + e^{-\frac{\tilde{x}_n^{(m)}}{\alpha} \zeta} \right] J_m\left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi\right) + A_{400}^3 \delta_m^1 \quad (46)$$

In Eqs (33), (34), (38)–(40), (42), (45) and (46), A_{imn}^q ($i = 1, 2, 3, 4$, $q = 1, 2, 3$, $m = 0, 1, 2, 3, \dots$, $n = 1, 2, 3, \dots$) are the unknown coefficients which can be determined by the conditions (9), (11), (16), (17). Obviously, the ordered pair (i, q) should belong to the set $\{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 3)\}$.

5. Eigen-frequency equation

Substituting Eqs (18) and (31) into Eqs (9) and (11), the interface conditions between Ω_i and Ω_j can be rewritten as follows:

$$\Phi_i = \Phi_j \text{ on } \Gamma_k \quad (47)$$

$$\frac{\partial \Phi_i}{\partial \vec{\eta}_k} = \frac{\partial \Phi_j}{\partial \vec{\eta}_k} \text{ on } \Gamma_k \quad (48)$$

Introducing Eqs (18) and (31) into Eqs (16) and (17), the free surface conditions can be rewritten into the following mode shape formulations:

$$\left. \frac{\partial \Phi_1}{\partial \zeta} \right|_{\zeta=\beta} - \Lambda^2 \Phi_1|_{\zeta=\beta} = 0 \quad (49)$$

$$\left. \frac{\partial \Phi_2}{\partial \zeta} \right|_{\zeta=\beta} - \Lambda^2 \Phi_2|_{\zeta=\beta} = 0 \quad (50)$$

Substituting Eq. (32) into Eqs (47)–(50) gives

$$\sum_{q=1}^{K_i} \Phi_{im}^q = \sum_{q=1}^{K_j} \Phi_{jm}^q \text{ on } \Gamma_k \quad (51)$$

$$\sum_{q=1}^{K_i} \frac{\partial \Phi_{im}^q}{\partial \vec{\eta}_k} = \sum_{q=1}^{K_j} \frac{\partial \Phi_{jm}^q}{\partial \vec{\eta}_k} \text{ on } \Gamma_k \quad (52)$$

$$\sum_{q=1}^{K_1} \left. \frac{\partial \Phi_{1m}^q}{\partial \zeta} \right|_{\zeta=\beta} - \Lambda^2 \Phi_{1m}^1|_{\zeta=\beta} = 0 \quad (53)$$

$$\sum_{q=1}^{K_1} \left. \frac{\partial \Phi_{2m}^q}{\partial \zeta} \right|_{\zeta=\beta} - \Lambda^2 \Phi_{2m}^1|_{\zeta=\beta} = 0 \quad (54)$$

The series equations with unknown coefficients A_{imn}^q ($q = 1, 2, 3, i = 1, 2, 3, 4, n = 1, 2, \dots$) can be obtained by substituting Eqs (33), (34), (38)–(40), (42), (45), (46) into Eqs (51)–(54). Truncating n in the series up to N gives

$$\begin{aligned} & \sum_{n=1}^N A_{1mn}^1 \left[e^{\lambda_n^m \zeta} + e^{\lambda_n^m (2\beta_2 - \zeta)} \right] \left[N_m'(\lambda_n^m \alpha) J_m(\lambda_n^m \alpha) - J_m'(\lambda_n^m \alpha) N_m(\lambda_n^m \alpha) \right] + A_{100}^1 \delta_m^1 + \\ & \sum_{n=1}^N A_{1mn}^2 \cos \left(\frac{(2n-1)\pi}{2\beta_1} (\zeta - \beta_2) \right) \left[I_m \left(\frac{(2n-1)\pi \alpha}{2\beta_1} \right) + k_1^b K_m \left(\frac{(2n-1)\pi \alpha}{2\beta_1} \right) \right] = \\ & \sum_{n=1}^N A_{2mn}^1 \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha} \zeta} + e^{\frac{\tilde{x}_n^{(m)}}{\alpha} (2\beta_2 - \zeta)} \right] J_m \left(\tilde{x}_n^{(m)} \right) + A_{200}^1 \delta_m^1 + \\ & \sum_{n=1}^N A_{2mn}^2 \cos \left(\frac{(2n-1)\pi}{2\beta_1} (\zeta - \beta_2) \right) I_m \left(\frac{(2n-1)\pi \alpha}{2\beta_1} \right) + \\ & \sum_{n=1}^N A_{2mn}^3 \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha} \zeta} - e^{\frac{\tilde{x}_n^{(m)}}{\alpha} (2\beta - \zeta)} \right] J_m \left(\tilde{x}_n^{(m)} \right) + A_{200}^3 \delta_m^1 (\zeta - \beta) \end{aligned} \quad (55)$$

$$\begin{aligned} & \frac{\pi}{2\beta_1} \sum_{n=1}^N A_{1mn}^2 (2n-1) \cos \left(\frac{(2n-1)\pi}{2\beta_1} (\zeta - \beta_2) \right) \left[I_m' \left(\frac{(2n-1)\pi \alpha}{2\beta_1} \right) + k_1^b K_m' \left(\frac{(2n-1)\pi \alpha}{2\beta_1} \right) \right] = \\ & \frac{\pi}{2\beta_1} \sum_{n=1}^N A_{2mn}^2 (2n-1) \cos \left(\frac{(2n-1)\pi}{2\beta_1} (\zeta - \beta_2) \right) I_m' \left(\frac{(2n-1)\pi \alpha}{2\beta_1} \right) \end{aligned} \quad (56)$$

$$\begin{aligned}
& 2 \sum_{n=1}^N A_{2mn}^1 e^{\frac{\tilde{x}_n^{(m)}}{\alpha} \beta_2} J_m \left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi \right) + A_{200}^1 \delta_m^1 + \sum_{n=1}^N A_{2mn}^2 I_m \left(\frac{(2n-1)\pi\xi}{2\beta_1} \right) + \\
& \sum_{n=1}^N A_{2mn}^3 \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha} \beta_2} - e^{\frac{\tilde{x}_n^{(m)}}{\alpha} (2\beta_1 + \beta_2)} \right] J_m \left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi \right) - A_{200}^3 \delta_m^1 \beta_1 = \\
& \sum_{n=1}^N A_{4mn}^2 (-1)^n I_m \left(\frac{n\pi}{\beta_2} \xi \right) + A_{4m0}^2 \xi^m \delta_m^2 + A_{400}^2 \delta_m^1 + \\
& \sum_{n=1}^N A_{4mn}^3 \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha} \beta_2} + e^{-\frac{\tilde{x}_n^{(m)}}{\alpha} \beta_2} \right] J_m \left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi \right) + A_{400}^3 \delta_m^1
\end{aligned} \tag{57}$$

$$\begin{aligned}
& \frac{1}{\alpha} \sum_{n=1}^N A_{2mn}^3 \tilde{x}_n^{(m)} \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha} \beta_2} + e^{\frac{\tilde{x}_n^{(m)}}{\alpha} (2\beta_1 + \beta_2)} \right] J_m \left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi \right) + A_{200}^3 \delta_m^1 = \\
& \frac{1}{\alpha} \sum_{n=1}^N A_{4mn}^3 \tilde{x}_n^{(m)} \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha} \beta_2} - e^{-\frac{\tilde{x}_n^{(m)}}{\alpha} \beta_2} \right] J_m \left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi \right)
\end{aligned} \tag{58}$$

$$\begin{aligned}
& \sum_{n=1}^N A_{3mn}^2 \cos \left(\frac{n\pi}{\beta_2} \zeta \right) \left[I_m \left(\frac{\alpha n \pi}{\beta_2} \right) + k_3^b K_m \left(\frac{\alpha n \pi}{\beta_2} \right) \right] + A_{3m0}^2 (\alpha^m + \alpha^{-m}) \delta_m^2 + A_{300}^2 \delta_m^1 = \\
& \sum_{n=1}^N A_{4mn}^2 \cos \left(\frac{n\pi}{\beta_2} \zeta \right) I_m \left(\frac{\alpha n \pi}{\beta_2} \right) + A_{4m0}^2 \alpha^m \delta_m^2 + A_{400}^2 \delta_m^1 + \\
& \sum_{n=1}^N A_{4mn}^3 \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha} \zeta} + e^{-\frac{\tilde{x}_n^{(m)}}{\alpha} \zeta} \right] J_m \left(\tilde{x}_n^{(m)} \right) + A_{400}^3 \delta_m^1
\end{aligned} \tag{59}$$

$$\begin{aligned}
& \frac{\pi}{\beta_2} \sum_{n=1}^N A_{3mn}^2 n \cos \left(\frac{n\pi}{\beta_2} \zeta \right) \left[I'_m \left(\frac{\alpha n \pi}{\beta_2} \right) + k_3^b K'_m \left(\frac{\alpha n \pi}{\beta_2} \right) \right] + A_{3m0}^2 m (\alpha^{m-1} - \alpha^{-m-1}) \delta_m^2 = \\
& \frac{\pi}{\beta_2} \sum_{n=1}^N A_{4mn}^2 n \cos \left(\frac{n\pi}{\beta_2} \zeta \right) I'_m \left(\frac{\alpha n \pi}{\beta_2} \right) + A_{4m0}^2 m \alpha^{m-1} \delta_m^2
\end{aligned} \tag{60}$$

$$\begin{aligned}
& \frac{1}{\alpha} \sum_{n=1}^N A_{2mn}^1 \tilde{x}_n^{(m)} \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha} \beta} - e^{\frac{\tilde{x}_n^{(m)}}{\alpha} (\beta_2 - \beta_1)} \right] J_m \left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi \right) + \\
& \frac{\pi}{2\beta_1} \sum_{n=1}^N A_{2mn}^2 (-1)^n (2n-1) I_m \left(\frac{(2n-1)\pi\xi}{2\beta_1} \right) + \frac{2}{\alpha} \sum_{n=1}^N A_{2mn}^3 \tilde{x}_n^{(m)} e^{\frac{\tilde{x}_n^{(m)}}{\alpha} \beta} J_m \left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi \right) + A_{200}^3 \delta_m^1 - \\
& \Lambda_m^2 \left\{ \sum_{n=1}^N A_{2mn}^1 \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha} \beta} + e^{\frac{\tilde{x}_n^{(m)}}{\alpha} (\beta_2 - \beta_1)} \right] J_m \left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi \right) + A_{200}^1 \delta_m^1 \right\} = 0
\end{aligned} \tag{61}$$

$$\sum_{n=1}^N A_{1mn}^1 \lambda_n^m \left[e^{\lambda_n^m \beta} - e^{\lambda_n^m (\beta_2 - \beta_1)} \right] [N'_m (\lambda_n^m \alpha) J_m (\lambda_n^m \xi) - J'_m (\lambda_n^m \alpha) N_m (\lambda_n^m \xi)] +$$

$$\frac{\pi}{2\beta_1} \sum_{n=1}^N A_{1mn}^2 (-1)^n (2n-1) \left[I_m \left(\frac{(2n-1)\pi\xi}{2\beta_1} \right) + k_1^b K_m \left(\frac{(2n-1)\pi\xi}{2\beta_1} \right) \right] - \Lambda_m^2 \left\{ \sum_{n=1}^N A_{1mn}^1 \left[e^{\lambda_n^m \beta} + e^{\lambda_n^m (\beta_2 - \beta_1)} \right] \left[N_m'(\lambda_n^m \alpha) J_m(\lambda_n^m \xi) - J_m'(\lambda_n^m \alpha) N_m(\lambda_n^m \xi) \right] + A_{100}^1 \delta_m^1 \right\} = 0 \quad (62)$$

where Λ_m is the non-dimensional frequency parameter corresponding to the circumferential wave number m . Space coordinate ζ in Eqs (55), (56) and (59), (60) can be eliminated by using the Fourier series expansion and the space coordinate ξ in Eqs (57), (58) and (61), (62) can be eliminated by using the Bessel series expansion. The total system of linear equations about the unknown coefficients $\{A_m\}$ can be obtained by truncating these expansions up to a proper number of terms N :

$$[D_m(\Lambda_m)] \{A_m\} = 0 \quad (63)$$

in which, the matrix $[D_m(\Lambda_m)]$ and the vector $\{A_m\}$ are, respectively,

$$[D_m(\Lambda_m)] = \begin{bmatrix} [0] & [0] & [0] & [0] & [0] & [d_{mn\bar{n}}^{16}] & [d_{mn\bar{n}}^{17}] & [0] \\ [0] & [0] & [0] & [0] & [0] & [d_{mn\bar{n}}^{26}] & [d_{mn\bar{n}}^{27}] & [d_{mn\bar{n}}^{28}] \\ [0] & [d_{mn\bar{n}}^{32}] & [0] & [d_{mn\bar{n}}^{34}] & [0] & [0] & [0] & [0] \\ [d_{mn\bar{n}}^{41}] & [d_{mn\bar{n}}^{42}] & [d_{mn\bar{n}}^{43}] & [d_{mn\bar{n}}^{44}] & [d_{mn\bar{n}}^{45}] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [d_{mn\bar{n}}^{55}] & [0] & [0] & [d_{mn\bar{n}}^{58}] \\ [d_{mn\bar{n}}^{61}] & [0] & [0] & [d_{mn\bar{n}}^{64}] & [d_{mn\bar{n}}^{65}] & [0] & [d_{mn\bar{n}}^{67}] & [d_{mn\bar{n}}^{68}] \\ [0] & [0] & [d_{mn\bar{n}}^{73}] & [d_{mn\bar{n}}^{74}] & [d_{mn\bar{n}}^{75}] & [0] & [0] & [0] \\ [d_{mn\bar{n}}^{81}] & [d_{mn\bar{n}}^{82}] & [0] & [0] & [0] & [0] & [0] & [0] \end{bmatrix} \quad (64)$$

$$\{A_m\} = [\{A_{1mn}^1\}, \{A_{1mn}^2\}, \{A_{2mn}^1\}, \{A_{2mn}^2\}, \{A_{2mn}^3\}, \{A_{3mn}^2\}, \{A_{4mn}^2\}, \{A_{4mn}^3\}]^T \quad (65)$$

Nonzero elements in $[D_m(\Lambda_m)]$ are the Fourier series expansion coefficients and the Bessel series expansion coefficients, which can be numerically calculated by the Gaussian integrals if the non-dimensional frequency parameter Λ_m is given. The detailed expressions of $\{A_m\}$ and nonzero elements in $[D_m(\Lambda_m)]$ are given in Appendix A. As Eq. (63) should have a nonzero solution, one obtains the eigen-frequency equation as follows

$$|D_m(\Lambda_m)| = 0 \quad (66)$$

The non-dimensional frequency parameter Λ_{ml} (The subscript l denotes the radial wave number of the modal, $l = 1, 2, 3, \dots$) can be obtained by using the searching root approach to Eq. (66). Substituting Λ_{ml} into Eq. (63) and solving the group of linear equations give the corresponding coefficients $\{A_m\}$. Substituting the coefficients $\{A_m\}$ into Eqs (33), (34), (38)–(40), (42), (45), (46) gives the solution of each sub-main velocity potential.

Introducing the sloshing height function f , one can rewrite the equation of the free surface in the form of

$$f + \frac{1}{g} \frac{\partial \phi}{\partial t} = 0 \text{ on } S \quad (67)$$

When liquid is undergoing free sloshing in the container, f is a harmonic function of time, therefore

$$f(r, \theta, t) = j\omega e^{j\omega t} \frac{F(r, \theta)}{R_2} \quad (68)$$

where $F(r, \theta)$ is the free surface mode shape. According to Eqs (32) and (67), $F(r, \theta)$ should be in the form of

$$F(r, \theta) = \sum_{m=0}^{\infty} F_m(r) \cos m\theta \quad (69)$$

in which, $F_m(r)$ is the radial surface profile of liquid corresponding to the mode Φ_m . Substituting Eqs (18), (31), (32), (68) and (69) into Eq. (67), one can obtain

Table 1
Convergence of Λ_{ml}^2 ($m = 0, 1, l = 1, 2, 3, 4$) versus truncated terms number for $\beta_1 = 0.2$

α	m	l	$N = 5$	$N = 7$	$N = 9$	$N = 11$	$N = 13$	$N = 15$	$N = 17$	$N = 19$
0.3	0	1	2.491	2.491	2.491	2.491	2.491	2.492	2.492	2.492
		2	6.317	6.318	6.318	6.318	6.318	6.318	6.318	6.318
		3	9.932	9.933	9.933	9.933	9.933	9.933	9.933	9.933
		4	13.237	13.237	13.238	13.238	13.238	13.238	13.238	13.238
	1	1	0.708	0.709	0.71	0.71	0.71	0.711	0.711	0.711
		2	4.508	4.513	4.515	4.517	4.518	4.519	4.52	4.52
		3	8.208	8.207	8.207	8.207	8.207	8.207	8.207	8.207
		4	11.618	11.583	11.572	11.567	11.564	11.562	11.56	11.559
0.7	0	1	3.104	3.113	3.118	3.121	3.123	3.125	3.126	3.126
		2	6.768	6.77	6.77	6.769	6.769	6.769	6.769	6.769
		3	10.088	10.089	10.09	10.09	10.09	10.091	10.091	10.091
		4	13.281	13.287	13.288	13.287	13.287	13.287	13.286	13.286
	1	1	1.266	1.273	1.277	1.279	1.281	1.283	1.284	1.284
		2	4.949	4.948	4.947	4.947	4.947	4.947	4.947	4.947
		3	8.412	8.411	8.411	8.411	8.411	8.411	8.411	8.411
		4	11.649	11.649	11.649	11.649	11.649	11.649	11.649	11.649

$$\sum_{m=0}^{\infty} F_m(\xi) \cos m\theta + \sum_{m=0}^{\infty} \Lambda_m^2 \Phi_m(\xi, \beta) \cos m\theta = 0 \quad (70)$$

From the above equation, we obtain the radial surface profile of liquid for circumferential wave number m :

$$F_m(\xi) = -\Lambda_m^2 \Phi_m(\xi, \beta) \quad (71)$$

6. Convergence and comparison

6.1. Convergence study

In order to demonstrate the high accuracy and small computational cost of the proposed method, a convergence study on non-dimensional frequency parameters is carried out. Two different baffle-container radius ratios are considered: $\alpha = 0.3, 0.7$. The ratio of fluid height and container radius is taken as $\beta = 1$ and two different positions of the baffle are considered: $\beta_1 = 0.2, 0.4$. Table 1 corresponding to $\beta_1 = 0.2$ and Table 2 corresponding to $\beta_1 = 0.4$ give the first four non-dimensional frequency parameters Λ_{ml}^2 ($l = 1, 2, 3, 4$), respectively, for $m = 0$ and $m = 1$ with respect to eight different truncated terms: $N = 5, 7, 9, 11, 13, 15, 17, 19$. From these two tables, it is seen that the non-dimensional frequency parameters converge quickly with the increase of the terms number. Both accuracy and convergent speed are affected by the position of the baffle and the baffle-container radius ratio. However, these effects are very small. One can obtain three significant figures when the truncated terms number is greater than 9. This ensures a small-size eigen-frequency equation to be solved for all cases. A lot of numerical experiments with various α , β_1 and N have shown that this analytical method can guarantee at least four significant figures for the first four non-dimensional frequency parameters when the truncated terms number $N \geq 17$. Therefore, in all the following calculations, 17 terms of series are fixed to be taken.

6.2. Comparison study

Biswal et al. [22] evaluated the natural frequencies of liquid sloshing in a liquid-filled cylindrical rigid container with a rigid annular baffle by the finite elements method. In order to make a comparison with Biswal's results, the fundamental non-dimensional frequency parameters for $m = 1$ are computed for the parameters: $\alpha = 0.2, 0.4, 0.6, 0.8$ and $\beta_1 = 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$, which are given in Table 3. It can be seen from Table 3 that the present results are in good agreement with those of Biswal et al. [22]. Gavriluk et al. [26] used the Green function method to develop a semi-analytical approach and provided accurate approximations to natural frequencies and modes of liquid sloshing in a rigid vertical circular cylindrical container with thin rigid-ring horizontal baffles. The first four non-dimensional frequency parameters for $m = 0, 1$ are presented for the parameters: $\alpha = 0.7, \beta_2 = 0.5, \beta_1 = 0.1, 0.5$, as given in Table 4. It can be seen from Table 4 that the present results are in good agreement with the results of Gavriluk et al. [26].

Table 2
Convergence of Λ_{ml}^2 ($m = 0, 1, l = 1, 2, 3, 4$) versus truncated terms number for $\beta_1 = 0.4$

α	m	l	$N = 5$	$N = 7$	$N = 9$	$N = 11$	$N = 13$	$N = 15$	$N = 17$	$N = 19$
0.3	0	1	3.485	3.491	3.493	3.494	3.495	3.495	3.496	3.496
		2	6.863	6.92	6.942	6.952	6.958	6.961	6.964	6.964
		3	10.058	10.116	10.139	10.15	10.156	10.159	10.162	10.162
		4	13.31	13.318	13.32	13.321	13.322	13.322	13.323	13.323
	1	1	1.196	1.197	1.198	1.198	1.199	1.199	1.199	1.199
		2	5.226	5.228	5.228	5.229	5.229	5.229	5.229	5.229
		3	8.482	8.505	8.513	8.517	8.519	8.521	8.522	8.522
		4	11.427	11.571	11.628	11.655	11.67	11.679	11.685	11.689
0.7	0	1	3.646	3.66	3.666	3.669	3.671	3.673	3.674	3.674
		2	6.919	6.96	6.976	6.983	6.987	6.99	6.992	6.992
		3	10.169	10.171	10.171	10.171	10.172	10.172	10.172	10.172
		4	13.05	13.203	13.257	13.282	13.295	13.303	13.308	13.311
	1	1	1.539	1.542	1.545	1.546	1.547	1.548	1.548	1.549
		2	5.223	5.251	5.262	5.267	5.27	5.272	5.274	5.274
		3	8.479	8.507	8.517	8.522	8.525	8.527	8.528	8.528
		4	11.668	11.689	11.696	11.699	11.701	11.702	11.703	11.704

Table 3
Comparison study of Λ_{11} for $\beta = 1$

β_1	Present	Biswal et al. [22]	relative error	Present	Biswal et al. [22]	relative error
			$\alpha = 0.2$			
0.05	0.4296	0.43293	0.77%	0.50664	0.51788	2.17%
0.1	0.59499	0.60075	0.96%	0.67657	0.69277	2.34%
0.2	0.81724	0.8222	0.60%	0.88617	0.9003	1.57%
0.3	0.97015	0.97426	0.42%	1.02286	1.034	1.08%
0.4	1.08056	1.08397	0.31%	1.11931	1.12789	0.76%
0.5	1.1606	1.16349	0.25%	1.18834	1.19493	0.55%
0.6	1.21823	1.22075	0.21%	1.23754	1.24267	0.41%
0.7	1.25934	1.26161	0.18%	1.27227	1.27635	0.32%
0.8	1.2884	1.29054	0.17%	1.29637	1.29979	0.26%
			$\alpha = 0.6$			
0.05	0.681725	0.70242	2.95%	1.047903	1.10269	4.97%
0.1	0.857999	0.88887	3.47%	1.155597	1.19425	3.24%
0.2	1.034904	1.05742	2.13%	1.228239	1.24909	1.67%
0.3	1.132431	1.14969	1.50%	1.261549	1.27612	1.14%
0.4	1.197593	1.20985	1.01%	1.283635	1.29339	0.75%
0.5	1.242441	1.25116	0.70%	1.298455	1.30502	0.50%
0.6	1.273556	1.27977	0.49%	1.308499	1.31295	0.34%
0.7	1.294907	1.29938	0.34%	1.315202	1.31825	0.23%
0.8	1.309111	1.31249	0.26%	1.319472	1.32168	0.17%

7. Numerical results

7.1. Natural frequencies versus R_1/R_2

The curves of non-dimensional frequency parameters versus baffle-container radius ratio $\alpha = R_1/R_2$ for $m = 0, 1, 2$ are shown in Fig. 3. In order to illustrate the effect of the baffle position on frequency parameters, four different positions of the baffle are considered: $\beta_1 = 0.2, 0.4, 0.6, 0.8$. For $\beta_1 = 0.2, 0.4$, the first three non-dimensional frequency parameters are illustrated. For $\beta_1 = 0.6$, the first two non-dimensional frequency parameters are illustrated and for $\beta_1 = 0.8$, only the fundamental non-dimensional frequency parameters are illustrated. From Fig. 3, two important points can be found. One is that the baffle placed near to the free surface has a great influence on the non-dimensional frequency parameters while the effect of the baffle is gradually reduced when it is moved towards the bottom of the container. The second is that the non-dimensional frequency parameter closes to a constant with the increase of α when β_1 is relatively large (for example, $\beta_1 \geq 0.6$). And the differences among the curves for

Table 4
Comparison study of Λ_{ml}^2 ($m = 0, 1, 2, l = 1, 2, 3, 4$) for $\alpha = 0.7, \beta_2 = 0.5$

β_1	l	$m = 0$			$m = 1$			$m = 2$		
		Present	Gavrilyuk et al. [26]	relative error	Present	Gavrilyuk et al. [26]	relative error	Present	Gavrilyuk et al. [26]	relative error
0.1	1	2.2656	2.2866	0.92%	0.9279	0.9403	1.31%	1.5166	1.5354	1.23%
	2	6.2014	6.1973	0.06%	4.1898	4.1851	0.11%	5.9411	5.9344	0.11%
	3	9.5923	9.6079	0.16%	7.915	7.9223	0.09%	9.3794	9.3976	0.19%
	4	12.8247	12.8083	0.13%	11.144	11.1509	0.06%	12.6638	12.6491	0.12%
0.5	1	3.7468	3.7591	0.33%	1.6191	1.624	0.30%	2.9009	2.9062	0.18%
	2	6.9802	7.0109	0.44%	5.2891	5.3119	0.43%	6.6678	6.7008	0.49%
	3	10.1723	10.1733	0.01%	8.5148	8.5354	0.24%	9.9637	9.9692	0.06%
	4	13.229	13.3237	0.71%	11.6923	11.706	0.12%	13.0979	13.1704	0.55%

Fig. 3. Non-dimensional frequency parameters Λ_{ml}^2 ($m = 0, 1, 2, l = 1, 2, 3$) versus α for $\beta = 1$ and four different β_1 (— $\beta_1 = 0.2$, --- $\beta_1 = 0.4$, $\beta_1 = 0.6$, - · - $\beta_1 = 0.8$).

relatively large β_1 are reduced with the increase of the radial wave number. These curves also show the quite different quantitative behavior of non-dimensional frequency parameters for different radial wave numbers when β_1 is relatively small. The curves for $\Lambda_{01}^2, \Lambda_{11}^2$ and Λ_{21}^2 when $\beta_1 = 0.2$ only have one steep stage. The curves for $\Lambda_{02}^2, \Lambda_{12}^2$ and Λ_{22}^2 when $\beta_1 = 0.2$ have two steep stages and the curves for $\Lambda_{03}^2, \Lambda_{13}^2$ and Λ_{23}^2 when $\beta_1 = 0.2$ have three steep stages. A lot of calculations demonstrate that the number of the steep stages is just equal to the radial wave number.

7.2. Natural frequencies versus h_1/R_2

Figure 4 illustrates the first three non-dimensional frequency parameters versus the position of the baffle $\beta_1 = h_1/R_2$ for $m = 0, 1, 2$. Three different baffle-container radius ratios are considered: $\alpha = 0.3, 0.6, 0.75$. From Fig. 4 one can see that the baffle with a small inner diameter has more influence on the non-dimensional frequency

Fig. 4. Non-dimensional frequency parameters Λ_{ml}^2 ($m = 0, 1, 2, l = 1, 2, 3$) versus β_1 for $\beta = 1$ and three different α (— $\alpha = 0.3$, $\cdots$ $\alpha = 0.6$, --- $\alpha = 0.75$).

parameters. The effect of the baffle is gradually reduced as its inner diameter increases. It is also seen that, for different baffle-container radius ratios, the non-dimensional frequency parameters converge rapidly to the same value as the distance between the free-surface and the baffle increases. Moreover, one can see from Fig. 4 that, for a given α the speed convergent to the constant value of non-dimensional frequency parameters increases with the increase of the radial wave number.

7.3. Radial surface profiles of natural modes

The free surface mode shapes have been investigated according to Eq. (71). The radial surface profiles responding to Λ_{ml}^2 have been written as $F_{ml}(\xi)$. In Fig. 5, nine radial surface profiles $F_{ml}(\xi)$ ($m = 0, 1, 2, l = 1, 2, 3$) are plotted for parameters: $\beta = 1$ and $\beta_1 = 0.1$ when $\alpha = 0.2, 0.4, 0.6, 0.8$. Nine radial surface profiles for parameters: $\beta = 1$ and $\beta_1 = 0.3$ when $\alpha = 0.2, 0.4, 0.6, 0.8$ are shown in Fig. 6. It is seen that the differences among curves for different α in Fig. 5 are more obvious than those in Fig. 6 and the baffle placed near to the free surface has more influence on the mode shape. Moreover, one can find from Figs 5 and 6 that for a constant β_1 , the differences among the mode shapes for different α reduced as the radial wave number increases.

8. Conclusions

Pursuing an analytical method to study the linear natural sloshing problems in rigid containers with a rigid baffle, an efficient analytical approach was developed to obtain the natural frequencies and their corresponding modes. The complicated liquid domain is divided into several simple sub-domains by introducing the artificial interfaces. Analytical solutions of velocity potential corresponding to every sub-domain are obtained by using the method of separation of variables and the superposition principle. The Eigen-frequency equation is precisely obtained by using

the Fourier-Bessel expansion on the free surface and the artificial interfaces. The natural frequencies can be obtained by using the numerically searching root approach.

High accuracy and small computational cost are the advantage of the present method over the traditional numerical methods. This analytical method can be directly applied to deal with the fluid sloshing in cylindrical containers with rigid multi-baffles. The application range of this method can be expanded to solve the linear liquid sloshing in flexible containers with flexible baffles. The response of liquid sloshing can also be solved easily by using the present method, which will be discussed in a forthcoming paper.

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Appendix A

The detailed expressions of $\{A_m\}$ and nonzero elements in $[D_m(\Lambda_m)]$

$$\{A_{imn}^q\} = [A_{im1}^q, A_{im2}^q, A_{im3}^q, \dots, A_{imN}^q], q = 1, 2, 3, i = 1, 2, 3, 4 \quad (72)$$

$$d_{mn\bar{n}}^{17} = \delta_{n\bar{n}} I'_m \left(\frac{n\pi}{\beta_2} \alpha \right) \quad (73)$$

$$d_{mn\bar{n}}^{16} = -\delta_{n\bar{n}} \left[I'_m \left(\frac{n\pi}{\beta_2} \alpha \right) + k_3^b K'_m \left(\frac{n\pi}{\beta_2} \alpha \right) \right] \quad (74)$$

$$d_{mn\bar{n}}^{27} = \delta_{n\bar{n}} \int_0^{\beta_2} \cos^2 \left(\frac{n\pi}{\beta_2} \zeta \right) d\zeta I_m \left(\frac{n\pi}{\beta_2} \alpha \right) \quad (75)$$

$$d_{mn\bar{n}}^{26} = -\delta_{n\bar{n}} \int_0^{\beta_2} \cos^2 \left(\frac{n\pi}{\beta_2} \zeta \right) d\zeta \left[I_m \left(\frac{n\pi}{\beta_2} \alpha \right) + k_3^b K_m \left(\frac{n\pi}{\beta_2} \alpha \right) \right] \quad (76)$$

$$d_{mn\bar{n}}^{28} = \int_0^{\beta_2} \left[e^{\frac{\tilde{x}_{\bar{n}}^{(m)}}{\alpha} \zeta} + e^{-\frac{\tilde{x}_{\bar{n}}^{(m)}}{\alpha} \zeta} \right] \cos \left(\frac{n\pi}{\beta_2} \zeta \right) d\zeta J_m \left(\tilde{x}_{\bar{n}}^{(m)} \right) \quad (77)$$

$$d_{mn\bar{n}}^{34} = \delta_{n\bar{n}} I'_m \left(\frac{(2n-1)\pi\alpha}{2\beta_1} \right) \quad (78)$$

$$d_{mn\bar{n}}^{32} = -\delta_{n\bar{n}} \left[I'_m \left(\frac{(2n-1)\pi\alpha}{2\beta_1} \right) + k_1^b K'_m \left(\frac{(2n-1)\pi\alpha}{2\beta_1} \right) \right] \quad (79)$$

$$a_{mn\bar{n}}^{44} = \begin{cases} \frac{(2\bar{n}-1)(-1)^{\bar{n}}\pi}{\beta_1 \Lambda_m^2 \alpha^2} \int_0^\alpha \xi I_0 \left(\frac{(2\bar{n}-1)\pi\xi}{2\beta_1} \right) d\xi \int_{\beta_2}^\beta \cos \left(\frac{(2n-1)(\zeta-\beta_2)\pi}{2\beta_1} \right) d\zeta & (m=0) \\ 0 & (m \neq 0) \end{cases} \quad (80)$$

$$b_{mn\bar{n}}^{44} = \delta_{n\bar{n}} \int_{\beta_2}^\beta \cos^2 \left(\frac{(2n-1)(\zeta-\beta_2)\pi}{2\beta_1} \right) d\zeta I_m \left(\frac{(2n-1)\pi\xi}{2\beta_1} \right) \quad (81)$$

$$d_{mn\bar{n}}^{44} = a_{mn\bar{n}}^{44} + d_{mn\bar{n}}^{44} \quad (82)$$

$$a_{mn\bar{n}}^{42} = \begin{cases} \frac{(2\bar{n}-1)(-1)^{\bar{n}+1}\pi}{\beta_1\Lambda_m^2(1-\alpha^2)} \int_{\alpha}^1 \xi \left[I_0 \left(\frac{(2\bar{n}-1)\pi\xi}{2\beta_1} \right) + k_1^b K_0 \left(\frac{(2\bar{n}-1)\pi\xi}{2\beta_1} \right) \right] d\xi \times \\ \int_{\beta_2}^{\beta} \cos \left(\frac{(2n-1)(\zeta-\beta_2)\pi}{2\beta_1} \right) d\zeta & (m=0) \\ 0 & (m \neq 0) \end{cases} \quad (83)$$

$$b_{mn\bar{n}}^{42} = -\delta_{n\bar{n}} \int_{\beta_2}^{\beta} \cos^2 \left(\frac{(2n-1)(\zeta-\beta_2)\pi}{2\beta_1} \right) d\zeta \times \left[I_m \left(\frac{(2n-1)\pi\alpha}{2\beta_1} \right) + k_1^b K_m \left(\frac{(2n-1)\pi\alpha}{2\beta_1} \right) \right] \quad (84)$$

$$d_{mn\bar{n}}^{42} = a_{mn\bar{n}}^{42} + d_{mn\bar{n}}^{42} \quad (85)$$

$$d_{mn\bar{n}}^{45} = \int_0^{\beta_1} \cos \left(\frac{(2n-1)(\zeta-\beta_2)\pi}{2\beta_1} \right) \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha}\zeta} - e^{\frac{\tilde{x}_n^{(m)}}{\alpha}(2\beta-\zeta)} \right] d\zeta J_m \left(\tilde{x}_n^{(m)} \right) \quad (86)$$

$$d_{mn\bar{n}}^{43} = \int_{\beta_2}^{\beta} \cos \left(\frac{(2n-1)(\zeta-\beta_2)\pi}{2\beta_1} \right) \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha}\zeta} + e^{\frac{\tilde{x}_n^{(m)}}{\alpha}(2\beta_2-\zeta)} \right] d\zeta J_m \left(\tilde{x}_n^{(m)} \right) \quad (87)$$

$$d_{mn\bar{n}}^{41} = - \int_{\beta_2}^{\beta} \cos \left(\frac{(2n-1)(\zeta-\beta_2)\pi}{2\beta_1} \right) \left[e^{\lambda_n^m \zeta} + e^{\lambda_n^m (2\beta_2-\zeta)} \right] d\zeta \times [N'_m(\lambda_n^m \alpha) J_m(\lambda_n^m \alpha) - J'_m(\lambda_n^m \alpha) N_m(\lambda_n^m \alpha)] \quad (88)$$

$$d_{mn\bar{n}}^{58} = \delta_{n\bar{n}} \left[e^{-\frac{\tilde{x}_n^{(m)}}{\alpha}\beta_2} - e^{\frac{\tilde{x}_n^{(m)}}{\alpha}\beta_2} \right] \quad (89)$$

$$d_{mn\bar{n}}^{55} = \delta_{n\bar{n}} \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha}\beta_2} + e^{\frac{\tilde{x}_n^{(m)}}{\alpha}(2\beta_1+\beta_2)} \right] \quad (90)$$

$$d_{mn\bar{n}}^{67} = \int_0^{\alpha} \xi J_m \left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi \right) I_m \left(\frac{\bar{n}\pi}{\beta_2} \xi \right) d\xi \quad (91)$$

$$d_{mn\bar{n}}^{64} = - \int_0^{\alpha} \xi J_m \left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi \right) I_m \left(\frac{(2\bar{n}-1)\pi\xi}{2\beta_1} \right) d\xi \quad (92)$$

$$a_{mn\bar{n}}^{68} = \begin{cases} \frac{(1-\alpha^{-2m})}{2\beta_2\alpha^{-m}} \left\{ \int_0^{\beta_2} \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha}\zeta} + e^{-\frac{\tilde{x}_n^{(m)}}{\alpha}\zeta} \right] d\zeta J_m \left(\tilde{x}_n^{(m)} \right) \right\} \int_0^{\alpha} \xi^m J_m \left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi \right) d\xi & (m \neq 0) \\ 0 & (m = 0) \end{cases} \quad (93)$$

$$b_{mn\bar{n}}^{68} = \delta_{n\bar{n}} \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha}\beta_2} + e^{-\frac{\tilde{x}_n^{(m)}}{\alpha}\beta_2} \right] \int_0^{\alpha} \xi J_m^2 \left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi \right) d\xi \quad (94)$$

$$d_{mn\bar{n}}^{68} = a_{mn\bar{n}}^{68} + b_{mn\bar{n}}^{68} \quad (95)$$

$$d_{mn\bar{n}}^{65} = -\delta_{n\bar{n}} \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha}\beta_2} - e^{\frac{\tilde{x}_n^{(m)}}{\alpha}(2\beta_1+\beta_2)} \right] \int_0^{\alpha} \xi J_m^2 \left(\frac{\tilde{x}_n^{(m)}}{R_1} \xi \right) d\xi \quad (96)$$

$$d_{mn\bar{n}}^{61} = -2\delta_{n\bar{n}} e^{\frac{\tilde{x}_n^{(m)}}{\alpha}\beta_2} \int_0^{\alpha} \xi J_m^2 \left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi \right) d\xi \quad (97)$$

$$d_{mn\bar{n}}^{75} = 2\delta_{n\bar{n}} \frac{\tilde{x}_n^{(m)}}{\alpha} e^{\frac{\tilde{x}_n^{(m)}}{\alpha}\beta} \int_0^\alpha \xi J_m^2 \left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi \right) d\xi \quad (98)$$

$$d_{mn\bar{n}}^{74} = \frac{(2\bar{n}-1)(-1)^{\bar{n}}\pi}{2\beta_1} \int_0^\alpha \xi J_m \left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi \right) I_m \left(\frac{(2\bar{n}-1)\pi\xi}{2\beta_1} \right) d\xi \quad (99)$$

$$a_{mn\bar{n}}^{73} = -\delta_{n\bar{n}} \Lambda_m^2 \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha}\beta} + e^{\frac{\tilde{x}_n^{(m)}}{\alpha}(\beta_2-\beta_1)} \right] \int_0^\alpha \xi J_m^2 \left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi \right) d\xi \quad (100)$$

$$b_{mn\bar{n}}^{73} = \delta_{n\bar{n}} \frac{\tilde{x}_n^{(m)}}{\alpha} \left[e^{\frac{\tilde{x}_n^{(m)}}{\alpha}\beta} - e^{\frac{\tilde{x}_n^{(m)}}{\alpha}(\beta_2-\beta_1)} \right] \int_0^\alpha \xi J_m^2 \left(\frac{\tilde{x}_n^{(m)}}{\alpha} \xi \right) d\xi \quad (101)$$

$$d_{mn\bar{n}}^{73} = a_{mn\bar{n}}^{73} + b_{mn\bar{n}}^{73} \quad (102)$$

$$d_{mn\bar{n}}^{82} = \frac{(2\bar{n}-1)(-1)^{\bar{n}}\pi}{2\beta_1} \int_\alpha^1 \xi [N'_m(\lambda_n^m \alpha) J_m(\lambda_n^m \xi) - J'_m(\lambda_n^m \alpha) N_m(\lambda_n^m \xi)] \times \\ \left[I_m \left(\frac{(2\bar{n}-1)\pi\xi}{2\beta_1} \right) + k_1^b K_m \left(\frac{(2\bar{n}-1)\pi\xi}{2\beta_1} \right) \right] d\xi \quad (103)$$

$$a_{mn\bar{n}}^{81} = -\delta_{n\bar{n}} \Lambda_m^2 \left[e^{\lambda_n^m \beta} + e^{\lambda_n^m (\beta_2-\beta_1)} \right] \times \\ \int_\alpha^1 \xi [N'_m(\lambda_n^m \alpha) J_m(\lambda_n^m \xi) - J'_m(\lambda_n^m \alpha) N_m(\lambda_n^m \xi)]^2 d\xi \quad (104)$$

$$b_{mn\bar{n}}^{81} = \delta_{n\bar{n}} \lambda_n^m \left[e^{\lambda_n^m \beta} - e^{\lambda_n^m (\beta_2-\beta_1)} \right] \times \\ \int_\alpha^1 \xi [N'_m(\lambda_n^m \alpha) J_m(\lambda_n^m \xi) - J'_m(\lambda_n^m \alpha) N_m(\lambda_n^m \xi)]^2 d\xi \quad (105)$$

$$d_{mn\bar{n}}^{81} = a_{mn\bar{n}}^{81} + b_{mn\bar{n}}^{81} \quad (106)$$

in which, $n, \bar{n} = 1, 2, 3, \dots, N$. $\delta_{n\bar{n}}$ satisfies:

$$\delta_{n\bar{n}} = \begin{cases} 0 & (n \neq \bar{n}) \\ 1 & (n = \bar{n}) \end{cases} \quad (107)$$

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Fig. 5. Radial surface profiles F_{ml} ($m = 0, 1, 2, l = 1, 2, 3$) of the natural modes for $\beta = 1, \beta_1 = 0.1$ and four different α (— $\alpha = 0.2$, --- $\alpha = 0.4$, $\alpha = 0.6$, -.- $\alpha = 0.8$).

Fig. 6. Radial surface profiles F_{ml} ($m = 0, 1, 2, l = 1, 2, 3$) of the natural modes for parameters: $\beta = 1, \beta_1 = 0.3$ and four different α ($\alpha = 0.2, \alpha = 0.4, \alpha = 0.6, \alpha = 0.8$).