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Boundary characteristic orthogonal polynomials in the study of transverse vibrations of nonhomogeneous rectangular plates with bilinear thickness variation

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Abstract. The free transverse vibrations of thin nonhomogeneous rectangular plates of variable thickness have been studied using boundary characteristic orthogonal polynomials in the Rayleigh-Ritz method. Gram-Schmidt process has been used to generate these orthogonal polynomials in two variables. The thickness variation is bidirectional and is the cartesian product of linear variations along two concurrent edges of the plate. The nonhomogeneity of the plate is assumed to arise due to linear variations in Young's modulus and density of the plate material with the in-plane coordinates. Numerical results have been computed for four different combinations of clamped, simply supported and free edges. Effect of the nonhomogeneity and thickness variation with varying values of aspect ratio on the natural frequencies of vibration is illustrated for the first three modes of vibration. Three dimensional mode shapes for all the four boundary conditions have been presented. A comparison of results with those available in the literature has been made.

Keywords: Nonhomogeneous, rectangular, bilinear thickness, Rayleigh-Ritz

1. Introduction

Plates of various geometries are widely used as structural components in various technological situations ranging from household goods such as mixer, grinder etc. to modern space technology. By appropriate variation of plate thickness, these plates can have significantly greater efficiency for vibration as compared to the plates of uniform thickness and also provide the advantage of reduction in weight and size. Numerous studies have been made by researchers on free vibrations of rectangular plates with different types of thickness variations and boundary conditions. The work up to 1985 has been compiled by Leissa in his monograph [1] and a series of review articles [2–5]. The thickness variations are mainly unidirectional such as linear, parabolic, stepped, quadratic and exponential etc. The analysis has been presented using analytical, numerical or approximate methods. Notable contributions, made thereafter are listed in references [6–15]. In this context, authors have come across a limited number of papers in which the thickness of the plate varies in both the directions and reported in references [16–23], to mention a few. Out of these, Sakiyama and Huang [19] considered sinusoidal variation, Cheung and Zhou [20] assumed the power functions of both the coordinates and the rest deal with bilinear variation in thickness.

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Nonhomogeneous elastic plates find their applications in the design of space vehicle, modern missile and aircraft wings. Various models for the nonhomogeneity of the plate material are available in the literature and a brief review is given in reference [24]. Recently, a number of papers have appeared in the literature analyzing the effect of nonhomogeneity on the vibrational behavior of rectangular plates and listed in references [25–27]. In these studies, it has been assumed that nonhomogeneity of the plate material i.e. material properties are functions of only one variable. However, in many practical situations, particularly in modern missile technology and microelectronics, plate type structural components have to work under high temperature conditions, which cause nonhomogeneity. For such type of situations, it is more appropriate to take the variation in the mechanical properties of the material as the function of two variables instead of one. During the survey of literature, it has been found that (i) no work has been done dealing with the vibration of rectangular plates in which the nonhomogeneity of the plate material depends upon the in-plane variables (ii) the boundary characteristic orthogonal polynomials have been extensively used for analyzing the dynamic behavior of the plates of various geometries under different boundary conditions. The simplification of the eigenvlaue problem and the rapid convergence are the main characteristics of the approach.

Keeping in view the above fact, a study dealing with nonhomogeneous rectangular plates of varying thickness using two dimensional boundary characteristic orthogonal polynomials in the Rayleigh-Ritz method is presented employing classical plate theory. These orthogonal polynomials may be generated either by Gram-Schmidt process [28] or using the recurrence relation [29]. In the present work, these orthogonal polynomials are generated using the Gram-Schmidt process which has an advantage over the procedure proposed by Liew et al. [30] and Bhat [31] that all the polynomials will automatically satisfy the essential boundary conditions. Further, this procedure to generate orthogonal polynomials for a particular boundary condition need not be repeated for various values of aspect ratio of the plate. Thus the present methodology saves time and efforts. Nonhomogeneity is assumed to arise due to linear variations in Young's modulus and density of the plate material with both the in-plane variables. The value of Poisson ratio is assumed to remain constant. The thickness of the plate is varying bidirectionally and is the cartesian product of linear variations along the two concurrent edges of the plate. The above considerations give-rise seven plate parameters and study of their effects on the natural frequencies for all the possible 21 combinations of classical boundary conditions at the four edges generates the huge data. In the present work, only four different combinations of edge conditions namely: CCCC- all the four edges are clamped; SCSC- two opposite edges are clamped and other two are simply supported; FCFC- two opposite edges are clamped and other two are free; FSFS- two opposite edges are simply supported and the other two are free have been considered. Mode shapes for the first three frequencies of a specified plate have been plotted. A comparison of results has been presented.

2. Formulation and solution of the problem

Consider an isotropic nonhomogeneous rectangular plate of varying thickness $h(x, y)$ occupying the domain $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$ in $xy-$ plane, where a and b are the length and the breadth of the plate, respectively. The $x-$ and $y-$ axes are taken along the edges of the plate and axis of $z-$ is perpendicular to the $xy-$ plane. The middle surface being $z = 0$ and origin is at one of the corners of the plate as shown in Fig. 1(a).

The expressions for strain energy and kinetic energy of the plate are given by

$$V = \frac{1}{2}\int_0^a\int_0^b D[w_{xx}^2 + 2\,\upsilon w_{xx}w_{yy} + w_{yy}^2 + 2(1-\upsilon)w_{xy}^2]\,dx\,dy \tag{1}$$

$$T = \frac{1}{2}\int_0^a\int_0^b \rho\,h\left(\frac{\partial w}{\partial t}\right)^2 dx\,dy, \tag{2}$$

where $D = \frac{Eh^3}{12(1-\upsilon^2)}$ is the flexural rigidity, $E(x, y)$ is the Young's modulus , $\rho(x, y)$ is the density, υ is the Poisson's ratio, t is the time, $w(x, y, t)$ is the plate deflection at point (x, y) and subscript denotes the partial differentiation with respect to that variables.

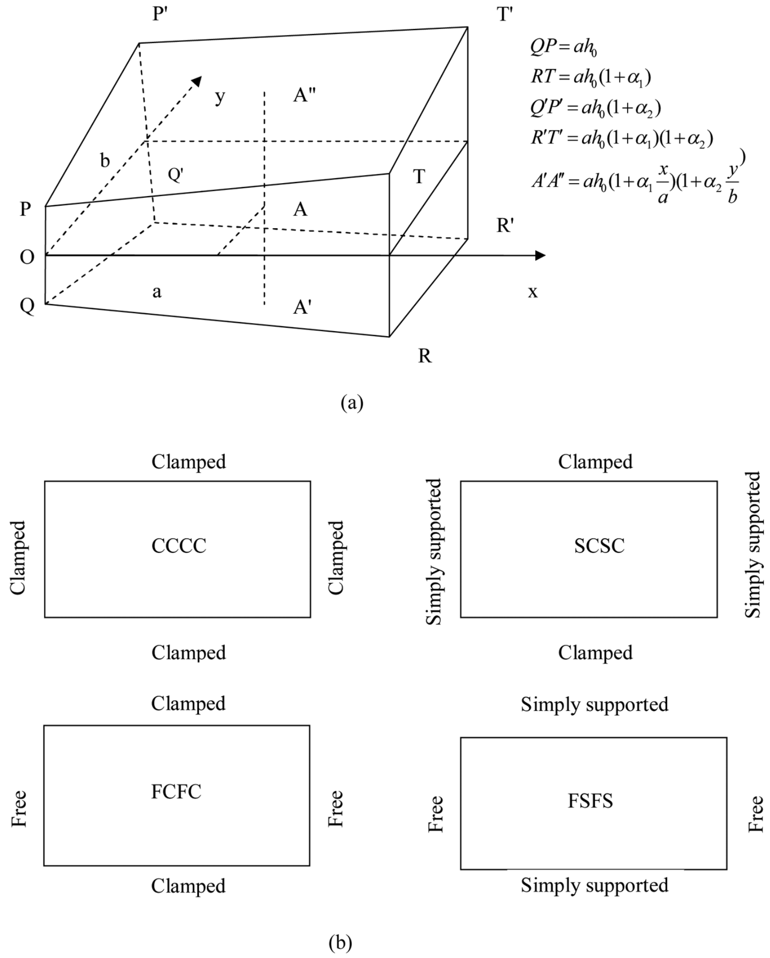

Fig. 1. (a) Geometry of the plate and (b) boundary conditions.

For harmonic solution, the deflection function is assumed to be

$$w(x,y,t) = \overline{W}(x,y)\sin\omega t, \tag{3}$$

where $\overline{W}(x,y)$ represents the maximum transverse displacement at the point (x,y) and ω is the circular frequency.

Substituting relation (3) in expressions (1) and (2), the expressions for maximum strain energy and kinetic energy of the plate become

$$V_{\max} = \frac{1}{2}\int_0^a \int_0^b D\ [\overline{W}_{xx}^2 + 2\,\upsilon\,\overline{W}_{xx}\overline{W}_{yy} + \overline{W}_{yy}^2 + 2(1-\upsilon)\bar{W}_{xy}^2]\,dx\,dy \tag{4}$$

$$T_{\max} = \frac{\omega^2}{2}\int_0^a \int_0^b \rho\,h\,\overline{W}^2\,dx\,dy. \tag{5}$$

According to the Raleigh-Ritz method equating Eqs (4) and (5), one gets the Rayleigh quotient as

$$\omega^2 = \frac{\int_0^a \int_0^b D[\overline{W}_{xx}^2 + 2\,\upsilon\overline{W}_{xx}\overline{W}_{yy} + \overline{W}_{yy}^2 + 2(1-\upsilon)\overline{W}_{xy}^2]\,dx\,dy}{\int_0^a \int_0^b \rho\,h\,\overline{W}^2\,dx\,dy}. \tag{6}$$

Introducing the non-dimensional variables $X = x/a, Y = y/b, W = \overline{W}/a$ and $H = h/a$ together with the assumption that the Young's modulus and density of the plate material vary with the in-plane coordinates by the functional relations

$$E = E_0(1 + \alpha_1 X + \alpha_2 Y) \text{ and } \rho = \rho_0(1 + \beta_1 X + \beta_2 Y) \tag{7}$$

and thickness of plate varies linearly in both $X-$ and $Y-$ directions given by

$$H(H,Y) = h_0(1 + \gamma_1 X)(1 + \gamma_2 Y), \tag{8}$$

where E_0, ρ_0 and h_0 are Young's modulus, density and thickness of the plate at $X = 0, Y = 0$, respectively.

Eq. (6) reduces to

$$\omega^2 = \frac{\int_0^1 \int_0^1 D_0[W_{XX}^2 + 2\,\upsilon(\frac{a}{b})^2 W_{XX}W_{YY} + (\frac{a}{b})^4 W_{YY}^2 + 2(1-\upsilon)\,(\frac{a}{b})^2 W_{XY}^2]\,dX\,dY}{\int_0^1 \int_0^1 \rho_0 h_0\,(1 + \beta_1 X + \beta_2 Y)\,(1 + \gamma_1 X)(1 + \gamma_2 Y)\,W^2\,dX\,dY}, \tag{9}$$

where $D_0 = \frac{E_0 h_0^3 (1+\alpha_1 X+\alpha_2 Y)(1+\gamma_1 X)^3(1+\gamma_2 Y)^3}{12a(1-\upsilon^2)}$.

Now satisfying the essential boundary conditions, let us assume

$$W(X,Y) = \sum_{k=1}^{N} d_k \hat{\phi}_k(X,Y), \tag{10}$$

where N is the order of approximation to get the desired accuracy, d_k's are unknowns and $\hat{\phi}_k$ are orthonormal polynomials which are generated using Gram-Schmidt process as follows:

Orthogonal polynomials ϕ_k over the region $0 \leqslant X \leqslant 1, 0 \leqslant Y \leqslant 1$ have been generated with the help of linearly independent set of functions $L_k = l\, l_k$, $k =$ 1,2,3, ..., with

$$l = X^{p_1}(1-X)^{p_2}Y^{p_3}(1-Y)^{p_4}, \quad l_k = \{1,\ X,\ Y,\ X^2,\ XY,\ Y^2,\ X^3,\ X^2Y,\ XY^2,\ Y^3, \ldots\},$$

where $p_1 =$ 0,1 or 2 as the edge $X = 0$ is free, simply supported or clamped. Same justification can be given to p_2, p_3 and p_4 for the edges $X = 1, Y = 0$ and $Y = 1$.

$$\phi_1 = L_1\ , \quad \phi_k = L_k - \sum_{j=1}^{k-1} \alpha_{kj}\phi_j\ ,$$

$$\alpha_{kj} = \frac{< L_k, \phi_j >}{< \phi_j, \phi_j >}\,, \qquad j = 1,2,3,\ldots,(k-1)\,,\ \ k = 2,3,4,\ldots,N. \tag{11}$$

The inner product of the functions say, ϕ_1 and ϕ_2 is defined as

$$< \phi_1, \phi_2 >= \int_0^1 \int_0^1 (1 + \beta_1 X + \beta_2 Y)\,(1 + \gamma_1 X)(1 + \gamma_2 Y)\ \phi_1(X,Y)\ \phi_2(X,Y)\,dX\,dY, \tag{12}$$

where $(1 + \beta_1 X + \beta_2 Y)\,(1 + \gamma_1 X)(1 + \gamma_2 Y)$ is the weight function and the norm of the function ϕ_1 is given by

$$||\phi_1|| =< \phi_1, \phi_1 >^{1/2} = \left[\int_0^1 \int_0^1 (1 + \beta_1 X + \beta_2 Y)\,(1 + \gamma_1 X)(1 + \gamma_2 Y)\,\phi_1^2(X,Y)\,dX\,dY\right]^{1/2}. \tag{13}$$

The normalization can be done by using

$$\hat{\phi}_k = \frac{\phi_k}{||\phi_k||}. \tag{14}$$

Using expression (10) into Eq. (9) and minimization of the resulting expression for ω^2 w.r.t. d_k's leads to the standard eigenvalue problem

$$\sum_{k=1}^{N}(a_{jk}-\Omega^2\delta_{jk})\,d_k = 0, j = 1,2,3,\ldots,N, \tag{15}$$

where

$$a_{jk} = \int_0^1\int_0^1 F[\hat{\phi}_j^{XX}\hat{\phi}_k^{XX} + \upsilon\mu^2(\hat{\phi}_j^{XX}\hat{\phi}_k^{YY} + \hat{\phi}_k^{XX}\hat{\phi}_j^{YY}) + 2(1-\upsilon)\mu^2\hat{\phi}_j^{XY}\hat{\phi}_k^{XY} + \mu^4\hat{\phi}_j^{YY}\hat{\phi}_k^{YY}]\,dX\,dY, \tag{16}$$

$$F = (1+\alpha_1 X+\alpha_2 Y)(1+\gamma_1 X)^3(1+\gamma_2 Y)^3, \quad \Omega^2 = \frac{12\rho_0 a^2(1-\upsilon^2)\omega^2}{E_0 h_0^2}, \quad \mu = \frac{a}{b}$$

and $\delta_{jk} = \begin{cases} 1, & if\ \ j = k \\ 0, & if\ \ j \neq k \end{cases}$.

The integrals involved in Eq. (16) have been evaluated using the formula

$$\int_0^1\int_0^1 X^{p_1}(1-X)^{p_2}Y^{p_3}(1-Y)^{p_4}\,dX\,dY = \frac{p_1!p_2!p_3!p_4!}{(p_1+p_2+1)!(p_3+p_4+1)!}.$$

3. Boundary conditions

The four boundary conditions namely CCCC, SCSC, FCFC and FSFS have been considered in which C stands for clamped edge, S for simply supported edge and F for free edge. The edge conditions are taken in anti-clockwise direction starting at the edge $x = 0$ (Fig. 1(b)) and obtained by assigning various values to p_1, p_2, p_3 and p_4 as 0, 1, 2 for free, simply supported and clamped edge conditions, respectively.

4. Results and discussion

The numerical values of the frequency parameter Ω have been obtained by solving Eq. (15) employing Jacobi method. The lowest three eigenvalues have been reported as the first three natural frequencies corresponding to different boundary conditions considered here. The values of various plate parameters for these three modes of vibration are taken as follows.

Nonhomogeneity parameters: $\alpha_1, \alpha_2, \beta_1, \beta_2 = -0.5, -0.4, -0.3, -0.2, -0.1, 0.0, 0.1, 0.2, 0.3, 0.4, 0.5$, thickness parameters: $\gamma_1, \gamma_2 = -0.5$, -0.4, -0.3, -0.2, -0.1, 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, aspect ratio: $a/b = 0.25$, 0.50, 0.75, 1.00, 1.25, 1.50, 1.75, 2.00 and $\upsilon = 0.3$.

To choose the appropriate value of the order of approximation N, a computer program developed in C++ for the evaluation of frequency parameter Ω was run for different values of N. The numerical values showed a consistent improvement with the increasing value of N for different sets of the values of plate parameters. Table 1 shows the convergence of frequency parameter Ω with N for a particular set of plate parameters for all the four boundary conditions where possibly maximum value of N was required for first three modes. The value of N may be taken as 45 but for the safer side of the accuracy of four decimal places for all possible combinations of the values of various plate parameters it has been taken $N = 47$. All the computations were carried out in double precision arithmetic without observing any kind of numerical instability as pointed out by [32].

A comparison of frequency parameter Ω for homogeneous ($\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$) square ($a/b = 1$) plate of uniform ($\gamma_1 = \gamma_2 = 0$) thickness obtained by Rayleigh-Ritz method [7,34,37], exact and approximate results using Ritz method [33], Differential quadrature method [11,36], Optimized Kantorovich method [35] and non-uniform thickness obtained by finite element method [6], Rayleigh-Ritz method [7,9], differential quadrature method [11], optimized Kantorovich method [35] has been presented in Table 2. A close agreement of results is obtained.

Table 1
Convergence of frequency parameter Ω with N for $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 0.5$ and $a/b = 1$

CCCC				SCSC			
	mode				Mode		
N	I	II	III	N	I	II	III
10	55.3971	112.4090	113.0890	10	44.7853	86.2823	106.5780
20	55.3880	111.9560	112.8570	20	44.7636	84.3675	105.8680
30	55.3837	111.9530	112.8320	30	44.7633	84.3289	105.8460
40	55.3834	111.9470	112.8270	40	44.7633	84.3276	105.8450
45	55.3822	111.9460	112.8270	45	44.7632	84.3274	105.8450
46	55.3822	111.9460	112.8270	46	44.7632	84.3274	105.8450
47	55.3822	111.9460	112.8270	47	44.7632	84.3274	105.8450
FCFC				FSFS			
	mode				mode		
N	I	II	III	N	I	II	III
10	33.3218	42.9404	73.0086	10	15.0639	26.1364	62.6908
20	33.2861	42.6022	68.4567	20	14.8862	25.6528	57.8629
30	33.2634	42.5609	68.0796	30	14.8847	25.6334	57.4212
40	33.2502	42.5341	68.0380	40	14.8845	25.6327	57.4015
45	33.2455	42.5301	68.0372	45	14.8845	25.6325	57.3994
46	33.2455	42.5301	68.0372	46	14.8845	25.6325	57.3994
47	33.2455	42.5301	68.0372	47	14.8845	25.6325	57.3994

To analyze the effect of various plate parameters on the frequencies, a huge amount of results are computed and all can not be given here. However, to observe their trends on the frequency parameter Ω, some of them with their minimum and maximum values are reported in Tables 3–6 and few of them with their varying values are presented in Figs 2–5. It is observed that the frequency parameter Ω decreases in the order of boundary conditions CCCC>SCSC>FCFC>FSFS for the same set of values of plate parameters. This may be attributed to the clamped edge condition which causes an additional constraint over simply supported and free edge conditions. Figure 2 shows the effect of nonhomogeneity parameter α_1 on the frequency parameter Ω for $\beta_1 = \pm 0.5$, $\gamma_1 = 0.5$, $\alpha_2 = \pm 0.5$, $\beta_2 = 0.5, \gamma_2 = 0.5$ and $a/b = 1$ for the first two modes of vibration. It is observed that the frequency parameter Ω increases with increasing values of α_1 for all the boundary conditions keeping other plate parameters fixed. This may be attributed to the increased stiffness of the plate towards the edge $X = 1$. The value of Ω decreases with the increasing values of β_1 while increases with the increasing value of α_2 for fixed values of other plate parameters. The effect of β_1 and α_2 is more pronounced for $\alpha_1 = -0.5$ as compared to $\alpha_1 = 0.5$. The rate of increase in Ω with α_1 is in the order of the boundary conditions CCCC>SCSC>FSFS>FCFC when β_1 changes from -0.5 to 0.5 and order becomes CCCC>SCSC>FCFC>FSFS when α_2 changes from -0.5 to 0.5, other parameters being fixed. This rate is higher in the second mode as compared to the first mode.

Figure 3 depicts the behavior of the frequency parameter Ω with the density parameter β_1 for $\alpha_1 = \pm\, 0.5, \gamma_1 = 0.5, \alpha_2 = 0.5, \beta_2 = \pm\, 0.5, \gamma_2 = 0.5$ and $a/b = 1$ for the first two modes of vibration. It is seen that the frequency parameter Ω decreases with increasing value of β_1 whatever be the values of other plate parameters. This may be due to the increased inertia of the plate towards the edge $X = 1$. The frequency parameter Ω is found to increase with the increasing value of α_1 keeping other plate parameters fixed. The value of Ω further decreases with the increasing value of β_2. The rate of increase in Ω with β_1 is in the order of the boundary conditions CCCC>SCSC>FSFS>FCFC when α_1 changes from -0.5 to 0.5 and order becomes CCCC>SCSC>FCFC>FSFS when β_2 changes from -0.5 to 0.5. This rate is more pronounced in the second mode as compared to the first mode.

Figure 4 demonstrates the effect of thickness parameter γ_1 on the frequency parameter Ω for $\alpha_1 = \pm\, 0.5, \alpha_2 = 0.5, \beta_1 = 0.5, \beta_2 = 0.5, \gamma_2 = \pm\, 0.5$ and $a/b = 1$ for the first two modes of vibration. It is noticed that the frequency parameter Ω increases with increasing value of γ_1 for all the boundary conditions for fixed values of all other plate parameters and this may happen due to further increase in the stiffness of the plate. The value of Ω increases with the increasing value of α_1 as well as γ_2. The effect of γ_2 is more pronounced for $\gamma_1 = 0.5$ as compared to $\gamma_1 = -0.5$ keeping all other parameters fixed. The rate of increase in Ω with γ_1 is in the order of the boundary conditions CCCC>SCSC>FSFS>FCFC when α_1 changes from -0.5 to 0.5 and order becomes CCCC>SCSC>FCFC>FSFS when γ_2 changes from -0.5 to 0.5, other parameters being fixed. This rate is higher in the second mode as compared to the first mode.

Table 2
Frequency parameter Ω for CCCC plate

β_2	γ_1	γ_2 \ α_1	-0.5				0.5			
		α_2	-0.5		0.5		-0.5		0.5	
		β_1	-0.5	0.5	-0.5	0.5	-0.5	0.5	-0.5	0.5
			19.3194	12.3382	28.0630	18.7744	28.0630	19.1214	34.4220	23.5406
		-0.5	38.3353	22.5533	56.1936	36.7619	56.1936	38.2334	69.8841	47.2181
			39.3757	26.0377	58.7348	39.0080	58.7348	39.1125	71.7825	48.5656
	-0.5									
			32.0325	21.1648	45.1533	30.8472	46.0628	32.0665	55.7056	38.8909
		0.5	64.6521	40.1646	87.9649	59.9417	92.9325	64.8018	110.3670	77.6594
			65.8430	44.5579	94.8250	63.6885	96.2981	65.9608	116.6890	79.7512
-0.5										
			32.0325	21.9697	46.0628	32.7571	45.1533	32.7648	55.7056	40.4841
		-0.5	64.6521	42.1908	92.9325	66.8537	87.9649	65.6756	110.3670	81.9796
			65.8431	46.1395	96.2981	67.2514	94.8250	66.6911	116.6890	82.9651
	0.5									
			53.1453	37.3332	74.4509	53.9204	74.4509	54.9434	90.2531	66.7977
		0.5	108.6610	74.7430	148.2670	107.4270	148.2670	111.4410	178.7160	133.1570
			110.7270	78.2965	154.1290	111.4810	154.1280	112.2990	186.1890	137.5560
			12.3379	9.7523	19.1209	15.2298	18.7740	15.2298	23.5400	18.9969
		-0.5	22.5209	17.6698	38.2330	29.9808	36.7614	29.9808	47.2177	38.0554
			26.0363	20.6061	39.1123	31.3547	39.0079	31.3548	48.5655	39.0199
	-0.5									
			21.9693	17.3014	32.7640	25.8783	32.7563	26.3967	40.4832	32.4387
		0.5	42.1881	32.6758	65.6750	50.6433	66.8537	53.5335	81.9795	65.3369
			46.1391	36.4905	66.6910	53.3204	67.2513	54.1700	82.9647	66.2725
0.5										
			21.1648	17.3015	32.0664	26.3968	30.8472	25.8785	38.8908	32.4387
		-0.5	40.1644	32.6853	64.8017	53.5361	59.9413	50.6436	77.6588	65.3387
			44.5577	36.4925	65.9608	54.1709	63.6883	53.3218	79.7511	66.2736
	0.5									
			37.3332	30.3760	54.9433	44.9271	53.9203	44.9270	66.7976	55.3822
		0.5	74.7429	60.6256	111.4400	90.5968	107.4270	90.5966	133.1570	111.9460
			78.2964	63.6673	112.2990	92.2379	111.4800	92.2372	137.5550	112.8270

Table 3
Frequency parameter Ω for SCSC plate

β_2	γ_1	γ_2 \ α_1	-0.5				0.5			
		α_2	-0.5		0.5		-0.5		0.5	
		β_1	-0.5	0.5	-0.5	0.5	-0.5	0.5	-0.5	0.5
			15.8288	10.0820	22.6522	15.1442	22.7096	15.5680	27.7044	19.0277
		-0.5	29.6139	18.7642	43.2138	29.5249	43.1587	29.3469	53.1713	36.4812
			36.5592	22.0575	54.1953	33.7457	54.6299	36.2081	67.0286	44.1533
	-0.5									
			25.9434	17.0580	36.9156	25.1153	37.1841	25.9706	45.3035	31.6600
		0.5	49.5691	33.5417	68.0709	47.7092	70.7054	49.5013	84.5400	59.6612
			60.7680	36.7313	88.7463	56.4503	90.4471	60.8994	110.1120	73.7508
-0.5										
			26.0508	17.8318	36.9415	26.2520	36.3509	26.5476	44.6119	32.5528
		-0.5	49.2968	32.9727	71.9098	50.2276	71.0651	49.7493	88.0896	61.9052
			61.8612	42.1987	87.3160	63.0403	81.9713	62.5889	101.6570	77.4281
	0.5									
			42.7285	29.8533	60.4435	43.6618	59.7970	44.2794	72.9738	54.1165
		0.5	82.4664	57.3753	113.9660	81.5611	117.3380	83.9148	140.1370	101.0990
			103.1190	71.3366	143.6540	105.0010	136.0170	105.2260	168.0180	129.1570
			10.1884	8.0128	15.4849	12.3110	15.3288	12.4427	19.0540	15.3862
		-0.5	18.1526	14.5748	29.2546	23.7039	28.2114	23.0690	35.8124	29.1627
			23.2897	17.8211	36.1624	27.7178	36.7800	29.2558	45.6831	36.0090
	-0.5									
			17.7971	13.9651	26.6568	21.0289	26.4646	21.3646	32.8436	26.3536
		0.5	33.6318	27.0180	50.1313	40.1460	50.1504	40.6564	61.7687	49.8520
			40.3862	30.4609	61.8154	47.1420	63.5424	50.2416	78.2919	61.3618
0.5										
			17.3446	14.1203	25.7997	21.2037	25.0479	21.0440	31.3080	26.1385
		-0.5	31.2610	25.6007	49.0711	40.2327	46.9529	38.9244	59.8922	49.3592
			41.7350	33.7074	61.5976	50.9978	57.8339	49.6774	72.6871	62.0621
	0.5									
			30.0080	24.3239	44.4238	36.3032	43.3360	36.1801	53.8839	44.7632
		0.5	56.9179	46.4323	84.0571	68.2662	83.8002	68.7383	103.1850	84.3274
			72.5726	58.2973	105.3290	86.8637	99.5057	85.3685	124.3190	105.8450

Table 4
Frequency parameter Ω for FCFC plate

		α_1	-0.5				0.5			
		α_2	-0.5		0.5		-0.5		0.5	
		β_1	-0.5	0.5	-0.5	0.5	-0.5	0.5	-0.5	0.5
β_2	γ_1	γ_2								
-0.5	-0.5		11.5938	5.9495	17.1726	9.5506	17.1904	10.9074	21.1602	13.0431
		-0.5	15.8680	11.7064	21.6663	16.8238	21.6613	15.6651	26.0459	19.4423
			24.8016	13.6796	35.1407	23.6275	35.5142	24.1670	43.2027	29.8098
			18.3658	9.7841	28.0636	16.0122	28.1795	18.0746	34.9588	21.8494
		0.5	25.8790	19.7354	35.6276	28.0021	34.8492	25.9548	42.4493	32.4135
			40.4169	23.2072	56.2973	39.3190	57.4089	40.1568	69.3879	49.1784
	0.5		19.1213	12.3241	25.7368	19.7460	23.0993	19.8592	28.6621	24.4745
		-0.5	24.9502	17.1471	37.8406	24.1452	41.1680	25.2953	49.6063	30.4764
			40.5032	26.9577	58.9624	40.0693	57.9644	40.5369	73.5290	49.6994
			31.1564	20.2531	42.7561	33.1750	37.8281	32.8156	47.4944	40.8913
		0.5	39.7283	28.1449	61.7380	40.3451	67.0782	41.9643	80.8927	50.9164
			65.5782	45.2538	94.7201	65.9686	96.8160	67.4040	120.5530	81.9310
0.5	-0.5		6.6863	4.7972	10.9923	7.9145	11.7465	8.9028	14.4298	10.7474
		-0.5	10.8111	9.1740	15.3778	13.4101	14.0625	12.2173	17.5677	15.3714
			15.0069	11.1011	23.8493	19.6690	23.0939	18.9916	28.9155	23.7650
			11.3965	8.0962	19.0827	13.6040	20.1149	15.1186	24.9737	18.4486
		0.5	18.6459	15.9146	26.5452	23.0212	24.0450	20.8516	30.3209	26.4127
			26.6934	19.2283	41.0383	33.3718	40.1307	32.7542	49.8699	40.6886
	0.5		13.0711	9.9391	18.8500	16.0805	16.7944	15.4351	21.1870	19.3335
		-0.5	15.3395	13.2940	24.5523	19.2691	26.6514	20.3939	32.5960	24.7882
			25.4455	21.0139	39.4285	32.1096	39.5623	31.9591	49.3156	39.7811
			22.2239	16.7170	32.5837	27.6989	28.6241	26.2737	36.5184	33.2455
		0.5	25.9767	22.4715	42.5456	33.1621	45.9160	34.7706	56.3468	42.5301
			44.4380	36.4946	67.9009	54.8482	68.9300	55.1751	85.0350	68.0372

Table 5
Frequency parameter Ω for FSFS plate

		α_1	-0.5				0.5			
		α_2	-0.5		0.5		-0.5		0.5	
		β_1	-0.5	0.5	-0.5	0.5	-0.5	0.5	-0.5	0.5
β_2	γ_1	γ_2								
-0.5	-0.5		5.3099	3.3374	7.7566	5.0410	7.1622	5.0330	9.1055	6.2883
		-0.5	9.8325	6.7026	13.4069	9.5383	13.7134	9.2545	16.3665	11.3041
			20.2783	10.0078	29.4520	16.6270	30.0505	18.9737	36.5468	22.9268
			9.3075	5.7147	12.8212	8.3982	12.4715	8.6960	15.1708	10.5291
		0.5	15.5058	11.2755	22.1137	15.9341	21.8513	15.1796	26.9390	18.8537
			33.3579	17.0209	47.0111	27.5858	48.5233	31.8239	58.5448	38.1395
	0.5		8.0912	5.4484	12.0274	8.6637	11.4294	8.6924	14.2882	10.8825
		-0.5	15.5343	10.5776	22.2108	14.9299	23.6342	15.5461	28.4194	18.6649
			33.6039	21.5310	46.0512	33.9170	40.1199	34.3520	50.6633	42.0169
			14.4275	9.8394	19.9430	14.3626	19.4733	15.0950	23.8108	18.2112
		0.5	24.6488	16.8355	36.7103	25.1086	38.7266	25.2899	46.9184	31.1591
			55.6164	36.9152	75.6068	55.6992	66.7955	56.7574	83.4162	69.0018
0.5	-0.5		3.5333	2.7133	5.3766	4.1621	5.0421	4.0951	6.4427	5.1598
		-0.5	6.3172	5.2718	9.1480	7.6557	9.0061	7.3612	11.0054	9.0683
			11.3260	8.1488	19.4097	13.8143	19.6299	15.5317	24.5240	18.9235
			6.2411	4.7079	9.0325	6.9963	8.8701	7.1419	10.8700	8.7041
		0.5	10.6292	9.0195	15.7013	13.0311	14.9452	12.2685	18.8121	15.4048
			19.9846	14.0979	33.0706	23.3376	33.9711	26.6035	42.3084	32.1125
	0.5		5.5644	4.4254	8.6419	7.0922	8.2168	6.9998	10.3932	8.8285
		-0.5	10.0513	8.3345	14.8911	12.0308	15.5220	12.4090	18.9402	15.0647
			21.7161	17.3004	33.2483	27.1987	29.2432	26.8715	37.4233	33.5956
			10.1256	8.0532	14.4781	11.8381	14.2649	12.2325	17.5547	14.8845
		0.5	16.4051	13.5070	25.5220	20.6059	26.1263	20.6179	32.3152	25.6325
			37.6140	30.1741	57.0221	46.4876	50.3322	46.0973	63.6347	57.3994

Table 6
Comparison of frequency parameter Ω for homogeneous ($\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$) square ($a/b = 1$) plate for $\gamma_2 = 0$ and $\upsilon = 0.3$

Boundary Conditions	Ref.	γ_1	Mode		
			I	II	III
CCCC	33	0.0	35.992	73.413	73.413
	34	0.0	35.986	73.395	73.395
	7	0.0	35.986	73.395	73.395
	30	0.0	35.99	73.41	–
	present		35.9855	73.3954	73.3954
	7	0.2	39.5097	80.5194	80.5857
	present		39.5097	80.5201	80.5859
	7	0.4	42.9088	87.2835	87.5259
	6	0.4	43.92	–	–
	present		42.9087	87.2843	87.5252
	35	0.5	44.696	–	–
	present		44.5698	90.5565	90.9178
	9	−0.4	28.375	57.523	57.887
	6	−0.4	28.424	–	–
	present		28.3740	57.5240	57.8850
SCSC	33	0.0	28.951	54.743	69.327
	7	0.0	28.9509	54.7432	69.3270
	36	0.0	28.9551	54.7466	69.3393
	35	0.0	28.95	54.88	69.34
	present		28.9509	54.7431	69.3270
	35	0.5	36.095	–	–
	present		36.9633	68.0145	85.2326
	6	0.4	34.176	–	–
	present		34.9521	65.4164	82.2399
	6	−0.4	24.052	–	–
	present		22.9351	43.3774	54.0400
FCFC	37	0.0	22.03	26.05	43.20
	33	0.0	22.27	26.53	43.66
	present		22.1922	26.4510	43.6205
FSFS	37	0.0	9.631	16.13	36.73
	33	0.0	9.6314	–	–
	11	0.0	9.631385	16.134778	36.725643
	present		9.6314	16.1348	36.7256
	11	0.2	10.588088	17.775345	40.383104
	present		10.5881	17.7753	40.3831
	11	0.4	11.535708	19.460527	44.015354
	present		11.5357	19.4605	44.0154

The effect of aspect ratio a/b on the frequency parameter Ω for $\alpha_1 = -0.5$, $\beta_1 = -0.5$, $\gamma_1 = -0.5$, $\alpha_2 = -0.5$, $\beta_2 = \pm\, 0.5$ and $\gamma_2 = \pm\, 0.5$ for the first two modes of vibration has been shown in Fig. 5. It is clear that frequency parameter Ω increases with the increasing value of a/b for all the boundary conditions, other plate parameters being fixed. The values of Ω is found to decrease with the increasing value of β_2 and increases with the increasing value of γ_2. The rate of increase in Ω with a/b is in the order of the boundary conditions FCFC>SCSC>CCCC>FSFS when β_2 changes from -0.5 to 0.5 i.e. the plate is becoming more and more dense towards the edge $Y = 1$ and becomes SCSC>CCCC>FCFC>FSFS when γ_2 changes from -0.5 to 0.5 i.e. the plate is becoming more and more stiff towards the edge $Y = 1$, keeping all other parameters fixed. This rate of increase is much higher for $a/b > 1$ as compared to $a/b < 1$ and increases with the increase in the number of modes.

No special feature was observed from the graphs for third mode of vibration (figures not given here) except that the rate of increase/decrease in the frequency parameter Ω with a specific parameter is higher than that for second mode. Three dimensional mode shapes for specified plate have been plotted using METLAB and shown in Fig. 6.

5. Conclusions

The effect of nonhomogeneity arising due to the dependence of Young's modulus and density of the plate material on both the in-plane variables, on the natural frequencies of isotropic rectangular plates of bidirectionally varying

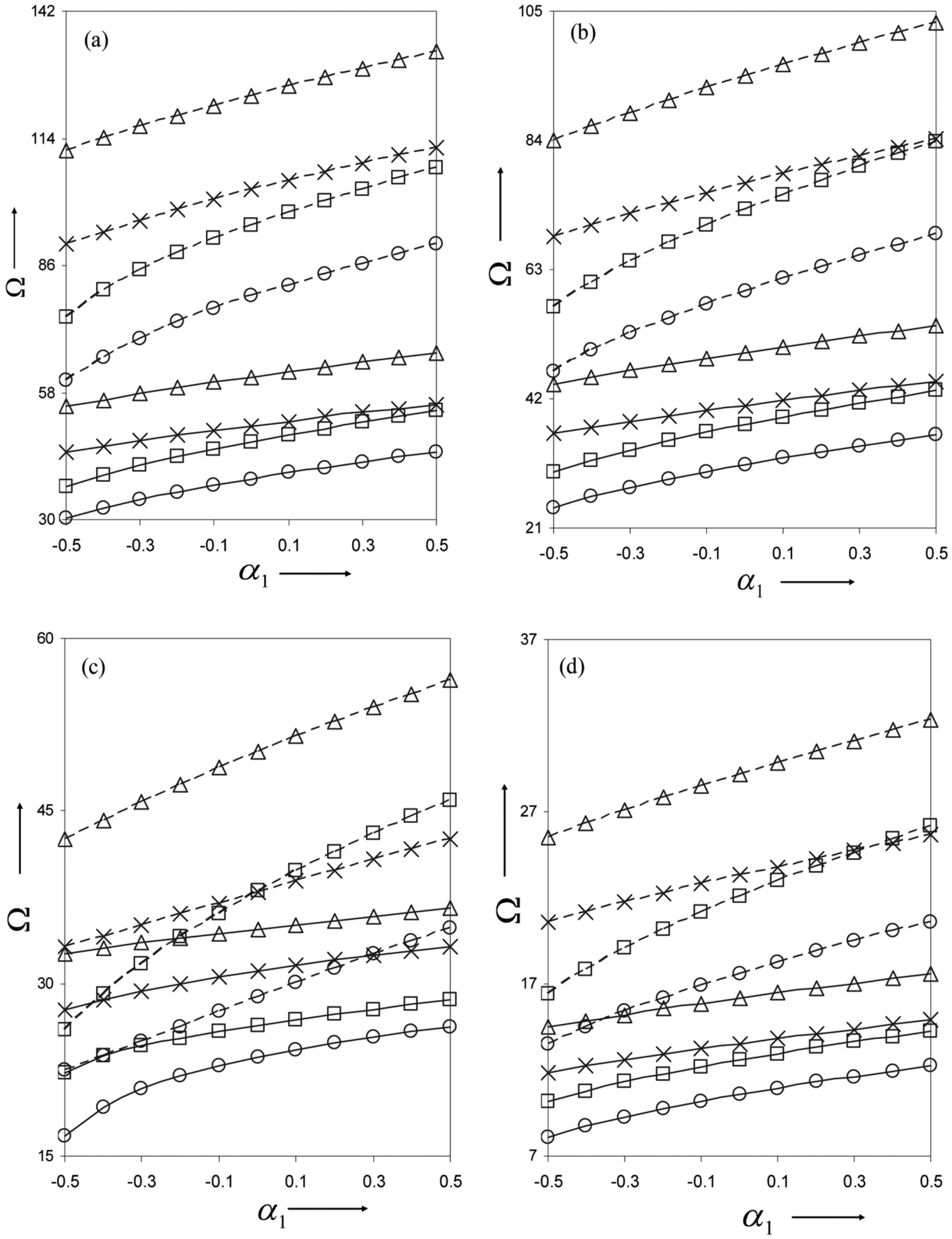

Fig. 2. Frequency parameter Ω for (a) CCCC, (b) SCSC, (c) FCFC and (d) FSFS plate: for $\beta_2 = 0.5$, $\gamma_1 = 0.5$, $\gamma_2 = 0.5$. −, first mode; - - - -, second mode; □, $\alpha_2 = -0.5$, $\beta_1 = -0.5$; o, $\alpha_2 = -0.5$, $\beta_1 = 0.5$; Δ, $\alpha_2 = 0.5$, $\beta_1 = -0.5$; ×, $\alpha_2 = 0.5$, $\beta_1 = 0.5$.

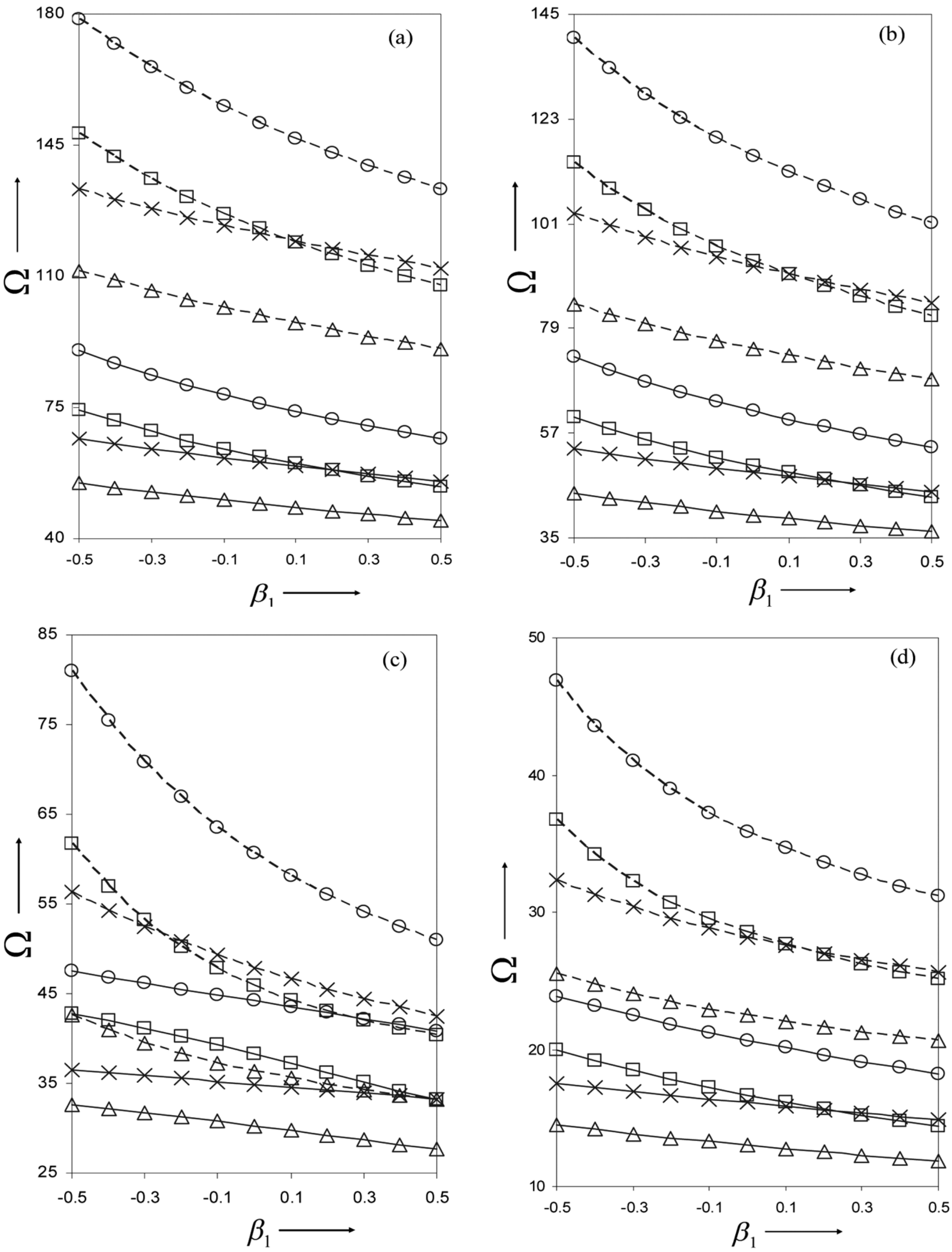

Fig. 3. Frequency parameter Ω for (a) CCCC, (b) SCSC, (c) FCFC and (d) FSFS plate: for $\alpha_2 = 0.5$, $\gamma_1 = 0.5$, $\gamma_2 = 0.5$. − , first mode; - - - -, second mode; □, $\beta_2 = -0.5$, $\alpha_1 = -0.5$; o, $\beta_2 = -0.5$, $\alpha_1 = 0.5$; Δ, $\beta_2 = 0.5$, $\alpha_1 = -0.5$; ×, $\beta_2 = 0.5$, $\alpha_1 = 0.5$.

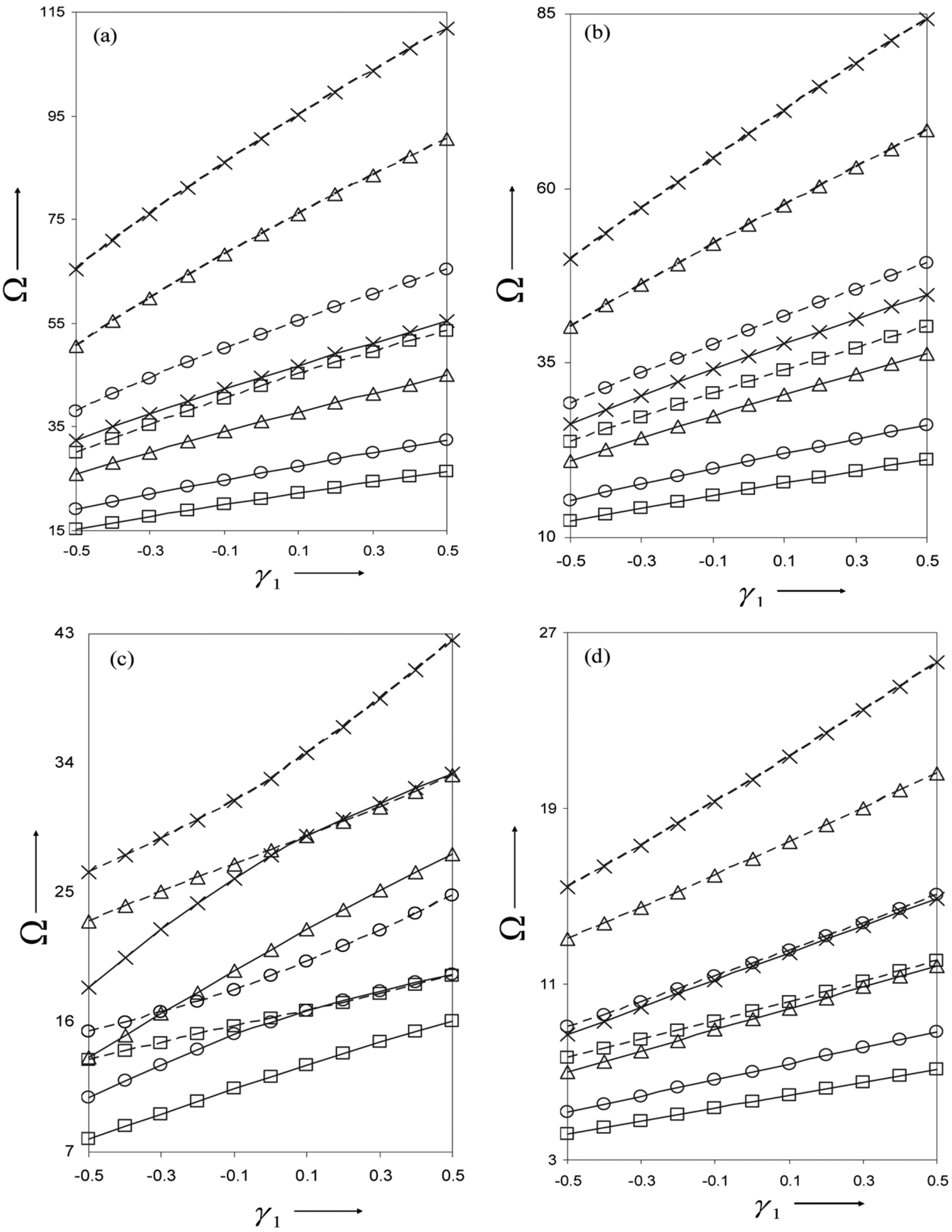

Fig. 4. Frequency parameter Ω for (a) CCCC, (b) SCSC, (c) FCFC and (d) FSFS plate: for $\alpha_2 = 0.5$, $\beta_1 = 0.5$, $\beta_2 = 0.5$. $-$, first mode; - - - -, second mode; $\square$, $\gamma_2 = -0.5$, $\alpha_1 = -0.5$; o, $\gamma_2 = -0.5$, $\alpha_1 = 0.5$; Δ, $\gamma_2 = 0.5$, $\alpha_1 = -0.5$; $\times$, $\gamma_2 = 0.5$, $\alpha_1 = 0.5$.

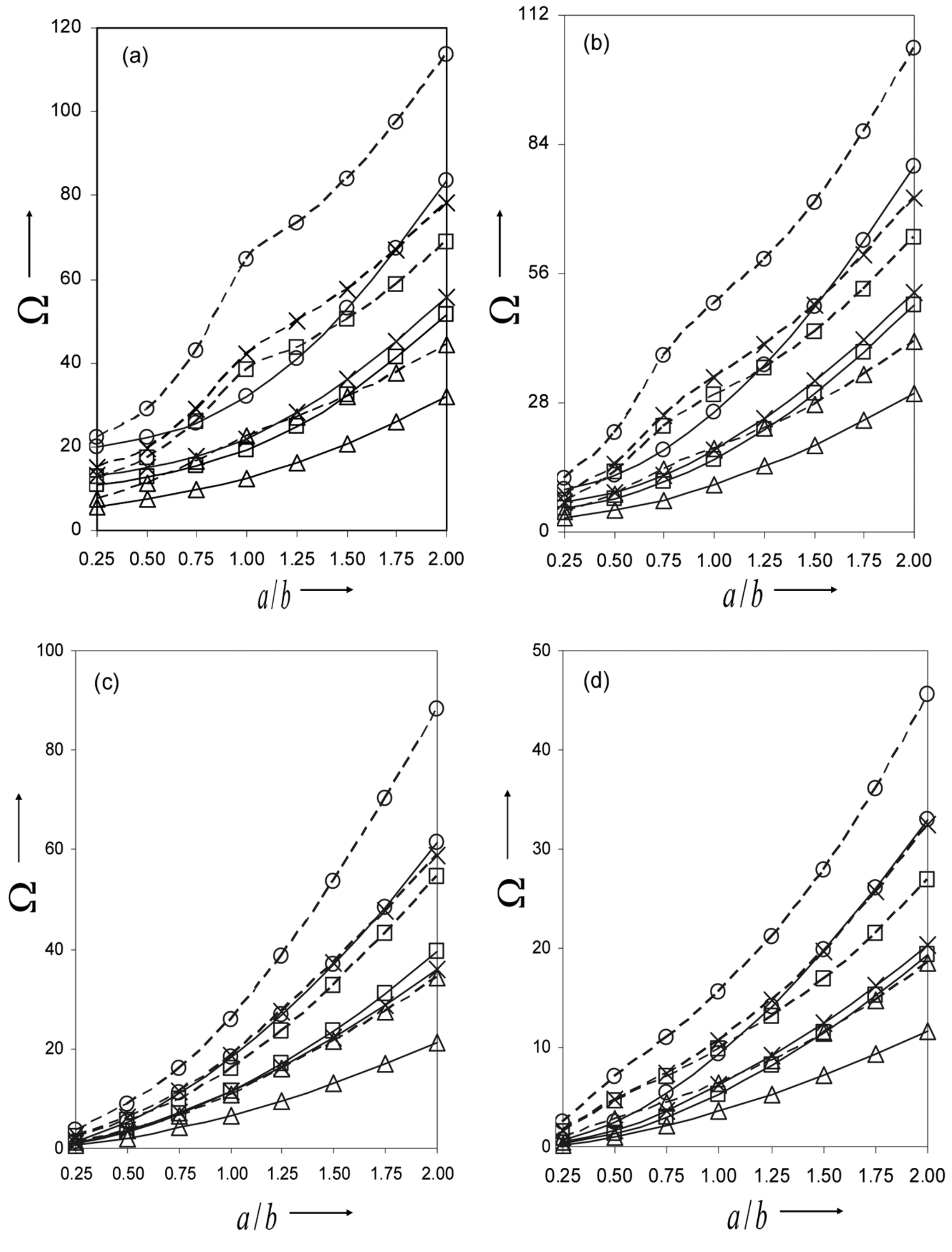

Fig. 5. Frequency parameter Ω for (a) CCCC,(b) SCSC ,(c) FCFC plate,(d) FSFS plate: for $\alpha_1 = \alpha_2 = \beta_1 = \gamma_1 = -0.5$. –, first mode; - - - -, second mode; □, $\beta_2 = -0.5$, $\gamma_2 = -0.5$; o, $\beta_2 = -0.5$, $\gamma_2 = 0.5$; Δ, $\beta_2 = 0.5$, $\gamma_2 = -0.5$; ×, $\beta_2 = 0.5$, $\gamma_2 = 0.5$.

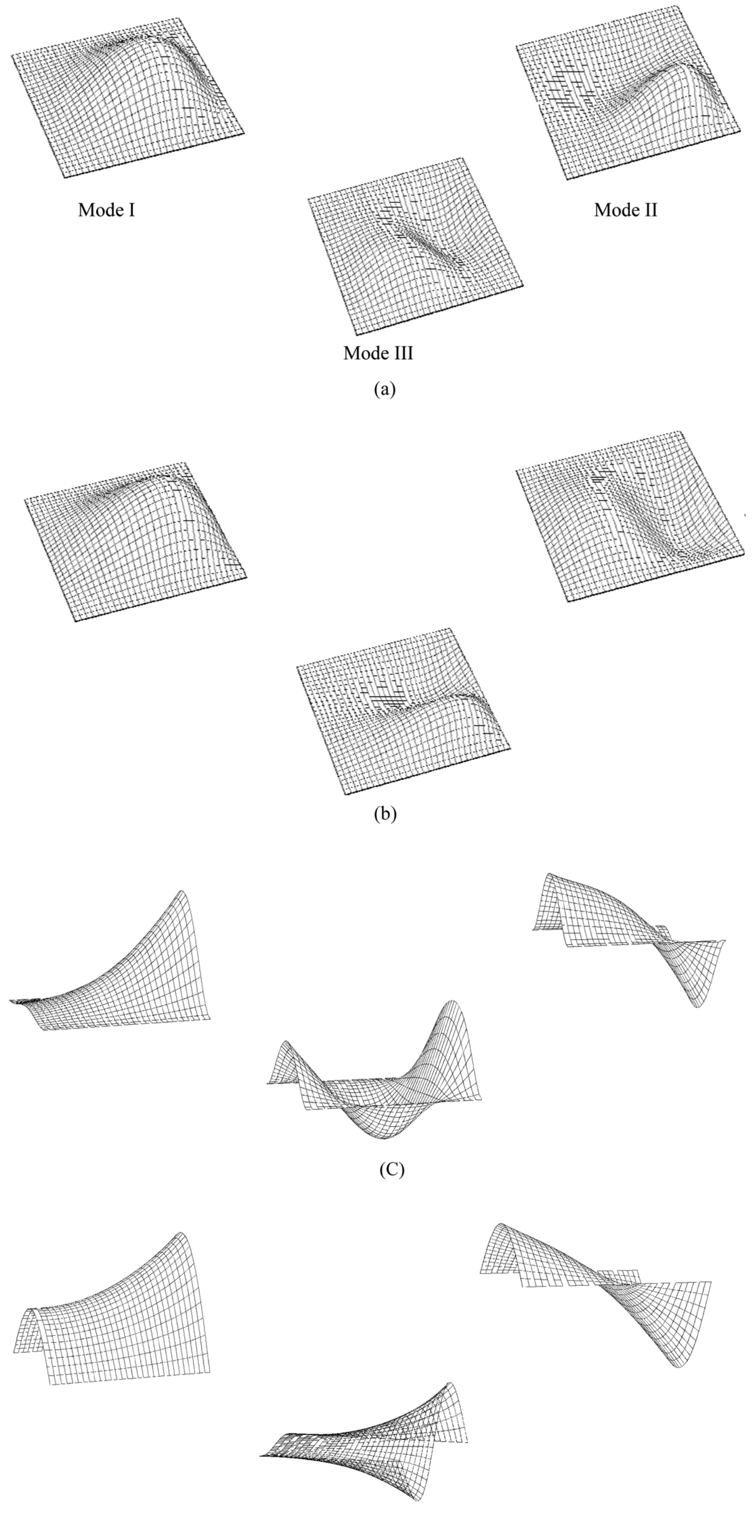

Fig. 6. First three mode shapes of (a) CCCC, (b) SCSC, (c) FCFC and (d) FSFS square plates for $\alpha_1 = \beta_1 = \gamma_1 = -0.5$ and $\alpha_2 = \beta_2 = \gamma_2 = 0.5$.

thickness has been studied using boundary characteristic orthogonal polynomials in Rayleigh-Ritz method on the basis of classical plate theory. It is observed that the frequency parameter Ω increases as the plate becomes stiffer and stiffer towards both the edges $X = 1$ and $Y = 1$ due to an increase in the values of nonhomogeneity parameters α_1 and α_2, thickness parameters γ_1 and γ_2, while it decreases as the plate becomes more and more dense towards both the edges $X = 1$ and $Y = 1$ due to an increase in the values of density parameters β_1 and β_2 for all the four boundary conditions, keeping all other plate parameters fixed. The frequency parameter also increases with the increasing values of aspect ratio a/b. The percentage variations in the value of frequency parameter Ω for the first mode of vibration are -13.9436 to 11.04898, -13.7756 to 11.1362, -10.9496 to 7.0636 and -14.6117 to 11.2559 for CCCC, SCSC, FCFC and FSFS boundary conditions, respectively when the nonhomogeneity arises due to the change in only α_1 from -0.5 to 0.5. The corresponding variations are -14.1588 to 10.0688, -14.0086 to 10.0976, -6.6299 to 7.0057 and -12.3385 to 9.8556 when β_1 changes from -0.5 to 0.5. These variations decrease by almost 0.5% with the increase in the number of modes. The present analysis will be of great use to the design engineers dealing with nonhomogeneous plates in obtaining the desired frequency by varying one or more plate parameters considered here.

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