Dynamic effects of embedded macro-fiber composite actuators on ultra-light flexible structures of repeated pattern- a homogenization approach

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Abstract. Motivated by deployable satellite technology, this article presents a homogenization model of an inflatable, rigidized lattice structure with distributed macro-fiber composite (MFC) actuation. The model is based upon a general expression for the strain and kinetic energy of a fundamental repeated element of the structure. These expressions are reduced in order and expressed in terms of the strain and displacement components of an equivalent one-dimensional vibration model. The resulting model is used to analyze changes in the structural natural frequencies introduced by the local effects of the added macro-fiber composite actuators for several configurations. A finite element solution is used as a comparison for the homogenization model, and the two are shown to be in good agreement, although the latter requires significantly less computational effort.

Nomenclature

A_L, A_d, A_b	=	cross-sectional area of longerons, diagonals and battens
$A^{(k)}$	=	cross-sectional area of member k
A_{MFC}	=	cross-sectional area of <i>MFC</i>
m_1, m_2	=	coefficients of expansion of element strain energy
n_L	=	number of actuators at cross section of the longeron
n_d	=	number of actuators at cross section of diagonal
$C_1, C_2, C_3,$ C_4, C_5, C_6	=	coefficients of expansion of truss element strain energy
E_L, E_d, E_b	=	modulus of elasticity of longerons, diagonals and battens
$E^{(k)}$	=	modulus of elasticity of member k
E_{MFC}	=	modulus of elasticity of <i>MFC</i>
f	=	frequency (rad/sec)
k	=	element number
K_L, K_d, K_b	=	stiffness of longerons, diagonals and battens
K_L^*	=	stiffness of actuated length of longeron

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K_d^*	=	stiffness of actuated length of diagonal
$K^{(k)}$	=	stiffness of member k
K_P	=	stiffness of non-actuated length of longeron
L_L, L_d, L_b	=	length of longerons, diagonals and battens
$L^{(k)}$	=	length of member k
L_t	=	total length of the truss
L_P	=	non-actuated length of longeron
L_{MFC}	=	length of <i>MFC</i>
m_L	=	number of actuators along length of longeron
m_d	=	number of actuators along length of diagonal
T	=	kinetic energy
$T^{(k)}$	=	kinetic energy of member k
T_e	=	kinetic energy of truss element
T	=	time
U	=	potential energy
U_L, U_d, U_b	=	potential energy of longerons, diagonals and battens
U_e	=	potential energy of truss element
u_1, u_2, u_3	=	displacement components along (x_1, x_2, x_3) directions
$u_{1,0}, u_{2,0}, u_{3,0}$	=	displacement components evaluated at center truss element
x_1, x_2, x_3	=	Cartesian coordinates
$\varepsilon_{1,0}, \varepsilon_{2,0}, \varepsilon_{3,0}$	=	strain components evaluated at center of truss element
$\varepsilon_{12,0}, \varepsilon_{13,0}, \varepsilon_{23,0}$	=	strain components of a point in truss element
$\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$	=	strain components of a point in truss element
$\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}$	=	strain components of a point in truss element
$\varepsilon_{ij}^{(k)}$	=	strain components of member k
$\varepsilon^{(k)}$	=	strain in member k
$\varphi_1, \varphi_2, \varphi_3$	=	rotational degrees of freedom of truss element
$\kappa_{1,0}, \kappa_{2,0}, \kappa_{3,0}$	=	curvatures evaluated at center of truss element
ρ_L, ρ_d, ρ_b	=	density of longerons, diagonals and battens
$\rho^{(k)}$	=	density of member k

1. Introduction

We consider the vibration analysis of an inflatable satellite, which deploys and rigidizes in the form of a large truss structure. This type of deployable satellite belongs to a class of *space-inflatable structures*, or *gossamer structures*, characterized primarily by a low weight and packing volume. This technology enables placing larger metrology systems in space which results in improved coverage of antenna systems and enables them to keep track of smaller moving objects on Earth. A broad collection of research and development efforts in gossamer technology can be found in the two volume series edited by Jenkins [1,2], and the review articles of Ruggiero and Inman [3] and Ruggiero et al. [4].

Due to the large size of inflatable structures, significant time is required for their FEA vibration analysis. On the other hand, analysis of such structures can be significantly simplified, relative to the computational efforts required of finite discretization methods, by a process generally referred to as *continuum modeling*, whereby an equivalent beam or plate model is derived from careful assumptions relating the overall geometry and local strain properties of the equivalent continuum model and the truss structure. This simplified modeling technique is particularly attractive for control applications. As discussed in Noor et al. [5,6] and Salehian et al. [7–9], there are a few ways of going about this discrete to continuum transformation process [10–17]. The method of interest here, developed by Noor et al. [5], involves equating kinetic and potential energy expressions of the lattice and its intended beam equivalent. From these expressions and by making necessary assumptions for local free deformations, the equivalent beam-like equations of motion can be derived [7,8]. In addition to the computational savings involved with the continuum

modeling approach, also the mathematical structure of the equations of motion is preserved in that it is possible to directly relate changes in structural design (i.e., geometry, material properties, etc.) to changes in the vibration response. As satellites are subject to various sources of disturbance (e.g. repositioning, temperature changes, etc.), vibration mitigation is an important feature of the structural design. The macro-fiber composite (MFC) actuator, a new generation of the piezoelectric actuators, is generally considered ideal for this purpose owing to its capacity for large force and displacement, and especially the ability to conform to curved structures. These actuators are essentially piezo-ceramic fibers and interdigitated electrodes embedded in a Kapton membrane.

The MFC has been considered in several recent studies for gossamer structures [18–22]: Park et al. [18] and Sodano et al. [19] demonstrated the effectiveness of the MFC as an actuator and sensor, respectively, for an inflatable torus; Jha and Inman [20] demonstrated sliding mode vibration control of the same structure; Tarazaga et al. [21] demonstrated vibration control of a space-rigidizable deployable composite boom using positive position feedback; in modeling efforts, Renno and Inman [22] derived the dynamic response for a MFC-actuated membrane strip.

In this article we apply this dynamic modeling approach to the aforementioned inflatable satellite truss structure motivated by the Innovative Space-based radar Antenna Technology (ISAT), a DARPA program to place a large antenna in the Medium Earth Orbit (MEO). We derive the homogenized equations of motion for a satellite truss structure with a distributed array of macro-fiber composite (MFC) actuators, thereby extending the authors' previous work [7,8]. The resulting model is used to analyze the vibration response of the composite satellite structure and to quantify the dynamic effects of the MFC actuators.

2. Modeling

2.1. Strain energy

In this section, the equations of motion are derived for an inflatable antenna truss with integrated MFC actuators. A single element of an inflatable antenna truss by ILC Dover is shown in Fig. 1. This element consists of battens, longerons and diagonal members as shown in the figure. The truss element used in this study is very similar to the figure shown except for that the diagonals have a symmetric pattern in Fig. 2 which is motivated by ISAT structure. The truss structure is made of this element and is the support for the ISAT satellite antenna.

Fig. 1. Fundamental truss element of an inflatable truss (courtesy of ILC-Dover).

The general equations of strain are first derived by assumptions on the displacement field. The added stiffness of the MFC actuators is then evaluated and integrated into the strain energy expressions. Hamilton's principle is then applied to derive the partial differential equations of motion similar to [7,8].

To derive the kinetic and strain energy expressions for the resultant continuum, certain assumptions for the displacement components within an element are made. The individual members of the truss are modeled as bars, and the connections by pin joints. This results in a linearly varying displacement field where all the second order displacement components (bending and nonlinear terms) are neglected. Therefore, the three displacement components (u_1 , u_2 and u_3) along the coordinates (x_1 , x_2 , x_3) shown in Fig. 2 can be written as [5,7,8]:

Fig. 2. Fundamental element of the ISAT truss.

$$\begin{aligned}
 u_1(x_1, x_2, x_3) &= u_{1,0}(x_1) - x_2\phi_3(x_1) + x_3\phi_2(x_1) \\
 u_2(x_1, x_2, x_3) &= u_{2,0}(x_1) + x_2\varepsilon_{2,0}(x_1) + x_3 \left[-\phi_1(x_1) + \frac{1}{2} (2\varepsilon_{23,0}(x_1)) \right] \\
 u_3(x_1, x_2, x_3) &= u_{3,0}(x_1) + x_2 \left[\phi_1(x_1) + \frac{1}{2} (2\varepsilon_{23,0}(x_1)) \right] + x_3\varepsilon_{3,0}(x_1)
 \end{aligned} \tag{1}$$

where, $u_{1,0}$, $u_{2,0}$ and $u_{3,0}$ are the displacement components at the center of a cross-section located at coordinate x_1 and $\varepsilon_{2,0}$, $\varepsilon_{3,0}$ and $\varepsilon_{23,0}$ are the extensional and shear strain components evaluated at the center of that cross section respectively. Also, ϕ_1 , ϕ_2 and ϕ_3 are the rotational components of the cross section around x_1 , x_2 and x_3 . For example, the displacement along x_1 coordinate for a point at a cross section located at x_1 is denoted as u_1 . This can be found as the summation of the displacement $u_{1,0}$ at the center of the cross section and the projection of the components of the rotational degrees of freedom. Having the displacement field in Eq. (1), one can easily obtain the spatial derivatives of the displacement components to find the strain values ε_{11} , ε_{22} , ε_{33} , ε_{12} , ε_{13} and ε_{23} . These strain components are considered the extensional and shear strain components and can be obtained at every node in the truss element. Equation (A.1) in Appendix-A presents a complete derivation for these strain components. A Taylor series expansion around the x_1 coordinate is considered to account for variations of strain components within an element. Finally, the strain components take the form:

$$\begin{aligned}
 \varepsilon_{11} &= \varepsilon_{1,0} - x_2\kappa_{2,0} + x_3\kappa_{3,0} + x_1 \left(\frac{\partial \varepsilon_{1,0}}{\partial x_1} - x_2 \frac{\partial \kappa_{2,0}}{\partial x_1} + x_3 \frac{\partial \kappa_{3,0}}{\partial x_1} \right) \\
 \varepsilon_{22} &= \varepsilon_{2,0} + x_1 \frac{\partial \varepsilon_{2,0}}{\partial x_1} + \frac{1}{2} (x_1)^2 \frac{\partial^2 \varepsilon_{2,0}}{\partial x_1^2} \\
 \varepsilon_{33} &= \varepsilon_{3,0} + x_1 \frac{\partial \varepsilon_{3,0}}{\partial x_1} + \frac{1}{2} (x_1)^2 \frac{\partial^2 \varepsilon_{3,0}}{\partial x_1^2} \\
 \varepsilon_{12} &= \frac{1}{2} \left(2\varepsilon_{12,0} + x_2 \frac{\partial \varepsilon_{2,0}}{\partial x_1} + x_3 \left[-\kappa_{1,0} + \frac{1}{2} \left(2 \frac{\partial \varepsilon_{23,0}}{\partial x_1} \right) \right] \right. \\
 &\quad \left. + x_1 \left(2 \frac{\partial \varepsilon_{12,0}}{\partial x_1} + x_2 \frac{\partial^2 \varepsilon_{2,0}}{\partial x_1^2} + x_3 \left[-\frac{\partial \kappa_{1,0}}{\partial x_1} + \frac{1}{2} \left(2 \frac{\partial^2 \varepsilon_{23,0}}{\partial x_1^2} \right) \right] \right) \right) \\
 \varepsilon_{13} &= \frac{1}{2} \left(2\varepsilon_{13,0} + x_2 \left[\kappa_{1,0} + \frac{1}{2} \left(2 \frac{\partial \varepsilon_{23,0}}{\partial x_1} \right) \right] + x_3 \frac{\partial \varepsilon_{3,0}}{\partial x_1} \right. \\
 &\quad \left. + x_1 \left(2 \frac{\partial \varepsilon_{13,0}}{\partial x_1} + x_2 \left[\frac{\partial \kappa_{1,0}}{\partial x_1} + \frac{1}{2} \left(2 \frac{\partial^2 \varepsilon_{23,0}}{\partial x_1^2} \right) \right] + x_3 \frac{\partial^2 \varepsilon_{3,0}}{\partial x_1^2} \right) \right) \\
 \varepsilon_{23} &= \varepsilon_{23,0} + x_1 \left(\frac{\partial \varepsilon_{23,0}}{\partial x_1} \right) + \frac{1}{2} (x_1)^2 \frac{\partial^2 \varepsilon_{23,0}}{\partial x_1^2}
 \end{aligned} \tag{2}$$

where, $\kappa_{2,0}$ and $\kappa_{3,0}$ are the curvatures along x_2 and x_3 and $\kappa_{1,0}$ is the torsional strain. The strain components shown in Eq. (2) are evaluated at the center of each of the bar members in a truss element to calculate the strain energy of the bar members and subsequently the fundamental truss element. The detailed derivations are presented in Appendix A.

2.2. Stiffness of an inflatable boom with embedded MFCs

Figures 3 and 4 depict the carbon fiber inflatable rigidized boom and schematic of the boom and the integrated MFC actuators. This boom is considered the constituent element of the truss structure of Fig. 1. The material and geometrical properties of the inflatable rigidized boom and the MFC are presented in Table 1.

Fig. 3. Schematic of an inflatable boom and attached MFC.

Fig. 4. Carbon fiber inflatable rigidized boom and the end mass made by ILC Dover.

The calculations in following section present the overall stiffness of a boom member with the attached MFCs for a longeron member shown in Fig. 3. The same derivations will be valid for battens and diagonal members. Figure 5 shows schematic of a segment of the carbon fiber longeron boom member and the attached MFCs.

Fig. 5. Carbon fiber boom segment attached to MFCs.

The longitudinal stiffness of this segment is the summation of the boom segment stiffness and the MFC stiffness. Therefore, the overall stiffness for the actuated part (with MFC) can be found as:

$$K_L^* = \frac{n_L E_{MFC} A_{MFC}}{L_{MFC}} + \frac{E_L A_L}{L_{MFC}} \quad (3)$$

where n_L is the number of MFCs at the cross section as shown in the Fig. 3. The total stiffness of the longeron is:

$$\frac{1}{K_L} = \frac{1}{K_P} + \frac{m_L}{K_L^*} \quad (4)$$

where m_L is the number of MFC actuators along the length of the boom and the stiffness of the non-actuated length (passive length) is given as:

$$K_P = \frac{E_L A_L}{L_P} \quad (5)$$

Here, L_P denotes the passive length of the longeron and can be found as

$$L_P = L_L - m_L L_{MFC} \quad (6)$$

The total stiffness of the longeron is then:

$$K_L = \frac{1}{\frac{L_P}{E_L A_L} + \frac{m_L}{K_L^*}} = \frac{E_L A_L K_L^*}{(L_L - m_L L_{MFC}) K_L^* + m_L E_L A_L} \quad (7)$$

Similar derivations are used to find the stiffness values for diagonal and battens. For example for the diagonal members we find:

$$K_d = \frac{E_d A_d K_d^*}{(L_d - m_d L_{MFC}) K_d^* + m_d E_d A_d} \quad (8)$$

where,

$$K_d^* = \frac{n_d E_{MFC} A_{MFC} + E_d A_d}{L_{MFC}} \quad (9)$$

Here, m_d and n_d are the number of MFCs along the length of the diagonal and at the cross section respectively.

Table 1
Material and geometrical properties of the inflatable truss element and MFC

Property	Value
Length of longerons and battens	3 m
Diameter of longerons, battens and diagonals	0.0979 m
Elastic modulus of longerons, battens and diagonals	$6.408 \times 10^{10} \text{ N/m}^2$
Density of longerons, battens and diagonals	$1.4293 \times 10^3 \text{ Kg/m}^3$
Length of the MFC actuator	0.085 m
Width of the MFC actuator	0.028 m
Thickness of the MFC actuator	0.3 mm
Elastic modulus of the MFC actuator	$3.0336 \times 10^{10} \text{ N/m}^2$
Tip mass of the longeron, diagonal and batten booms	0.220 Kg

2.3. Equations of motion

The strain energy of the fundamental truss element is found using the stiffness values for the inflatable rigidized diagonals, battens and longerons. The detailed derivations are presented in the Appendix. The strain energy of the fundamental element of Fig. 2 is:

$$U_e = 3L_L^2 K_L \left(\frac{\partial u_{1,0}}{\partial x_1} \right)^2 + \frac{6L_L^2 L_b^2 K_L K_d}{K_d L_L^2 + 4K_L L_d^2} \left(\frac{\partial u_{2,0}}{\partial x_1} - \phi_3 \right)^2 + \frac{6L_L^2 L_b^2 K_L K_d}{K_d L_L^2 + 4K_L L_d^2} \left(\frac{\partial u_{3,0}}{\partial x_1} + \phi_2 \right)^2 \\ + \frac{1}{2} L_L^2 L_b^2 K_L \left(\frac{\partial \phi_3}{\partial x_1} \right)^2 + \frac{1}{2} L_L^2 K_L L_b^2 \left(\frac{\partial \phi_2}{\partial x_1} \right)^2 + \frac{L_L^2 L_b^4 K_L K_d}{4(K_d L_L^2 + K_L L_d^2)} \left(\frac{\partial \phi_1}{\partial x_1} \right)^2 \quad (10)$$

The kinetic energy of this element is:

$$T_e = 3 (\rho_b A_b L_b + \rho_L A_L L_L + \rho_d A_d L_d) \left(\left(\frac{\partial u_{1,0}}{\partial t} \right)^2 + \left(\frac{\partial u_{2,0}}{\partial t} \right)^2 + \left(\frac{\partial u_{3,0}}{\partial t} \right)^2 \right) + \frac{L_b^2 (\rho_b A_b L_b + 2\rho_L A_L L_L + \rho_d A_d L_d)}{4} \left(2 \left(\frac{\partial \phi_{1,0}}{\partial t} \right)^2 + \left(\frac{\partial \phi_{2,0}}{\partial t} \right)^2 + \left(\frac{\partial \phi_{3,0}}{\partial t} \right)^2 \right) \quad (11)$$

The structure of our interest has a repeated pattern configuration. Therefore, having the kinetic and strain energy expressions for the fundamental truss element, the partial differential equations (PDE) of motion can be found through application of the Hamilton's principle. The displacement components in the kinetic and strain energy expressions are the same as those in a one-dimensional beam specimen and are found as $u_{1,0}$, $u_{2,0}$, $u_{3,0}$, $\phi_{1,0}$, $\phi_{2,0}$ and $\phi_{3,0}$. The assumptions employed to solve for the strain components of the 3-dimensionanl truss in terms of those for a continuum model are presented in detail in the Appendix. The PDEs are expressed in terms of the stated displacement components (a total of six displacement components). Free-Free boundary conditions are considered for the solution which agrees with conditions of space orbit. The PDEs and their boundary conditions are:

$$\begin{cases} 4 (m_1 + 6m_{tip}) \frac{\partial^2 u_2}{\partial t^2} - C_2 \frac{\partial}{\partial x} \left(\frac{\partial u_2}{\partial x} - \phi_3 \right) = 0 \\ 4 (m_2 + L_b^2 m_{tip}) \frac{\partial^2 \phi_3}{\partial t^2} - 4C_4 \frac{\partial^2 \phi_3}{\partial x^2} - C_2 \left(\frac{\partial u_2}{\partial x} - \phi_3 \right) = 0 \end{cases} \quad (12a)$$

$$\left(\frac{\partial u_2}{\partial x} - \phi_3 \right) \Big|_{x=0} = \left(\frac{\partial u_2}{\partial x} - \phi_3 \right) \Big|_{x=L_t} = \frac{\partial \phi_3}{\partial x} \Big|_{x=0} = \frac{\partial \phi_3}{\partial x} \Big|_{x=L_t} = 0 \quad (12b)$$

$$\begin{cases} 4 (m_1 + 6m_{tip}) \frac{\partial^2 u_3}{\partial t^2} - C_3 \frac{\partial \phi_2}{\partial x} \left(\frac{\partial u_3}{\partial x} + \phi_2 \right) = 0 \\ 4 (m_2 + L_b^2 m_{tip}) \frac{\partial^2 \phi_2}{\partial t^2} - 4C_5 \frac{\partial^2 \phi_2}{\partial x^2} + C_3 \left(\frac{\partial u_3}{\partial x} + \phi_2 \right) = 0 \end{cases} \quad (13a)$$

$$\left(\frac{\partial u_3}{\partial x} + \phi_2 \right) \Big|_{x=0} = \left(\frac{\partial u_3}{\partial x} + \phi_2 \right) \Big|_{x=L_t} = \frac{\partial \phi_2}{\partial x} \Big|_{x=0} = \frac{\partial \phi_2}{\partial x} \Big|_{x=L_t} = 0 \quad (13b)$$

$$2 (m_2 + L_b^2 m_{tip}) \frac{\partial^2 \phi_1}{\partial t^2} - C_6 \frac{\partial^2 \phi_1}{\partial x^2} = 0 \quad (14a)$$

$$\frac{\partial \phi_1}{\partial x} \Big|_{x=0} = \frac{\partial \phi_1}{\partial x} \Big|_{x=L_t} = 0 \quad (14b)$$

$$(m_1 + 6m_{tip}) \frac{\partial^2 u_1}{\partial t^2} - C_1 \frac{\partial^2 u_1}{\partial x^2} = 0 \quad (15a)$$

$$\frac{\partial u_1}{\partial x} \Big|_{x=0} = \frac{\partial u_1}{\partial x} \Big|_{x=L_t} = 0 \quad (15b)$$

As shown, each of the bending and the cross section rotation component coordinates are coupled in the form of two equations. This is similar to the bending equations of motion for a Timoshenko beam. Also, the torsional and longitudinal equations are similar to those of a bar member. Solution of these PDEs results in the natural frequencies of the system for each of the vibration coordinates. The solution procedure for these PDEs is very similar to [23] where the solution is expanded in terms of mode shape parameters (spatial terms) and frequencies (temporal solution). The boundary conditions are applied to the expanded solution to relate the mode shape parameters to the frequencies. Subsequently the frequencies are found from the roots of the determinant of the resultant matrix from the boundary conditions. The numerical results are presented in the following sections. Ref. [23] presents the detailed solution for PDEs of this form and the natural frequencies.

Table 2
Number of MFCs along the length of members for various configurations

Property	Value
longeron (configuration 2)	5
diagonal (configuration 2)	7
longeron (configuration 3)	10
diagonal (configuration 3)	14

Table 3
Bending natural frequencies of a radar truss
for $t_L = t_b = t_d = 4.8 \times 10^{-4} m$ and
 $L_{tot} = 120 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC ¹
6.258	6.301	0.687
17.049	17.167	0.692
32.834	33.063	0.697
53.029	53.398	0.696
77.041	77.577	0.696
104.260	104.985	0.695
134.110	135.042	0.695
166.073	167.226	0.694
199.703	201.089	0.694
234.626	236.254	0.694
270.536	272.411	0.693
307.186	309.315	0.693
344.380	346.766	0.693
381.962	384.608	0.693
419.812	422.719	0.692
457.836	461.006	0.692
495.964	499.397	0.692
534.139	537.836	0.692
572.320	576.281	0.692
610.480	614.702	0.692

¹PFC: Percentage Frequency Change.

Table 4
Torsional natural frequencies of a radar truss
for $t_L = t_b = t_d = 4.8 \times 10^{-4} m$ and
 $L_{tot} = 120 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC
26.294	26.476	0.692
52.589	52.952	0.690
78.884	79.429	0.691
105.178	105.905	0.691
131.473	132.381	0.691
157.767	158.857	0.691
184.062	185.334	0.691
210.356	211.810	0.691
236.651	238.286	0.691
262.945	264.762	0.691
289.240	291.238	0.691
315.534	317.715	0.691
341.829	344.191	0.691
368.123	370.667	0.691
394.418	397.143	0.691
420.713	423.619	0.691
447.007	450.096	0.691
473.302	476.572	0.691
499.596	503.048	0.691
525.891	529.524	0.691

3. Numerical results

The geometry and material properties of the truss, inflatable boom and the MFCs are given in Table 1. Different configurations for MFCs are considered in this work for the purpose of comparison. The first configuration considers a structure made of the booms shown in Fig. 4 with no MFCs attached. The second and third configurations have 4 MFCs along the circumference as shown in Fig. 3 for each of the diagonal and the longeron members. The number of MFCs along the length of each of these members is shown in Table 2. MFCs are equally spaced on all these members. The result for the frequencies of lattice structures with various lengths and thickness values are presented in this section. As shown in Eqs (12) and (13), the coordinates for bending and rotation of the cross section are coupled. Therefore, the frequencies for these two coordinates are the same. Also, the results are identical for bending along x_2 and x_3 axes due to the symmetric diagonal pattern for this structure. Figures 6 and 7 show the bending frequencies of a lattice structure made of the boom member shown in Fig. 4. The results are presented for various truss lengths and boom thickness values.

A Finite Element Analysis (FEA) code was developed in MATLAB using the method presented in [24] to validate the accuracy of the natural frequencies of the developed model. The FEA code consists of 903 linear elements and 810 degrees of freedom. The comparison with the FEA results is presented in Figs 6, 7 and 8; the results are shown to be in excellent agreement. As expected and also clear from these figures, the frequencies decrease as the length of the truss increases. The same bar lengths were used in the construction of the elements for all truss structures. Also, the results show higher accuracy for longer lengths. This is also expected as the overall length to the element size ratio is directly related to the accuracy of frequency estimations. Another noteworthy point about these figures is that the FEA overestimates the frequencies. This is because FEA solvers find the higher bound of a Rayleigh Quotient in

Fig. 6. Frequency comparison of DPM and FEA for satellite truss of various lengths (no actuators- configuration 1).

Fig. 7. Frequency comparison of DPM and FEA for satellite truss of various lengths (actuator configuration 2).

Fig. 8. Frequency comparison of DPM and FEA for satellite truss of various longeron thickness (no actuators- configuration 1).

Table 5

Longitudinal natural frequencies of a radar truss for $t_L = t_b = t_d = 4.8 \times 10^{-4} m$ and $L_{tot} = 120 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC
86.474	87.076	0.696
172.948	174.152	0.696
259.422	261.228	0.696
345.895	348.303	0.696
432.369	435.379	0.696
518.843	522.455	0.696
605.317	609.531	0.696
691.791	696.607	0.696
778.265	783.683	0.696
864.738	870.758	0.696
951.212	957.834	0.696
1037.690	1044.910	0.696
1124.160	1131.990	0.696
1210.630	1219.060	0.696
1297.110	1306.140	0.696
1383.580	1393.210	0.696
1470.060	1480.290	0.696
1556.530	1567.370	0.696
1643.000	1654.440	0.696
1729.480	1741.520	0.696

Table 6

Bending natural frequencies of a radar truss for $t_L = t_b = t_d = 4.8 \times 10^{-4} m$ and $L_{tot} = 210 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC
2.049	2.064	0.727
5.627	5.666	0.688
10.965	11.042	0.697
17.970	18.104	0.740
26.586	26.771	0.691
36.688	36.944	0.693
48.181	48.516	0.690
60.950	61.374	0.691
74.882	75.403	0.691
89.864	90.488	0.690
105.785	106.520	0.690
122.542	123.393	0.690
140.038	141.010	0.689
158.183	159.281	0.689
176.895	178.123	0.689
196.100	197.460	0.689
215.730	217.227	0.689
235.727	237.362	0.689
256.037	257.813	0.689
276.615	278.533	0.689

Table 7

Torsional natural frequencies of a radar truss for $t_L = t_b = t_d = 4.8 \times 10^{-4} m$ and $L_{tot} = 210 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC
15.025	15.129	0.692
30.051	30.258	0.689
45.076	45.388	0.692
60.102	60.517	0.690
75.127	75.646	0.691
90.153	90.776	0.691
105.178	105.905	0.691
120.204	121.034	0.691
135.229	136.163	0.691
150.254	151.293	0.692
165.28	166.422	0.691
180.305	181.551	0.691
195.331	196.680	0.691
210.356	211.810	0.691
225.382	226.939	0.691
240.407	242.068	0.691
255.433	257.198	0.691
270.458	272.327	0.691
285.484	287.456	0.691
300.509	302.585	0.691

Table 8

Longitudinal natural frequencies of a radar truss for $t_L = t_b = t_d = 4.8 \times 10^{-4} m$ and $L_{tot} = 210 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC
49.414	49.758	0.696
98.827	99.515	0.696
148.240	149.273	0.697
197.654	199.030	0.696
247.068	248.788	0.696
296.482	298.546	0.696
345.895	348.303	0.696
395.309	398.061	0.696
444.723	447.819	0.696
494.136	497.576	0.696
543.550	547.334	0.696
592.963	597.091	0.696
642.377	646.849	0.696
691.791	696.607	0.696
741.204	746.364	0.696
790.618	796.122	0.696
840.032	845.880	0.696
889.445	895.637	0.696
938.859	945.395	0.696
988.272	995.152	0.696

Table 9

Bending natural frequencies of a radar truss for $t_L = t_b = t_d = 4.8 \times 10^{-4} m$ and $L_{tot} = 300 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC	Configuration 3	PFC
1.005	1.012	0.696	1.019	1.393
2.765	2.784	0.687	2.804	1.411
5.404	5.442	0.703	5.480	1.406
8.897	8.959	0.697	9.022	1.405
13.222	13.314	0.696	13.408	1.407
18.354	18.482	0.697	18.612	1.406
24.263	24.432	0.696	24.604	1.405
30.915	31.131	0.699	31.350	1.407
38.275	38.542	0.698	38.814	1.408
46.306	46.628	0.695	46.957	1.406
54.968	55.350	0.695	55.741	1.406
64.222	64.669	0.696	65.125	1.406
74.030	74.545	0.696	75.070	1.405
84.353	84.939	0.695	85.538	1.405
95.152	95.813	0.695	96.489	1.405
106.392	107.131	0.695	107.886	1.404
118.037	118.857	0.695	119.695	1.405
130.054	130.958	0.695	131.880	1.404
142.410	143.400	0.695	144.411	1.405
155.079	156.156	0.694	157.256	1.404

Table 10

Torsional natural frequencies of a radar truss for $t_L = t_b = t_d = 4.8 \times 10^{-4} m$ and $L_{tot} = 300 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC	Configuration 3	PFC
10.518	10.590	0.684	10.665	1.398
21.036	21.181	0.689	21.329	1.393
31.553	31.772	0.694	31.994	1.398
42.071	42.362	0.692	42.659	1.398
52.589	52.952	0.690	53.323	1.396
63.107	63.543	0.691	63.988	1.396
73.625	74.133	0.690	74.653	1.396
84.142	84.724	0.692	85.318	1.398
94.660	95.314	0.691	95.982	1.397
105.178	105.905	0.691	106.647	1.397
115.696	116.495	0.691	117.312	1.397
126.214	127.086	0.691	127.976	1.396
136.732	137.676	0.690	138.641	1.396
147.249	148.267	0.691	149.306	1.397
157.767	158.857	0.691	159.970	1.396
168.285	169.448	0.691	170.635	1.396
178.803	180.038	0.691	181.300	1.396
189.321	190.629	0.691	191.964	1.396
199.838	201.219	0.691	202.629	1.397
210.356	211.810	0.691	213.294	1.397

Table 11

Longitudinal natural frequencies of a radar truss for $t_L = t_b = t_d = 4.8 \times 10^{-4} m$ and $L_{tot} = 300 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC	Configuration 3	PFC
34.589	34.830	0.697	35.076	1.408
69.179	69.661	0.697	70.152	1.406
103.770	104.491	0.695	105.229	1.406
138.358	139.321	0.696	140.305	1.407
172.948	174.152	0.696	175.381	1.407
207.537	208.982	0.696	210.457	1.407
242.127	243.812	0.696	245.534	1.407
276.716	278.643	0.696	280.610	1.407
311.306	313.473	0.696	315.686	1.407
345.895	348.303	0.696	350.762	1.407
380.485	383.134	0.696	385.839	1.407
415.074	417.964	0.696	420.915	1.407
449.664	452.794	0.696	455.991	1.407
484.253	487.625	0.696	491.067	1.407
518.843	522.455	0.696	526.144	1.407
553.433	557.285	0.696	561.220	1.407
588.022	592.116	0.696	596.296	1.407
622.612	626.946	0.696	631.372	1.407
657.201	661.776	0.696	666.449	1.407
691.791	696.607	0.696	701.525	1.407

Table 12

Bending natural frequencies of a truss for $t_b = t_d = 4.8 \times 10^{-4} m$ $t_L = 3.6 \times 10^{-4} m$ and $L_{tot} = 300 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC	Configuration 3	PFC
0.898	0.906	0.891	0.914	1.782
2.470	2.494	0.972	2.517	1.903
4.835	4.878	0.889	4.923	1.820
7.966	8.038	0.904	8.112	1.833
11.852	11.959	0.903	12.068	1.823
16.473	16.620	0.892	16.772	1.815
21.807	22.002	0.894	22.202	1.811
27.830	28.100	0.970	28.334	1.811
34.516	34.823	0.889	35.138	1.802
41.835	42.207	0.889	42.588	1.800
49.759	50.200	0.886	50.652	1.795
58.258	58.772	0.882	59.300	1.789
67.300	67.892	0.880	68.500	1.783
76.854	77.528	0.877	78.220	1.777
86.891	87.650	0.873	88.430	1.771
97.379	98.227	0.871	99.098	1.765
108.290	109.230	0.868	110.195	1.759
119.595	120.629	0.865	121.690	1.752
131.266	132.396	0.861	133.557	1.745
143.276	144.506	0.859	145.768	1.739

Table 13

Torsional natural frequencies of a radar truss for $t_b = t_d = 4.8 \times 10^{-4} m$, $t_L = 3.6 \times 10^{-4} m$ and $L_{tot} = 300 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC	Configuration 3	PFC
10.553	10.633	0.758	10.714	1.526
21.105	21.265	0.758	21.428	1.530
31.658	31.897	0.755	32.143	1.532
42.210	42.530	0.758	42.857	1.533
52.763	53.162	0.756	53.571	1.531
63.315	63.795	0.758	64.285	1.532
73.868	74.427	0.757	74.999	1.531
84.420	85.060	0.758	85.714	1.533
94.973	95.692	0.757	96.428	1.532
105.526	106.325	0.757	107.142	1.531
116.078	116.957	0.757	117.856	1.532
126.631	127.589	0.756	128.570	1.531
137.183	138.222	0.757	139.284	1.532
147.736	148.854	0.757	149.999	1.532
158.288	159.487	0.757	160.713	1.532
168.841	170.119	0.757	171.427	1.532
179.393	180.752	0.758	182.141	1.532
189.946	191.384	0.757	192.855	1.531
200.499	202.017	0.757	203.570	1.532
211.051	212.649	0.757	214.284	1.532

Table 14

Longitudinal natural frequencies of a radar truss for $t_b = t_d = 4.8 \times 10^{-4} m$, $t_L = 3.6 \times 10^{-4} m$ and $L_{tot} = 300 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC	Configuration 3	PFC
30.910	31.189	0.903	31.476	1.831
61.821	62.378	0.901	62.951	1.828
92.731	93.568	0.903	94.427	1.829
123.642	124.757	0.902	125.903	1.829
154.552	155.946	0.902	157.378	1.828
185.463	187.135	0.902	188.854	1.828
216.373	218.324	0.902	220.329	1.828
247.283	249.513	0.901	251.805	1.828
278.194	280.703	0.901	283.281	1.828
309.104	311.892	0.902	314.756	1.828
340.015	343.081	0.902	346.232	1.828
370.925	374.270	0.901	377.708	1.828
401.836	405.459	0.901	409.183	1.828
432.746	436.649	0.901	440.659	1.828
463.656	467.838	0.902	472.134	1.828
494.567	499.027	0.901	503.610	1.828
525.477	530.216	0.901	535.086	1.828
556.388	561.405	0.901	566.561	1.828
587.298	592.595	0.901	598.037	1.828
618.209	623.784	0.901	629.513	1.828

Table 15

Bending natural frequencies of a radar truss for $t_b = 4.8 \times 10^{-4} m$, $t_L = t_d = 3.6 \times 10^{-4} m$ and $L_{tot} = 300 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC	Configuration 3	PFC
0.942	0.951	0.955	0.959	1.805
2.592	2.616	0.926	2.639	1.813
5.067	5.112	0.888	5.159	1.816
8.342	8.417	0.899	8.494	1.822
12.397	12.509	0.903	12.624	1.831
17.209	17.364	0.901	17.524	1.830
22.749	22.954	0.901	23.165	1.829
28.987	29.248	0.900	29.516	1.825
35.888	36.211	0.900	36.543	1.825
43.417	43.808	0.901	44.210	1.826
51.539	52.003	0.900	52.480	1.826
60.216	60.758	0.900	61.316	1.827
69.412	70.037	0.900	70.769	1.955
79.090	79.803	0.902	80.534	1.826
89.216	90.019	0.900	90.845	1.826
99.755	100.653	0.900	101.575	1.824
110.674	111.670	0.900	112.693	1.824
121.941	123.038	0.900	124.166	1.825
133.528	134.729	0.899	135.963	1.824
145.405	146.713	0.900	148.057	1.824

Table 16

Torsional natural frequencies of a radar truss for $t_b = 4.8 \times 10^{-4} m$, $t_L = t_d = 3.6 \times 10^{-4} m$ and $L_{tot} = 300 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC	Configuration 3	PFC
9.863	9.951	0.892	10.042	1.815
19.725	19.902	0.897	20.083	1.815
29.588	29.853	0.895	30.125	1.814
39.451	39.804	0.894	40.167	1.814
49.313	49.755	0.896	50.208	1.814
59.176	59.706	0.895	60.245	1.806
69.039	69.657	0.895	70.292	1.814
78.902	79.608	0.894	80.333	1.813
88.764	89.559	0.895	90.375	1.814
98.627	99.510	0.895	100.417	1.814
108.490	109.461	0.895	110.458	1.814
118.352	119.412	0.895	120.500	1.814
128.215	129.363	0.895	130.541	1.814
138.078	139.313	0.894	140.583	1.814
147.940	149.264	0.895	150.625	1.814
157.803	159.215	0.894	160.666	1.814
167.666	169.166	0.894	170.708	1.814
177.528	179.117	0.895	180.750	1.814
187.391	189.068	0.894	190.791	1.814
197.254	199.019	0.894	200.833	1.814

Table 17

Longitudinal natural frequencies of a radar truss for $t_b = 4.8 \times 10^{-4} m$, $t_L = t_d = 3.6 \times 10^{-4} m$ and $L_{tot} = 300 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC	Configuration 3	PFC
32.432	32.724	0.900	33.025	1.828
64.863	65.448	0.901	66.049	1.828
97.295	98.172	0.901	99.074	1.828
129.727	130.900	0.904	132.099	1.828
162.158	163.621	0.902	165.123	1.828
194.590	196.345	0.901	198.148	1.828
227.022	229.069	0.901	231.173	1.828
259.453	261.793	0.901	264.197	1.828
291.885	294.517	0.901	297.222	1.828
324.317	327.241	0.901	330.247	1.828
356.748	359.965	0.901	363.271	1.828
389.180	392.690	0.902	396.296	1.828
421.611	425.414	0.902	429.321	1.828
454.043	458.138	0.902	462.345	1.828
486.475	490.862	0.902	495.370	1.828
518.906	523.586	0.902	528.395	1.828
551.338	556.310	0.902	561.419	1.828
583.770	589.034	0.902	594.444	1.828
616.201	621.758	0.902	627.469	1.828
648.633	654.483	0.902	660.493	1.828

Table 18

Bending natural frequencies of a radar truss for $t_b = t_d = 4.8 \times 10^{-4} m$, $t_L = 2.4 \times 10^{-4} m$ and $L_{tot} = 300 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC	Configuration 3	PFC
0.758	0.768	1.319	0.778	2.639
2.088	2.115	1.293	2.143	2.634
4.087	4.139	1.272	4.193	2.594
6.740	6.826	1.276	6.915	2.596
10.038	10.165	1.265	10.298	2.590
13.969	14.146	1.267	14.330	2.584
18.519	18.753	1.264	18.997	2.581
23.673	23.971	1.259	24.282	2.573
29.412	29.782	1.258	30.166	2.564
35.719	36.167	1.254	36.632	2.556
42.573	43.105	1.250	43.657	2.546
49.954	50.575	1.243	51.220	2.534
57.841	58.556	1.236	59.300	2.522
66.211	67.026	1.231	67.872	2.509
75.042	75.962	1.226	76.916	2.497
84.313	85.341	1.219	86.407	2.484
94.001	95.142	1.214	96.324	2.471
104.086	105.342	1.207	106.644	2.458
114.545	115.920	1.200	117.344	2.444
125.358	126.854	1.193	128.405	2.431

Table 19

Torsional natural frequencies of a radar truss for $t_b = t_d = 4.8 \times 10^{-4} m$, $t_L = 2.4 \times 10^{-4} m$ and $L_{tot} = 300 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC	Configuration 3	PFC
10.294	10.390	0.932	10.489	1.894
20.588	20.780	0.932	20.978	1.894
30.883	31.171	0.932	31.467	1.891
41.177	41.561	0.932	41.956	1.891
51.471	51.951	0.932	52.445	1.892
61.765	62.341	0.932	62.934	1.892
72.059	72.731	0.932	73.422	1.891
82.353	83.121	0.932	83.911	1.891
92.648	93.512	0.932	94.400	1.891
102.942	103.902	0.932	104.889	1.891
113.236	114.292	0.932	115.378	1.891
123.530	124.682	0.932	125.867	1.891
133.824	135.072	0.932	136.356	1.892
144.119	145.463	0.932	146.845	1.891
154.413	155.853	0.932	157.334	1.891
164.707	166.243	0.932	167.823	1.891
175.001	176.633	0.932	178.312	1.892
185.295	187.023	0.932	188.801	1.892
195.590	197.414	0.932	199.290	1.891
205.884	207.804	0.932	209.779	1.891

Table 20

Longitudinal natural frequencies of a radar truss for $t_b = t_d = 4.8 \times 10^{-4} m$, $t_L = 2.4 \times 10^{-4} m$ and $L_{tot} = 300 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC	Configuration 3	PFC
26.098	26.432	1.279	26.779	2.609
52.195	52.863	1.279	53.558	2.611
78.293	79.295	1.279	80.337	2.610
104.391	105.727	1.279	107.116	2.610
130.488	132.159	1.280	133.895	2.611
156.586	158.590	1.279	160.674	2.610
182.684	185.022	1.279	187.453	2.610
208.782	211.454	1.279	214.231	2.609
234.879	237.886	1.280	241.010	2.610
260.977	264.317	1.279	267.789	2.610
287.075	290.749	1.279	294.568	2.610
313.172	317.181	1.280	321.347	2.610
339.270	343.613	1.280	348.126	2.610
365.368	370.044	1.279	374.905	2.610
391.465	396.476	1.280	401.684	2.610
417.563	422.908	1.280	428.463	2.610
443.661	449.340	1.280	455.242	2.610
469.759	475.771	1.279	482.021	2.610
495.856	502.203	1.280	508.800	2.610
521.954	528.635	1.280	535.579	2.610

Table 21

Bending natural frequencies of a radar truss for $t_b = 4.8 \times 10^{-4} m$, $t_L = t_d = 2.4 \times 10^{-4} m$ and $L_{tot} = 300 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC	Configuration 3	PFC
0.845	0.856	1.302	0.867	2.604
2.326	2.356	1.290	2.387	2.623
4.547	4.605	1.276	4.666	2.617
7.486	7.582	1.282	7.681	2.605
11.125	11.268	1.285	11.416	2.616
15.443	15.641	1.282	15.846	2.610
20.415	20.676	1.279	20.948	2.611
26.012	26.345	1.280	26.691	2.610
32.205	32.617	1.279	33.045	2.608
38.962	39.460	1.278	39.978	2.608
46.250	46.842	1.280	47.456	2.608
54.037	54.728	1.279	55.446	2.608
62.290	63.086	1.278	63.913	2.606
70.975	71.882	1.278	72.825	2.607
80.062	81.850	2.233	82.148	2.606
89.519	90.663	1.278	91.852	2.606
99.318	100.586	1.277	101.905	2.605
109.429	110.827	1.277	112.279	2.604
119.827	121.357	1.277	122.947	2.604
130.485	132.151	1.277	133.882	2.603

Table 22

Torsional natural frequencies of a radar truss for $t_b = 4.8 \times 10^{-4} m$, $t_L = t_d = 2.4 \times 10^{-4} m$ and $L_{tot} = 300 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC	Configuration 3	PFC
8.852	8.964	1.265	9.0813	2.590
17.704	17.929	1.271	18.163	2.592
26.556	26.893	1.269	27.244	2.590
35.408	35.858	1.271	36.325	2.589
44.260	44.822	1.270	45.406	2.589
53.112	53.787	1.271	54.488	2.590
61.964	62.751	1.270	63.569	2.590
70.816	71.715	1.269	72.650	2.589
79.668	80.680	1.270	81.732	2.590
88.520	89.644	1.270	90.813	2.590
97.372	98.609	1.270	99.894	2.590
106.224	107.573	1.270	108.975	2.589
115.076	116.538	1.270	118.057	2.590
123.928	125.502	1.270	127.138	2.590
132.780	134.467	1.270	136.219	2.590
141.632	143.431	1.270	145.300	2.589
150.484	152.395	1.269	154.382	2.590
159.336	161.360	1.270	163.463	2.590
168.188	170.324	1.270	172.544	2.590
177.040	179.289	1.270	181.626	2.590

Table 23

Longitudinal natural frequencies of a radar truss for $t_b = 4.8 \times 10^{-4} m$, $t_L = t_d = 2.4 \times 10^{-4} m$ and $L_{tot} = 300 m$ in (rad/sec)

Configuration 1	Configuration 2	PFC	Configuration 3	PFC
29.104	29.476	1.278	29.864	2.611
58.208	58.953	1.280	59.727	2.610
87.312	88.429	1.279	89.591	2.610
116.415	117.906	1.281	119.454	2.611
145.519	147.382	1.280	149.318	2.611
174.623	176.858	1.280	179.181	2.610
203.727	206.335	1.280	209.045	2.610
232.831	235.811	1.280	238.909	2.611
261.935	265.287	1.280	268.772	2.610
291.039	294.764	1.280	298.636	2.610
320.143	324.240	1.280	328.499	2.610
349.246	353.717	1.280	358.363	2.611
378.350	383.193	1.280	388.226	2.610
407.454	412.669	1.279	418.090	2.610
436.558	442.146	1.280	447.954	2.610
465.662	471.622	1.279	477.817	2.610
494.766	501.098	1.279	507.681	2.610
523.870	530.575	1.279	537.544	2.610
552.973	560.051	1.280	567.408	2.610
582.077	589.528	1.280	597.271	2.610

the calculation of the frequencies. This results in having larger values compared to the Distributed Parameter Model (DPM) method. The difference between the FEA and the analytical DPM diminishes for larger lengths. Figure 7 indicates a noticeable increase of the frequencies when compared to Fig. 6. This is due to the addition of the MFCs in the structure. Figure 8 presents the results for the bending frequencies for a truss with various longeron and diagonal thickness values and no actuators. As shown, increasing the thickness of the longeron members result in increasing the frequency values. This is also expected as these members contribute significantly to the bending stiffness of the truss compared to the diagonal and batten members. Therefore, increasing their thicknesses will have a larger impact on the overall stiffness of the truss compared to the overall mass which results in larger frequency values. This is in contrast to the diagonal members for which increasing their thickness results in a decrease of the frequencies as they add more mass than stiffness to the structure.

Tables 3 to 11 present the numerical results for bending, torsional, longitudinal frequencies and their PFC (Percentage Frequency Change) for various MFC configurations. The PFC compares the frequencies for configurations 2 and 3.

The results are shown for various lengths of the truss: 120 and 210 and 300 meters. It is shown that the frequencies change up to 3% as a result of the addition of the MFCs for these structures. In Tables 12 to 23 the frequency results for various thickness values are compared for the three MFC configurations. The effects of the MFC actuators are also clearly shown in these tables. A noticeable increase of the frequencies is shown due to the addition of MFCs. It is clear from these tables that the smaller the thicknesses of longerons and diagonals, the more important the effects of the MFCs on the natural frequencies become. This, indeed, emphasizes the importance of developing a modeling technique that accounts for the effects of the MFCs on the dynamics of inflatable and gossamer structures due to being ultra-light-weight and flexible.

4. Conclusion

An approximate beam-like dynamic model for an inflatable rigidized lattice structure with embedded MFC actuators was derived using a continuum modeling approach. A comparison of the natural frequencies of this model with those of finite element analysis demonstrated the precision of the continuum model and thus its usefulness in analyzing complex truss structures. The natural frequencies of lattice structures with two distributed actuator configurations, one with more actuators per member, were compared with the case of a structure with no integrated MFC actuation. The change in natural frequency as a result of adding the MFCs was shown to vary within 3%, most significantly for the bending frequencies. These frequency changes are particularly important and must be considered when designing controllers for such lightweight structures. The developed model has the advantage of providing a closed form solution which provides the designer with great insight about the dominant physics and the underlying mathematics of the problem. In addition, closed form solutions are much faster; in the case for this structure the CPU usage was 180 times less than a MATLAB FEA solver. The numerical results are presented for various MFC configurations.

Appendix A

Strain energy

Taking spatial derivatives of the displacement field in Eq. (1), one can easily find the strain field for the fundamental truss element as follows [5,7,8]:

$$\begin{aligned}\varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} = \frac{\partial u_{1,0}}{\partial x_1} - x_2 \frac{\partial \phi_3}{\partial x_1} + x_3 \frac{\partial \phi_2}{\partial x_1} = \varepsilon_{1,0} - x_2 \kappa_{2,0} + x_3 \kappa_{3,0} \\ \varepsilon_{22} &= \frac{\partial u_2}{\partial x_2} = \varepsilon_{2,0} \\ \varepsilon_{33} &= \frac{\partial u_3}{\partial x_3} = \varepsilon_{3,0}\end{aligned}$$

$$\begin{aligned}
\varepsilon_{12} &= \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) = \frac{1}{2} \left(\left(\frac{\partial u_{2,0}}{\partial x_1} - \phi_3 \right) + x_2 \frac{\partial \varepsilon_{2,0}}{\partial x_1} + x_3 \left[-\frac{\partial \phi_1}{\partial x_1} + \frac{1}{2} \left(2 \frac{\partial \varepsilon_{23,0}}{\partial x_1} \right) \right] \right) \\
&= \frac{1}{2} \left(2\varepsilon_{12,0} + x_2 \frac{\partial \varepsilon_{2,0}}{\partial x_1} + x_3 \left[-\kappa_{1,0} + \frac{1}{2} \left(2 \frac{\partial \varepsilon_{23,0}}{\partial x_1} \right) \right] \right) \\
\varepsilon_{13} &= \frac{1}{2} \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) = \frac{1}{2} \left(\left(\frac{\partial u_{3,0}}{\partial x_1} + \phi_2 \right) + x_2 \left[\frac{\partial \phi_1}{\partial x_1} + \frac{1}{2} \left(2 \frac{\partial \varepsilon_{23,0}}{\partial x_1} \right) \right] + x_3 \frac{\partial \varepsilon_{3,0}}{\partial x_1} \right) = \\
&= \frac{1}{2} \left(2\varepsilon_{13,0} + x_2 \left[\kappa_{1,0} + \frac{1}{2} \left(2 \frac{\partial \varepsilon_{23,0}}{\partial x_1} \right) \right] + x_3 \frac{\partial \varepsilon_{3,0}}{\partial x_1} \right) \\
\varepsilon_{23} &= \frac{1}{2} \left(\left[-\phi_1 + \frac{1}{2} (2\varepsilon_{23,0}) \right] + \left[\phi_1 + \frac{1}{2} (2\varepsilon_{23,0}) \right] \right) = \varepsilon_{23,0}
\end{aligned} \tag{A-1}$$

Here, the strain components $\varepsilon_{1,0}$, $\varepsilon_{2,0}$, $\varepsilon_{3,0}$, $\varepsilon_{12,0}$, $\varepsilon_{13,0}$ and $\varepsilon_{23,0}$ are the shear and extensional strain components evaluated at the center of a cross section of the truss element shown in Fig. 2. These strain components depend on x_1 coordinate of the cross section only. Also $\kappa_{1,0}$, $\kappa_{2,0}$ and $\kappa_{3,0}$ are twist and curvature components for the strain field and they are all dependent on x_1 only. The strain components in Eq. (A-1) can be projected along each of the bar members to find the extensional strain energy of each member. This can be done for all the 18 bar members shown in Fig. 2. The strain energy of the truss element can then be found using the following relation:

$$U^{(k)} = \frac{1}{2} K^{(k)} \left(L^{(k)} \right)^2 \left(\varepsilon^{(k)} \right)^2 \tag{A-2}$$

where $\varepsilon^{(k)}$ is the axial strain along the length of each bar member which is evaluated at the center of the member. As an example, the strain energy of all the diagonal members can be found as:

$$\begin{aligned}
U_d &= \frac{L_d^2 K_d}{2 (L_L^2 + L_b^2)^2} \left(\left(L_L^2 \varepsilon_{11}^{(13)} - 2L_L L_b \varepsilon_{12}^{(13)} + L_b^2 \varepsilon_{22}^{(13)} \right)^2 + \left(L_L^2 \varepsilon_{11}^{(14)} + 2L_L L_b \varepsilon_{12}^{(14)} + L_b^2 \varepsilon_{22}^{(14)} \right)^2 + \right. \\
&\quad + \left(L_L^2 \varepsilon_{11}^{(15)} + L_L L_b \varepsilon_{12}^{(15)} - \sqrt{3} L_L L_b \varepsilon_{13}^{(15)} + \frac{L_b^2}{4} \varepsilon_{22}^{(15)} - \frac{\sqrt{3} L_b^2}{2} \varepsilon_{23}^{(15)} + \frac{3L_b^2}{4} \varepsilon_{33}^{(15)} \right)^2 \\
&\quad + \left(L_L^2 \varepsilon_{11}^{(16)} - L_L L_b \varepsilon_{12}^{(16)} + \sqrt{3} L_L L_b \varepsilon_{13}^{(16)} + \frac{L_b^2}{4} \varepsilon_{22}^{(16)} - \frac{\sqrt{3} L_b^2}{2} \varepsilon_{23}^{(16)} + \frac{3L_b^2}{4} \varepsilon_{33}^{(16)} \right)^2 \\
&\quad + \left(L_L^2 \varepsilon_{11}^{(17)} + L_L L_b \varepsilon_{12}^{(17)} + \sqrt{3} L_L L_b \varepsilon_{13}^{(17)} + \frac{L_b^2}{4} \varepsilon_{22}^{(17)} + \frac{\sqrt{3} L_b^2}{2} \varepsilon_{23}^{(17)} + \frac{3L_b^2}{4} \varepsilon_{33}^{(17)} \right)^2 \\
&\quad \left. + \left(L_L^2 \varepsilon_{11}^{(18)} - L_L L_b \varepsilon_{12}^{(18)} - \sqrt{3} L_L L_b \varepsilon_{13}^{(18)} + \frac{L_b^2}{4} \varepsilon_{22}^{(18)} + \frac{\sqrt{3} L_b^2}{2} \varepsilon_{23}^{(18)} + \frac{3L_b^2}{4} \varepsilon_{33}^{(18)} \right)^2 \right)
\end{aligned} \tag{A-3}$$

The strain energy of the longerons and battens can be found in a very similar manner and from there the strain energy of the fundamental truss element can be found using the following relation:

$$U_e = U_L + U_b + U_d \tag{A-4}$$

The above strain energy is expressed in terms of a total of 21 strain and curvature components and their gradients, i.e.,

$$\begin{aligned}
&\varepsilon_{1,0}, \varepsilon_{2,0}, \varepsilon_{3,0}, \varepsilon_{12,0}, \varepsilon_{23,0}, \varepsilon_{13,0}, \kappa_{1,0}, \kappa_{2,0}, \kappa_{3,0}, \frac{\partial \varepsilon_{1,0}}{\partial x_1}, \frac{\partial \varepsilon_{2,0}}{\partial x_1}, \frac{\partial \varepsilon_{3,0}}{\partial x_1}, \frac{\partial \varepsilon_{13,0}}{\partial x_1}, \frac{\partial \varepsilon_{23,0}}{\partial x_1}, \frac{\partial \varepsilon_{12,0}}{\partial x_1}, \frac{\partial^2 \varepsilon_{2,0}}{\partial x_1^2}, \\
&\frac{\partial^2 \varepsilon_{3,0}}{\partial x_1^2}, \frac{\partial^2 \varepsilon_{23,0}}{\partial x_1^2}, \frac{\partial \kappa_{1,0}}{\partial x_1}, \frac{\partial \kappa_{2,0}}{\partial x_1}, \text{ and } \frac{\partial \kappa_{3,0}}{\partial x_1}.
\end{aligned}$$

To find a beam equivalent of the structure, it is important that all these strain components be related to the strain components in a one-dimensional structure. Therefore, similar assumptions to those made in Refs. [5,7,8] should be made to find an equivalent reduced order model (1-dimensional). One of these assumptions includes the assumption of the local free deformations which indicates the condition that forces associated with local deformations should vanish. Since forces can be expressed in terms of the derivatives of the strain energy with respect to their strain components, this condition requires these derivatives vanish. Therefore,

$$\frac{\partial U_e}{\partial \frac{\partial \varepsilon_{1,0}}{\partial x_1}} = 0 \Rightarrow 3 \left(L_d^2 K_d L_L^5 + L_L^2 K_L L_L (L_b^2 + L_L^2)^2 \right) \frac{\partial \varepsilon_{1,0}}{\partial x_1} - \sqrt{3} L_d^2 K_d L_b^2 L_L^3 \kappa_{1,0} = 0 \quad (\text{A-5})$$

$$\frac{\partial U_e}{\partial \frac{\partial \kappa_{2,0}}{\partial x_1}} = 0 \Rightarrow \left(L_d^2 K_d L_L^4 + 4 L_L^2 K_L (L_b^2 + L_L^2)^2 \right) \frac{\partial \kappa_{2,0}}{\partial x_1} + 8 \sqrt{3} L_d^2 K_d L_L^2 \varepsilon_{13,0} = 0 \quad (\text{A-6})$$

$$\frac{\partial U_e}{\partial \frac{\partial \kappa_{3,0}}{\partial x_1}} = 0 \Rightarrow \left(L_d^2 K_d L_L^4 + 4 L_L^2 K_L (L_b^2 + L_L^2)^2 \right) \frac{\partial \kappa_{3,0}}{\partial x_1} + 8 \sqrt{3} L_d^2 K_d L_L^2 \varepsilon_{12,0} = 0 \quad (\text{A-7})$$

$$\begin{aligned} \frac{\partial U_e}{\partial \frac{\partial \varepsilon_{12,0}}{\partial x_1}} = 0 \Rightarrow & 3 L_b L_L^2 \frac{\partial^2 \varepsilon_{2,0}}{\partial x_1^2} - 3 L_b L_L^2 \frac{\partial^2 \varepsilon_{3,0}}{\partial x_1^2} + 4 L_b \left(\sqrt{3} L_L^2 \frac{\partial^2 \varepsilon_{23,0}}{\partial x_1^2} + 6 \varepsilon_{2,0} - 6 \varepsilon_{3,0} \right) \\ & - 8 L_L^2 \left(6 \frac{\partial \varepsilon_{12,0}}{\partial x_1} + \sqrt{3} \kappa_{3,0} \right) = 0 \end{aligned} \quad (\text{A-8})$$

$$\begin{aligned} \frac{\partial U_e}{\partial \frac{\partial \varepsilon_{13,0}}{\partial x_1}} = 0 \Rightarrow & \sqrt{3} L_b L_L^2 \frac{\partial^2 \varepsilon_{2,0}}{\partial x_1^2} - 24 L_b \varepsilon_{23,0} - L_L^2 \\ & \left(\sqrt{3} L_b \frac{\partial^2 \varepsilon_{3,0}}{\partial x_1^2} + 3 L_b \frac{\partial^2 \varepsilon_{23,0}}{\partial x_1^2} + 24 \frac{\partial \varepsilon_{13,0}}{\partial x_1} + 4 \sqrt{3} \kappa_{2,0} \right) = 0 \end{aligned} \quad (\text{A-9})$$

$$\begin{aligned} \frac{\partial U_e}{\partial \frac{\partial \kappa_{1,0}}{\partial x_1}} = 0 \Rightarrow & L_L^2 \left(L_b^2 \left(\sqrt{3} \frac{\partial^2 \varepsilon_{2,0}}{\partial x_1^2} + \sqrt{3} \frac{\partial^2 \varepsilon_{3,0}}{\partial x_1^2} - 4 \frac{\partial \kappa_{1,0}}{\partial x_1} \right) + 16 \sqrt{3} \varepsilon_{1,0} \right) \\ & + 8 \sqrt{3} L_b^2 (\varepsilon_{2,0} + \varepsilon_{3,0}) = 0 \end{aligned} \quad (\text{A-10})$$

$$\begin{aligned} \frac{\partial U_e}{\partial \frac{\partial^2 \varepsilon_{2,0}}{\partial x_1^2}} = 0 \Rightarrow & L_L^2 \left(13 L_d^2 K_d L_b^4 + 72 L_b^2 K_b (L_b^2 + L_L^2)^2 \right) \frac{\partial^2 \varepsilon_{2,0}}{\partial x_1^2} + L_L^2 \\ & \left(-L_d^2 K_d L_b^4 + 24 L_b^2 K_b (L_b^2 + L_L^2)^2 \right) \frac{\partial^2 \varepsilon_{3,0}}{\partial x_1^2} \\ & - 8 (6 L_d^2 K_d L_b^3 L_L^2 \frac{\partial \varepsilon_{12,0}}{\partial x_1} - 6 L_b^2 K_b (L_b^2 + L_L^2)^2 (3 \varepsilon_{2,0} + \varepsilon_{3,0}) \\ & + L_d^2 K_d L_b^2 \left(-12 L_L^2 \varepsilon_{1,0} + L_b^2 \left(\sqrt{3} L_L^2 \frac{\partial \kappa_{1,0}}{\partial x_1} - 9 \varepsilon_{2,0} - 3 \varepsilon_{3,0} + 4 \sqrt{3} \varepsilon_{23,0} \right) \right. \\ & \left. + L_b L_L^2 \left(4 \sqrt{3} \frac{\partial \varepsilon_{13,0}}{\partial x_1} + 2 \kappa_{2,0} + \sqrt{3} \kappa_{3,0} \right) \right) \end{aligned} \quad (\text{A-11})$$

$$\begin{aligned}
\frac{\partial U_e}{\partial \frac{\partial^2 \varepsilon_{3,0}}{\partial x_1^2}} = 0 \Rightarrow & L_L^2 \left(-L_d^2 K_d L_b^4 + 24 L_b^2 K_b (L_b^2 + L_L^2)^2 \right) \frac{\partial^2 \varepsilon_{2,0}}{\partial x_1^2} + L_L^2 \\
& \left(13 L_d^2 K_d L_b^4 + 72 L_b^2 K_b (L_b^2 + L_L^2)^2 \right) \frac{\partial^2 \varepsilon_{3,0}}{\partial x_1^2} \\
& + 8(6 L_d^2 K_d L_L^2 L_b^3 \frac{\partial \varepsilon_{12,0}}{\partial x_1} + 6 L_b^2 K_b (L_b^2 + L_L^2)^2 (\varepsilon_{2,0} + \varepsilon_{3,0}) \\
& + L_d^2 K_d L_b^2 \left(12 L_L^2 \varepsilon_{1,0} + L_b^2 \left(-\sqrt{3} L_L^2 \frac{\partial \kappa_{1,0}}{\partial x_1} + 3 \varepsilon_{2,0} + 9 \varepsilon_{3,0} + 4 \sqrt{3} \varepsilon_{23,0} \right) \right) \\
& + L_d^2 K_d L_b^3 L_L^2 \left(4 \sqrt{3} \frac{\partial \varepsilon_{13,0}}{\partial x_1} + 2 \kappa_{2,0} + \sqrt{3} \kappa_{3,0} \right) \Big) \quad (A-12)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial U_e}{\partial \frac{\partial^2 \varepsilon_{23,0}}{\partial x_1^2}} = 0 \Rightarrow & L_L^3 \left(7 L_d^2 K_d L_b^4 + 24 L_b^2 K_b (L_b^2 + L_L^2)^2 \right) \frac{\partial^2 \varepsilon_{23,0}}{\partial x_1^2} \\
& + 4 L_L \left(6 \left(L_d^2 K_d L_b^4 + 2 L_b^2 K_b (L_b^2 + L_L^2)^2 \right) \varepsilon_{23,0} \right. \\
& \left. + L_d^2 K_d L_b^3 \left(2 \sqrt{3} L_b (\varepsilon_{2,0} - \varepsilon_{3,0}) + L_L^2 \left(-4 \sqrt{3} \frac{\partial \varepsilon_{12,0}}{\partial x_1} + 6 \frac{\partial \varepsilon_{13,0}}{\partial x_1} + \sqrt{3} \kappa_{2,0} - 2 \kappa_{3,0} \right) \right) \right) \quad (A-13)
\end{aligned}$$

Similarly to get a shear deformation type beam the following should hold [5,7,8]:

$$\begin{aligned}
\frac{\partial U_e}{\partial \varepsilon_{2,0}} = 0 \Rightarrow & 9 L_L^2 \left(L_d^2 K_d L_b^4 + 2 L_b^2 K_b (L_b^2 + L_L^2)^2 \right) \frac{\partial^2 \varepsilon_{2,0}}{\partial x_1^2} + 3 L_L^2 \left(L_d^2 K_d L_b^4 + 2 L_b^2 K_b (L_b^2 + L_L^2)^2 \right) \frac{\partial^2 \varepsilon_{3,0}}{\partial x_1^2} \\
& + 24 L_b^2 K_b (L_L^2 + L_b^2)^2 (3 \varepsilon_{2,0} + \varepsilon_{3,0}) \\
& + 4 L_d^2 K_d L_b^2 \left(24 L_L^2 \varepsilon_{1,0} + L_b^2 (\sqrt{3} L_L^2 \left(\frac{\partial^2 \varepsilon_{23,0}}{\partial x_1^2} - 2 \frac{\partial \kappa_{1,0}}{\partial x_1} \right) + 6 (3 \varepsilon_{2,0} + \varepsilon_{3,0})) \right) \\
& - 2 L_b L_L^2 \left(6 \frac{\partial \varepsilon_{12,0}}{\partial x_1} + \sqrt{3} \kappa_{3,0} \right) \quad (A-14)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial U_e}{\partial \varepsilon_{3,0}} = 0 \Rightarrow & 3 L_L^2 \left(L_d^2 K_d L_b^4 + 2 L_b^2 K_b (L_L^2 + L_b^2)^2 \right) \frac{\partial^2 \varepsilon_{2,0}}{\partial x_1^2} + 9 L_L^2 \left(L_d^2 K_d L_b^4 + 2 L_b^2 K_b (L_b^2 + L_L^2)^2 \right) \frac{\partial^2 \varepsilon_{3,0}}{\partial x_1^2} \\
& + 4 L_d^2 K_d L_b^2 (24 L_L^2 \varepsilon_{1,0} + L_b^2 \left(-\sqrt{3} L_L^2 \left(\frac{\partial^2 \varepsilon_{23,0}}{\partial x_1^2} + 2 \frac{\partial \kappa_{1,0}}{\partial x_1} \right) + 6 (\varepsilon_{2,0} + 3 \varepsilon_{3,0}) \right) \\
& + 2 L_b L_L^2 \left(6 \frac{\partial \varepsilon_{12,0}}{\partial x_1} + \sqrt{3} \kappa_{3,0} \right) + 24 L_b^2 K_b (L_b^2 + L_L^2)^2 (\varepsilon_{2,0} + 3 \varepsilon_{3,0})) = 0 \quad (A-15)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial U_e}{\partial \varepsilon_{23,0}} = 0 \Rightarrow & 6 L_b^2 K_b (L_b^2 + L_L^2)^2 \left(L_L^2 \frac{\partial^2 \varepsilon_{23,0}}{\partial x_1^2} + 4 \varepsilon_{23,0} \right) \\
& + L_d^2 K_d L_b^3 \left(-\sqrt{3} L_b L_L^2 \frac{\partial^2 \varepsilon_{2,0}}{\partial x_1^2} + 24 L_b \varepsilon_{23,0} \right. \\
& \left. + L_L^2 \left(\sqrt{3} L_b \frac{\partial^2 \varepsilon_{3,0}}{\partial x_1^2} + 3 L_b \frac{\partial^2 \varepsilon_{23,0}}{\partial x_1^2} + 24 \frac{\partial \varepsilon_{13,0}}{\partial x_1} + 4 \sqrt{3} \kappa_{2,0} \right) \right) = 0 \quad (A-16)
\end{aligned}$$

Finally, the compatibility condition requires that the strain components on the plane of the mid- cross section satisfy the following [5,7,8]:

$$\frac{\partial \varepsilon_{2,0}}{\partial x_1} = 0, \quad \frac{\partial \varepsilon_{3,0}}{\partial x_1} = 0, \quad \frac{\partial \varepsilon_{23,0}}{\partial x_1} = 0 \quad (\text{A-17})$$

Equations (A-5) to (A-16) along with the 3 equations in Eq. (A-17) give us a total of 15 equations to solve for the 15 of the 21 variables as follow:

$$\begin{aligned} \frac{\partial \varepsilon_{12,0}}{\partial x_1} &= -\frac{\kappa_{3,0}}{2\sqrt{3}}, \quad \frac{\partial \kappa_{1,0}}{\partial x_1} = \frac{4\sqrt{3}\varepsilon_{1,0}}{L_b^2}, \quad \frac{\partial \varepsilon_{13,0}}{\partial x_1} = -\frac{\kappa_{2,0}}{2\sqrt{3}}, \quad \varepsilon_{2,0} = 0, \quad \varepsilon_{3,0} = 0, \quad \varepsilon_{23,0} = 0, \\ \frac{\partial \varepsilon_{1,0}}{\partial x_1} &= \frac{K_d L_b^2 \kappa_{1,0}}{\sqrt{3}(K_d L_L^2 + K_L(L_b^2 + L_L^2))}, \quad \frac{\partial \kappa_{3,0}}{\partial x_1} = -\frac{8\sqrt{3}K_d \varepsilon_{12,0}}{K_d L_L^2 + 4K_L(L_b^2 + L_L^2)} \\ \frac{\partial \kappa_{2,0}}{\partial x_1} &= -\frac{8\sqrt{3}K_d \varepsilon_{13,0}}{K_d L_L^2 + 4K_L(L_b^2 + L_L^2)}, \quad \frac{\partial^2 \varepsilon_{2,0}}{\partial x_1^2} = 0, \quad \frac{\partial^2 \varepsilon_{23,0}}{\partial x_1^2} = 0, \quad \frac{\partial^2 \varepsilon_{3,0}}{\partial x_1^2} = 0 \end{aligned} \quad (\text{A-18})$$

Therefore, we are left with 6 strain components $\varepsilon_{1,0}$, $\varepsilon_{12,0}$, $\varepsilon_{13,0}$, $\kappa_{2,0}$, $\kappa_{3,0}$ and $\kappa_{1,0}$ which are similar to those in a continuum beam structure (One-dimensional). Finally, the strain energy for a truss element can be expressed in a quadratic form in terms of the strain values as:

$$U_e = C_1 \varepsilon_{1,0}^2 + C_2 \varepsilon_{12,0}^2 + C_3 \varepsilon_{13,0}^2 + C_4 \kappa_{2,0}^2 + C_5 \kappa_{3,0}^2 + C_6 \kappa_{1,0}^2 \quad (\text{A-19})$$

where,

$$\begin{aligned} C_1 &= 3L_L^2 K_L \quad C_2 = C_3 = \frac{24L_L^2 L_b^2 K_L K_d}{K_d L_L^2 + 4K_L L_d^2} \\ C_4 &= C_5 = \frac{1}{2} L_L^2 L_b^2 K_L \quad C_6 = \frac{L_L^2 L_b^4 K_L K_d}{4(K_d L_L^2 + K_L L_d^2)} \end{aligned} \quad (\text{A-20})$$

Recalling the definition of $\varepsilon_{1,0}$, $\varepsilon_{12,0}$, $\varepsilon_{13,0}$, $\kappa_{2,0}$, $\kappa_{3,0}$ and $\kappa_{1,0}$ we get:

$$\begin{aligned} \varepsilon_{1,0} &= \frac{\partial u_{1,0}}{\partial x_1}, \quad \varepsilon_{12,0} = \frac{1}{2} \left(\frac{\partial u_{2,0}}{\partial x_1} - \phi_3 \right), \quad \varepsilon_{13,0} = \frac{1}{2} \left(\frac{\partial u_{3,0}}{\partial x_1} + \phi_2 \right), \\ \kappa_{2,0} &= \frac{\partial \phi_3}{\partial x_1}, \quad \kappa_{3,0} = \frac{\partial \phi_2}{\partial x_1}, \quad \kappa_{1,0} = \frac{\partial \phi_1}{\partial x_1} \end{aligned} \quad (\text{A-21})$$

Equations (A-19), (A-20) and (A-21) result in:

$$\begin{aligned} U_e &= 3L_L^2 K_L \left(\frac{\partial u_{1,0}}{\partial x_1} \right)^2 + \frac{6L_L^2 L_b^2 K_L K_d}{K_d L_L^2 + 4K_L L_d^2} \left(\frac{\partial u_{2,0}}{\partial x_1} - \phi_3 \right)^2 + \frac{6L_L^2 L_b^2 K_L K_d}{K_d L_L^2 + 4K_L L_d^2} \left(\frac{\partial u_{3,0}}{\partial x_1} + \phi_2 \right)^2 \\ &+ \frac{1}{2} L_L^2 L_b^2 K_L \left(\frac{\partial \phi_3}{\partial x_1} \right)^2 + \frac{1}{2} L_L^2 K_L L_b^2 \left(\frac{\partial \phi_2}{\partial x_1} \right)^2 + \frac{L_L^2 L_b^4 K_L K_d}{4(K_d L_L^2 + K_L L_d^2)} \left(\frac{\partial \phi_1}{\partial x_1} \right)^2 \end{aligned} \quad (\text{A-22})$$

Kinetic energy

The next step to help find PDEs of motion of the equivalent DPM is to calculate the kinetic energy of the fundamental truss element. It can be shown very simply that the kinetic energy of a bar member can be written in the form:

$$T = \frac{1}{2} \rho A l (V_{1x}^2 + V_{1y}^2 + V_{1z}^2 + V_{2x}^2 + V_{2y}^2 + V_{2z}^2 + V_{1x} V_{2x} + V_{1y} V_{2y} + V_{1z} V_{2z}) \quad (\text{A-23})$$

where the first subscript denotes each of the two end nodes of a bar and the second subscript denotes the projection of the velocity along the axis of the coordinate system. Using Eqs (1) and (A-23), we can easily find the kinetic energy of the truss element in the following form:

$$T_e = m_1 \left(\left(\frac{\partial u_{1,0}}{\partial t} \right)^2 + \left(\frac{\partial u_{2,0}}{\partial t} \right)^2 + \left(\frac{\partial u_{3,0}}{\partial t} \right)^2 \right) + m_2 \left(2 \left(\frac{\partial \phi_{1,0}}{\partial t} \right)^2 + \left(\frac{\partial \phi_{2,0}}{\partial t} \right)^2 + \left(\frac{\partial \phi_{3,0}}{\partial t} \right)^2 \right) \quad (\text{A-24})$$

where,

$$m_1 = 3(\rho_b A_b L_b + \rho_L A_L L_L + \rho_d A_d L_d) \quad (\text{A-25})$$

$$m_2 = \frac{L_b^2 (\rho_b A_b L_b + 2\rho_L A_L L_L + \rho_d A_d L_d)}{4}$$

These expressions are used to derive the PDE of motion for the overall structure.

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