Principal parametric resonance of axially accelerating viscoelastic beams: Multi-scale analysis and differential quadrature verification

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Abstract. Transverse non-linear vibration is investigated in principal parametric resonance of an axially accelerating viscoelastic beam. The axial speed is characterized as a simple harmonic variation about a constant mean speed. The material time derivative is used in the viscoelastic constitutive relation. The transverse motion can be governed by a non-linear partial-differential equation or a non-linear integro-partial-differential equation. The method of multiple scales is applied to the governing equations to determine steady-state responses. It is confirmed that the mode uninvolved in the resonance has no effect on the steady-state response. The differential quadrature schemes are developed to verify results via the method of multiple scales. It is demonstrated that the straight equilibrium configuration becomes unstable and a stable steady-state emerges when the axial speed variation frequency is close to twice any linear natural frequency. The results derived for two governing equations are qualitatively the same, but quantitatively different. Numerical simulations are presented to examine the effects of the mean speed and the variation of the amplitude of the axial speed, the dynamic viscosity, the non-linear coefficients, and the boundary constraint stiffness on the instability interval and the steady-state response amplitude.

Keywords: Vibration, nonlinearity, axially accelerating beam, viscoelasticity, method of multiple scales, differential quadrature method

1. Introduction

Many engineering devices can be modeled as an axially moving beam, which is a typical gyroscopic continuum. Under certain excitations, axially moving beams undergo transverse motion. If the motion amplitude is large, the non-linearity should be taken into account. Much research has been done on non-linear vibration of axially moving beams [1–26].

As exact solutions are usually unavailable for non-linear differential equations, approximate analytical methods are widely applied to investigate non-linear vibration of axially moving beams. The used methods includes the asymptotic method of Krylov, Bogoliubov, and Mitropolsky [1], the Lindstedt-Poincaré method [2,17], the method of normal forms [2,9], the method of non-linear, complex modes [4,5], the method of multiple scales [7,8,14,20–23],

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the incremental harmonic balance method [16], and asymptotic perturbation analysis [24]. In the analysis, the governing equations of axially moving beams are usually truncated into a discrete system containing a few modes involved the resonance. The effects of other modes have not been mathematically treated, although they were physically assumed to be neglectable because the resonance modes dominate the response.

In spite of the fact that there have been many approximate analytical investigations on non-linear vibration of axially moving beams, there are very limited researches on the topic via the direct numerical approaches. The Galerkin method is widely used in the numerical investigations [6,10–13,19]. Consequently, the results obtained via approximate analytical approaches were rarely verified numerically. The known exceptions are that Pellicano and Zirilli [2] and Chen and Ding [22] confirmed their analytical results via a finite difference scheme.

To address the lack of research in the above-mentioned two aspects, the present investigation studies the principal parametric resonance of an axially accelerating viscoelastic beam with the emphasis on understanding the effects of the mode uninvolved in the resonance and verifying the analytical results by the differential quadrature method. The differential quadrature method is an effective numerical technique for initial and boundary problems [24,25,27,28]. The present investigation also differentiates from the previous analytical study on axially accelerating viscoelastic beams [14] in the constitutive relation and the boundary conditions. The partial time derivative is used in the Kelvin model to describe the viscoelastic behavior of axially moving materials [29], and also used in [14]. Mockensturm and Guo [30] convincingly argued that the Kelvin model generalized to axially moving materials should contain the material time derivative to account for the energy dissipation in steady motion. Here the material time derivative is used in the Kelvin model. The beam treated analytically here is constrained by a rotating sleeve with a rotational spring at each end. The boundary conditions are more general than the simple supports used by Chen and Yang [14]. The generalization is necessary to understand some practical engineering problems. For example, if the axially moving beam models a belt on a pair of rotating pulleys, the boundary conditions at the contact points are usually not exactly a typical fixed or simply supported.

There are two kinds of non-linear transverse models of axially moving beams. One is a non-linear integro-partial-differential equation used in [1–6,8,9,12,14,15,18,22,25,26]. The other is a non-linear partial-differential equation used in [7,10,11,13,14,16–23,25,26]. Because there has no evidence to favor any models, both models are treated in the investigation.

The present paper is organized as follows. Section 2 establishes the governing equations, a non-linear partial-differential equation and a non-linear integro-partial-differential equation. Section 3 applies the method of multiple scales to the equations in Section 2. Section 4 presents numerical examples to demonstrate the effects of some parameters based on the analytical results. Section 5 develops a differential quadrature scheme to solve the equations in Section 2 and to predict the effects of some parameters. Section 6 compares the analytical results with numerical ones. Section 7 ends the paper with the concluding remarks.

2. Governing equations

A uniform axially moving viscoelastic beam, with density ρ , cross-sectional area A , moment of inertial I and initial tension P_0 , travels at time-dependent axial transport speed $\gamma(t)$ between two ends separated by distance l . Consider only the bending vibration described by the transverse displacement $v(x,t)$, where t is the time and x is the axial coordinate. The Newton second law of motion yields [26]

$$\rho A (v_{,tt} + \dot{\gamma} v_{,x} + 2\gamma v_{,xt} + \gamma^2 v_{,xx}) = \frac{\partial}{\partial x} [(P_0 + P) v_{,x}] - \frac{\partial^2 M}{\partial x^2}, \quad (1)$$

where $P(x,t)$ is the axial disturbed tension and $M(x,t)$ is the bending moment. The viscoelastic material of the beam obeys the Kelvin model, with the constitution relation using the material derivative [26]

$$P/A = E_0 \varepsilon_L + \eta (\varepsilon_{L,t} + \gamma \varepsilon_{L,x}), \quad (2)$$

where E_0 is the stiffness, η is the dynamic viscosity, and ε_L is the Lagrange strain

$$\varepsilon_L = \frac{1}{2} v_{,x}^2 \quad (3)$$

to account for geometric non-linearity due to small but finite stretching of the beam. For a slender beam (for example with $I/(Al^2) < 0.001$), the linear moment-curvature relationship [26]

$$\frac{\partial^2 M}{\partial x^2} = E_0 I v_{,xxxx} + \eta I (v_{,xxxxt} + \gamma v_{,xxxxxx}) \quad (4)$$

is sufficiently accurate. Substitution of Eqs (2), (3) and (4) into Eq. (1) yields a non-linear partial-differential equation

$$\begin{aligned} \rho A (v_{,tt} + 2\gamma v_{,xt} + \dot{\gamma} v_{,x} + \gamma^2 v_{,xx}) - P_0 v_{,xx} - \frac{3}{2} E_0 A v_{,x}^2 v_{,xx} - 2A\eta v_{,x} v_{,xx} v_{,xt} - A\eta v_{,x}^2 v_{,xxt} \\ - 2A\eta \gamma v_{,x} v_{,x}^2 - A\eta \gamma v_{,x}^2 v_{,xxx} + E_0 I v_{,xxxx} + \eta I v_{,xxxxt} + \eta I \gamma v_{,xxxxxx} = 0. \end{aligned} \quad (5)$$

Equation (5) is the extension of Eq. (10) in [11] to the time-dependent axial speed.

If the spatial variation of the tension is rather small, then one can use the averaged value of the disturbed tension $\frac{1}{l} \int_0^l P dx$ to replace the exact value P . In this case, Eq. (1) becomes

$$\rho A (v_{,tt} + \dot{\gamma} v_{,x} + 2\gamma v_{,xt} + \gamma^2 v_{,xx}) = P_0 v_{,xx} + \frac{v_{,xx}}{l} \int_0^l P dx - \frac{\partial^2 M}{\partial x^2}, \quad (6)$$

which leads to an integro-partial-differential equation

$$\begin{aligned} \rho A (v_{,tt} + 2\gamma v_{,xt} + \dot{\gamma} v_{,x} + \gamma^2 v_{,xx}) - P_0 v_{,xx} - \frac{A v_{,xx}}{l} \int_0^l \left[\frac{1}{2} E_0 v_{,x}^2 + \eta v_{,x} v_{,xt} + \eta \gamma v_{,x} v_{,xx} \right] dx \\ + E_0 I v_{,xxxx} + \eta I v_{,xxxxt} + \eta I \gamma v_{,xxxxxx} = 0. \end{aligned} \quad (7)$$

Consider the axially moving beam constrained by rotating sleeves with rotational springs [31]. The stiffness constant of two springs is the same, denoted as K . Nullifying the transverse displacements and balancing the bending moment at both ends lead to the boundary conditions

$$\begin{aligned} v(0, t) = 0, \quad E_0 I v_{,xx}(0, t) - K v_{,x}(0, t) = 0; \\ v(l, t) = 0, \quad E_0 I v_{,xx}(l, t) + K v_{,x}(l, t) = 0. \end{aligned} \quad (8)$$

3. Steady-state response via the method of multiple scales

Introduce the dimensionless variables and parameters as follows

$$\begin{aligned} v \leftrightarrow \frac{v}{\sqrt{\varepsilon l}}, \quad x \leftrightarrow \frac{x}{l}, \quad t \leftrightarrow t \sqrt{\frac{P_0}{\rho A l^2}}, \quad \gamma(t) \leftrightarrow \gamma(t) \sqrt{\frac{\rho A}{P_0}}, \\ k_f = \sqrt{\frac{E_0 I}{P_0 l^2}}, \quad \alpha = \frac{I \eta}{\varepsilon L^3 \sqrt{\rho A P_0}}, \quad k_1 = \sqrt{\frac{E_0 A}{P_0 l}}, \quad k_2 = \frac{\eta \sqrt{A}}{l \varepsilon \sqrt{\rho P_0}}, \quad k = \frac{K}{E_0 I l}. \end{aligned} \quad (9)$$

where bookkeeping device ε is a small dimensionless parameter accounting for the fact that both the transverse displacement and the dynamic viscosity are very small. Equations (5) and (7) can be respectively cast into the dimensionless form

$$\begin{aligned} v_{,tt} + 2\gamma v_{,xt} + \dot{\gamma} v_{,x} + (\gamma^2 - 1) v_{,xx} + k_f^2 v_{,xxxx} + \varepsilon \alpha v_{,xxxxt} + \varepsilon \alpha \gamma v_{,xxxxxx} \\ = \frac{3}{2} \varepsilon k_1^2 v_{,x}^2 v_{,xx} + \varepsilon^2 k_2 \alpha v_{,x} (2v_{,xx} v_{,xt} + v_{,x} v_{,xxt} + 2\gamma v_{,xx}^2 + \gamma v_{,x} v_{,xxx}), \end{aligned} \quad (10)$$

$$\begin{aligned} v_{,tt} + 2\gamma v_{,xt} + \dot{\gamma} v_{,x} + (\gamma^2 - 1) v_{,xx} + k_f^2 v_{,xxxx} + \varepsilon \alpha v_{,xxxxt} + \varepsilon \alpha \gamma v_{,xxxxxx} \\ = \varepsilon v_{,xx} \int_0^1 \left[\frac{1}{2} k_1^2 v_{,x}^2 + k_2 \varepsilon (v_{,x} v_{,xt} + \gamma v_{,x} v_{,xx}) \right] dx, \end{aligned} \quad (11)$$

Boundary conditions Eq. (8) become

$$v(0, t) = 0, v_{,xx}(0, t) - kv_{,x}(0, t) = 0; v(1, t) = 0, v_{,xx}(1, t) + kv_{,x}(1, t) = 0. \quad (12)$$

In the present investigation, the axial speed is assumed to be a small simple harmonic variation about the constant mean speed,

$$\gamma(t) = \gamma_0 + \varepsilon\gamma_1 \sin \omega t, \quad (13)$$

where γ_0 is the mean axial speed, and $\varepsilon\gamma_1$ and ω are respectively the amplitude and the frequency of the axial speed variation, all in the dimensionless form. Substitution of Eq. (13) into Eqs (10) and (11) respectively yields

$$v_{,tt} + 2\gamma_0 v_{,xt} + (\gamma_0^2 - 1)v_{,xx} + k_f^2 v_{,xxxx} = \varepsilon \left[\frac{3}{2} k_1^2 v_{,x}^2 v_{,xx} - 2\gamma_1 \sin \omega t v_{,xt} - 2\gamma_0 \gamma_1 \sin \omega t v_{,xx} - \omega \gamma_1 \cos \omega t v_{,x} - \alpha v_{,xxxxt} - \alpha \gamma_0 v_{,xxxx} \right] + O(\varepsilon^2), \quad (14)$$

$$v_{,tt} + 2\gamma_0 v_{,xt} + (\gamma_0^2 - 1)v_{,xx} + k_f^2 v_{,xxxx} = \varepsilon \left[\frac{1}{2} k_1^2 v_{,xx} \int_0^1 v_{,x}^2 dx - 2\gamma_1 \sin \omega t v_{,xt} - 2\gamma_0 \gamma_1 \sin \omega t v_{,xx} - \omega \gamma_1 \cos \omega t v_{,x} - \alpha v_{,xxxxt} - \alpha \gamma_0 v_{,xxxx} \right] + O(\varepsilon^2). \quad (15)$$

The method of multiple scales can be employed to solve Eqs (14) and (15). Consider Eq. (15) first. A uniform approximation is sought in the form

$$v(x, t; \varepsilon) = v_0(x, T_0, T_1) + \varepsilon v_1(x, T_0, T_1) + O(\varepsilon^2), \quad (16)$$

where $T_0 = t$ is a fast scale characterizing motions occurring at ω_k (one of the natural frequencies of the corresponding unperturbed linear system), and $T_1 = \varepsilon t$ is a slow scale characterizing the modulation of the amplitudes and phases due to non-linearity, viscoelasticity and possible resonance. Substitution of Eq. (16) and the following relationships

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + O(\varepsilon^2), \quad \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial T_0^2} + 2\varepsilon \frac{\partial^2}{\partial T_0 \partial T_1} + O(\varepsilon^2) \quad (17)$$

into Eq. (15) and then equalization of coefficients of ε^0 and ε in the resulting equation lead to

$$v_{0,T_0 T_0} + 2\gamma_0 v_{0,x T_0} + (\gamma_0^2 - 1)v_{0,xx} + k_f^2 v_{0,xxxx} = 0, \quad (18)$$

$$v_{1,T_0 T_0} + 2\gamma_0 v_{1,x T_0} + (\gamma_0^2 - 1)v_{1,xx} + k_f^2 v_{1,xxxx} = \frac{1}{2} k_1^2 v_{0,xx} \int_0^1 v_{0,x}^2 dx - 2v_{0,T_0 T_1} - 2\gamma_0 v_{0,x T_1} - 2\gamma_1 \sin \omega T_0 (v_{0,x T_0} + \gamma_0 v_{0,xx}) - \gamma_1 \omega \cos \omega T_0 v_{0,x} - \alpha v_{0,xxxx T_0} - \alpha \gamma_0 v_{0,xxxx}. \quad (19)$$

Substitution of Eq. (16) into Eq. (12) yields

$$v_0(0, t) = 0, v_{0,xx}(0, t) - kv_{0,x}(0, t) = 0; v_0(1, t) = 0, v_{0,xx}(1, t) + kv_{0,x}(1, t) = 0, \quad (20)$$

$$v_1(0, t) = 0, v_{1,xx}(0, t) - kv_{1,x}(0, t) = 0; v_1(1, t) = 0, v_{1,xx}(1, t) + kv_{1,x}(1, t) = 0. \quad (21)$$

The modal solution [32] to Eq. (18) is

$$v_0(x, T_0, T_1) = \sum_{n=1}^{\infty} \phi_n(x) A_n(T_1) e^{i\omega_n T_0} + cc \quad (22)$$

where ω_n and ϕ_n are respectively the n -th natural frequency and complex mode function of the corresponding linear homogeneous system, and cc denotes the complex conjugate of all preceding terms on the right hand of an equation. Under the boundary condition (20), the modal function corresponding to the n -th natural frequency ω_n is [31]

$$\begin{aligned}
\phi_n(x) = & e^{i\beta_{1n}x} - \frac{ik(e^{i\beta_{1n}} + e^{i\beta_{3n}})(\beta_{1n} - \beta_{3n}) + (e^{i\beta_{1n}} - e^{i\beta_{3n}})[k^2 + (\beta_{1n} + \beta_{4n})(\beta_{3n} + \beta_{4n})]}{ik(e^{i\beta_{2n}} + e^{i\beta_{3n}})(\beta_{2n} - \beta_{3n}) + (e^{i\beta_{2n}} - e^{i\beta_{3n}})[k^2 + (\beta_{2n} + \beta_{4n})(\beta_{3n} + \beta_{4n})]} \frac{(\beta_{1n} - \beta_{4n})}{(\beta_{2n} - \beta_{4n})} e^{i\beta_{2n}x} - \\
& \frac{ik(e^{i\beta_{1n}} + e^{i\beta_{2n}})(\beta_{1n} - \beta_{2n}) + (e^{i\beta_{1n}} - e^{i\beta_{2n}})[k^2 + (\beta_{1n} + \beta_{4n})(\beta_{2n} + \beta_{4n})]}{ik(e^{i\beta_{2n}} + e^{i\beta_{3n}})(\beta_{2n} - \beta_{3n}) + (e^{i\beta_{2n}} - e^{i\beta_{3n}})[k^2 + (\beta_{3n} + \beta_{4n})(\beta_{2n} + \beta_{4n})]} \frac{(\beta_{1n} - \beta_{4n})}{(\beta_{3n} - \beta_{4n})} e^{i\beta_{3n}x} - \quad (23) \\
& \left\{ 1 - \frac{ik(e^{i\beta_{1n}} + e^{i\beta_{3n}})(\beta_{1n} - \beta_{3n}) + (e^{i\beta_{1n}} - e^{i\beta_{3n}})[k^2 + (\beta_{1n} + \beta_{4n})(\beta_{3n} + \beta_{4n})]}{ik(e^{i\beta_{2n}} + e^{i\beta_{3n}})(\beta_{2n} - \beta_{3n}) + (e^{i\beta_{2n}} - e^{i\beta_{3n}})[k^2 + (\beta_{2n} + \beta_{4n})(\beta_{3n} + \beta_{4n})]} \frac{(\beta_{1n} - \beta_{4n})}{(\beta_{2n} - \beta_{4n})} - \right. \\
& \left. \frac{ik(e^{i\beta_{1n}} + e^{i\beta_{2n}})(\beta_{1n} - \beta_{2n}) + (e^{i\beta_{1n}} - e^{i\beta_{2n}})[k^2 + (\beta_{1n} + \beta_{4n})(\beta_{2n} + \beta_{4n})]}{ik(e^{i\beta_{2n}} + e^{i\beta_{3n}})(\beta_{2n} - \beta_{3n}) + (e^{i\beta_{2n}} - e^{i\beta_{3n}})[k^2 + (\beta_{3n} + \beta_{4n})(\beta_{2n} + \beta_{4n})]} \frac{(\beta_{1n} - \beta_{4n})}{(\beta_{3n} - \beta_{4n})} \right\} e^{i\beta_{4n}x},
\end{aligned}$$

where β_{jn} ($j = 1, 2, 3, 4$) are four roots of the following 4th order algebraic equation

$$k_f^4 \beta_{jn}^4 + (1 - \gamma^2) \beta_{jn}^2 - 2\omega_n \beta_{jn} - \omega_n^2 = 0 \quad (24)$$

that can be solved with ω_n from the frequency equation [31]

$$\begin{aligned}
& e^{i(\beta_{1n} + \beta_{2n})} (\beta_{1n} - \beta_{2n}) (-ik + \beta_{1n} + \beta_{2n}) (\beta_{3n} - \beta_{4n}) (ik + \beta_{3n} + \beta_{4n}) + e^{i(\beta_{1n} + \beta_{3n})} (\beta_{3n} - \beta_{1n}) \\
& (-ik + \beta_{1n} + \beta_{3n}) (\beta_{2n} - \beta_{4n}) (ik + \beta_{2n} + \beta_{4n}) + e^{i(\beta_{1n} + \beta_{4n})} (\beta_{2n} - \beta_{3n}) (ik + \beta_{2n} + \beta_{3n}) \\
& (\beta_{1n} - \beta_{4n}) (-ik + \beta_{1n} + \beta_{4n}) + e^{i(\beta_{2n} + \beta_{3n})} (\beta_{2n} - \beta_{3n}) (-ik + \beta_{2n} + \beta_{3n}) (\beta_{1n} - \beta_{4n}) \quad (25) \\
& (ik + \beta_{1n} + \beta_{4n}) + e^{i(\beta_{2n} + \beta_{4n})} (\beta_{3n} - \beta_{1n}) (ik + \beta_{3n} + \beta_{1n}) (\beta_{2n} - \beta_{4n}) (-ik + \beta_{2n} + \beta_{4n}) \\
& + e^{i(\beta_{3n} + \beta_{4n})} (\beta_{1n} - \beta_{2n}) (ik + \beta_{1n} + \beta_{2n}) (\beta_{3n} - \beta_{4n}) (-ik + \beta_{3n} + \beta_{4n}) = 0.
\end{aligned}$$

If the speed variation frequency ω approaches two times that of any natural frequency of the system Eq. (18), principal parametric resonance may occur [14, 19, 21, 22]. A detuning parameter σ is introduced to quantify the deviation of ω from ω_m , and ω is described by

$$\omega = 2\omega_m + \varepsilon\sigma \quad (26)$$

where ω_m is the m -th frequency of free vibration described by Eq. (18). To investigate the principal parametric resonance with the possible contributions of a mode not involved the resonance, the solution to Eq. (18) is expressed as

$$v_0(x, T_0, T_1) = \phi_m(x) A_m(T_1) e^{i\omega_m T_0} + \phi_n(x) A_n(T_1) e^{i\omega_n T_0} + cc \quad (27)$$

Substituting Eqs (26) and (27) into Eq. (19) and expressing the trigonometric functions in exponential form yield

$$\begin{aligned}
& v_{1, T_0 T_0} + 2\gamma_0 v_{1, x T_0} + (\gamma_0^2 - 1) v_{1, xx} + k_f^2 v_{1, xxx} \\
& = \left[k_1^2 |A_n|^2 A_m \left(\phi''_m \int_0^1 \bar{\phi}'_n \phi'_n dx + \phi''_n \int_0^1 \bar{\phi}'_m \phi'_m dx + \bar{\phi}''_n \int_0^1 \phi'_n \phi'_m dx \right) + \frac{1}{2} k_1^2 \left(\bar{\phi}''_m \int_0^1 \phi'^2_m dx + \right. \right. \\
& \left. \left. 2\phi''_m \int_0^1 \phi'_m \bar{\phi}'_m dx \right) A_m^2 \bar{A}_m - 2\dot{A}_m (i\omega_m \phi_m + \gamma_0 \phi'_m) + i\gamma_1 \gamma_0 \bar{\phi}''_m \bar{A}_m e^{i\sigma T_1} - \alpha i\omega_m \phi_m^{(4)} A_m - \alpha \gamma_0 \phi_m^{(5)} A_m \right] e^{i\omega_m T_0} \\
& + \left[k_1^2 |A_m|^2 A_n \left(\phi''_m \int_0^1 \bar{\phi}'_m \phi'_m dx + \phi''_m \int_0^1 \phi'_n \bar{\phi}'_m dx + \bar{\phi}''_m \int_0^1 \phi'_n \phi'_m dx \right) + \frac{1}{2} k_1^2 \left(\bar{\phi}''_n \int_0^1 \phi'^2_n dx \right. \right. \\
& \left. \left. + 2\phi''_n \int_0^1 \phi'_n \bar{\phi}'_n dx \right) A_n^2 \bar{A}_n - 2\dot{A}_n (i\omega_n \phi_n + \gamma_0 \phi'_n) - \alpha i\omega_n \phi_n^{(4)} A_n - \alpha \gamma_0 \phi_n^{(5)} A_n \right] e^{i\omega_n T_0} + NST \quad (28)
\end{aligned}$$

where the dot and the prime denote derivation with respect to the slow time variable T_1 and the dimensionless spatial variable x respectively, the over bar denotes complex conjugation, and *NST* stands for the terms that will not bring secular terms into the solution.

Equation (28) has a bounded solution only if a solvability condition holds. The solvability condition [33] demands that the possible secular term coefficient at the right hand of the Eq. (28) be orthogonal to every solution of the adjoint of the homogeneous problem. That is

$$\left\langle k_1^2 |A_m|^2 \left(\phi_m'' \int_0^1 \bar{\phi}_n' \phi_n' dx + \bar{\phi}_n'' \int_0^1 \phi_n' \phi_m' dx + \phi_n'' \int_0^1 \bar{\phi}_n' \phi_m' dx \right) - \alpha i \omega_m \phi_m^{(4)} A_m - \alpha \gamma_0 \phi_m^{(5)} A_m + \right. \\ \left. \frac{1}{2} k_1^2 |A_m|^2 A_m \left(\bar{\phi}_m'' \int_0^1 \phi_m'^2 dx + 2 \phi_m'' \int_0^1 \phi_m' \bar{\phi}_m' dx \right) - 2 \dot{A}_m (i \omega_m \phi_m + \gamma_0 \phi_m') + i \gamma_1 \gamma_0 \bar{\phi}_m'' \bar{A}_m e^{i \sigma T_1}, \phi_m \right\rangle = 0, \quad (29)$$

$$\left\langle k_1^2 |A_m|^2 A_n \left(\phi_n'' \int_0^1 \bar{\phi}_m' \phi_m' dx + \bar{\phi}_m'' \int_0^1 \phi_n' \phi_m' dx + \phi_m'' \int_0^1 \phi_n' \bar{\phi}_m' dx \right) - \alpha i \omega_n \phi_n^{(4)} A_n - \alpha \gamma_0 \phi_n^{(5)} A_n \right. \\ \left. + \frac{1}{2} k_1^2 \left(\bar{\phi}_n'' \int_0^1 \phi_n'^2 dx + 2 \phi_n'' \int_0^1 \phi_n' \bar{\phi}_n' dx \right) A_n |A_n|^2 - 2 \dot{A}_n (i \omega_n \phi_n + \gamma_0 \phi_n'), \phi_n \right\rangle = 0, \quad (30)$$

where the inner product is defined for complex functions $f(x)$ and $g(x)$ on $[0,1]$ as

$$\langle f, g \rangle = \int_0^1 f(x) \bar{g}(x) dx. \quad (31)$$

Application of the distributive law of the inner product to Eqs (29) and (30) respectively leads to

$$\dot{A}_m + (\alpha \mu_m - k_1^2 C_{mn} |A_n|^2) A_m + \gamma_1 e^{i \sigma T_1} \chi_m \bar{A}_m - k_1^2 \kappa_m A_m^2 \bar{A}_m = 0 \quad (32)$$

$$\dot{A}_n + (\alpha \mu_n - k_1^2 C_{nm} |A_m|^2) A_n - k_1^2 \kappa_n A_n^2 \bar{A}_n = 0, \quad (33)$$

where

$$\mu_j = \frac{i \omega_j \int_0^1 \phi_j^{(4)} \bar{\phi}_j dx + \gamma_0 \int_0^1 \phi_j^{(5)} \bar{\phi}_j dx}{2 \left(i \omega_j \int_0^1 \phi_j \bar{\phi}_j dx + \gamma_0 \int_0^1 \phi_j' \bar{\phi}_j' dx \right)} \quad (j = m, n), \quad (34)$$

$$\chi_m = - \frac{i \gamma_0 \int_0^1 \bar{\phi}_m'' \bar{\phi}_m dx}{2 \left(i \omega_m \int_0^1 \phi_m \bar{\phi}_m dx + \gamma_0 \int_0^1 \phi_m' \bar{\phi}_m' dx \right)},$$

$$C_{mn} = \frac{\int_0^1 \phi_m'' \bar{\phi}_m dx \int_0^1 \bar{\phi}_n' \phi_n' dx + \int_0^1 \phi_n'' \bar{\phi}_m dx \int_0^1 \bar{\phi}_n' \phi_m' dx + \int_0^1 \bar{\phi}_n'' \bar{\phi}_m dx \int_0^1 \phi_n' \phi_m' dx}{2 \int_0^1 (i \omega_m \phi_m + \gamma_0 \phi_m') \bar{\phi}_m dx}, \\ C_{nm} = \frac{\int_0^1 \phi_n'' \bar{\phi}_n dx \int_0^1 \bar{\phi}_m' \phi_m' dx + \int_0^1 \phi_m'' \bar{\phi}_n dx \int_0^1 \phi_n' \bar{\phi}_m' dx + \int_0^1 \bar{\phi}_m'' \bar{\phi}_n dx \int_0^1 \phi_n' \phi_m' dx}{2 \int_0^1 (i \omega_n \phi_n + \gamma_0 \phi_n') \bar{\phi}_n dx}, \quad (35)$$

$$\kappa_j = \frac{\int_0^1 \bar{\phi}_j \bar{\phi}''_j dx \int_0^1 \phi_j'^2 dx + 2 \int_0^1 \phi_j' \bar{\phi}_j' dx \int_0^1 \bar{\phi}_j \phi_j'' dx}{4 \left(i\omega_j \int_0^1 \bar{\phi}_j \phi_j dx + \gamma_0 \int_0^1 \bar{\phi}_j \phi_j' dx \right)} \quad (j = m, n). \quad (36)$$

The coefficients μ_j ($j = m, n$), χ_m , C_{mn} , C_{nm} , and κ_j ($j = m, n$) are determined by the natural frequencies and the modal function of linear system (18), which are dependent on γ_0 and k_f , and are independent of α , γ_1 and k_1 . It can be numerically demonstrated that under the given boundary conditions

$$Re(\mu_n) > 0, Im(\mu_n) = 0; Re(C_{jk}) = 0, Im(C_{jk}) > 0; Re(\kappa_n) = 0, Im(\kappa_n) < 0. \quad (37)$$

Express the solution of Eqs (32) and (33) in the polar form

$$A_m(T_1) = a_m(T_1) e^{i\varphi_m(T_1)}, A_n(T_1) = a_n(T_1) e^{i\varphi_n(T_1)}. \quad (38)$$

where real functions $a_m(T_1)$ and $\varphi_m(T_1)$ are respectively the amplitude and the phase angle of the response in the m -th principal parametric resonance, and real functions $a_n(T_1)$ and $\varphi_n(T_1)$ are possible contribution to the response in the m -th principal parametric resonance. Substitution of Eq. (38) into Eqs (32) and (33) and separation of real and imaginary parts in the resulting equations give

$$\begin{aligned} \dot{a}_m &= [-\alpha Re(\mu_m) + \gamma_1 Im(\chi_m) \sin(\sigma T_1 - 2\varphi_m) - \gamma_1 Re(\chi_m) \cos(\sigma T_1 - 2\varphi_m)] a_m, \\ a_m \dot{\varphi}_m &= -\gamma_1 [Re(\chi_m) \sin(\sigma T_1 - 2\varphi_m) + Im(\chi_m) \cos(\sigma T_1 - 2\varphi_m)] a_m \\ &\quad - k_1^2 Im(C_{mn}) a_n^2 a_m + k_1^2 Im(\kappa_m) a_m^3, \end{aligned} \quad (39)$$

$$\begin{aligned} \dot{a}_n &= -\alpha Re(\mu_n) a_n, \\ a_n \dot{\varphi}_n &= -k_1^2 Im(C_{nm}) |A_m|^2 a_n + k_1^2 Im(\kappa_n) a_n^3, \end{aligned} \quad (40)$$

From Eq. (40), a_n decays to zero exponentially. Therefore, the n -th mode has actually no effect on the m -th principal parametric resonance.

Inserting $a_n = 0$ into Eq. (39) leads to

$$\begin{aligned} \dot{a}_m &= [-\alpha Re(\mu_m) + \gamma_1 Im(\chi_m) \sin \theta_m - \gamma_1 Re(\chi_m) \cos \theta_m] a_m, \\ a_m \dot{\theta}_m &= \sigma a_m + 2\gamma_1 [Re(\chi_m) \sin \theta_m + Im(\chi_m) \cos \theta_m] a_m - 2k_1^2 Im(\kappa_m) a_m^3, \end{aligned} \quad (41)$$

where

$$\theta_m = \sigma T_1 - 2\varphi_m \quad (42)$$

For the steady-state response, the amplitude a_m and the new phase angle θ_m in Eq. (41) are constant [31,33]. Locating the fixed points, one can find the amplitudes of the steady-state responses. There is a fixed point at the origin (trivial zero solution), which represents the straight equilibrium configuration of the beam. In addition, there may exist nontrivial periodic solution with amplitudes defined by

$$a_{m1} = \frac{\sqrt{Im(\kappa_m)}}{k_1} \sqrt{2\sigma + 4\sqrt{\gamma_1^2 |\chi_m|^2 - \alpha^2 [Re(\mu_m)]^2}}, \quad (43)$$

$$a_{m2} = \frac{\sqrt{Im(\kappa_m)}}{k_1} \sqrt{2\sigma - 4\sqrt{\gamma_1^2 |\chi_m|^2 - \alpha^2 [Re(\mu_m)]^2}}. \quad (44)$$

From Eqs (43) and (44), it can be concluded that the nontrivial steady-state solutions exist only if the following conditions hold,

$$\alpha \leq \frac{\gamma_1 |\chi_m|}{Re(\mu_m)} \quad (45)$$

for both nontrivial solutions,

$$\sigma \geq \sigma_1 = -2\sqrt{\gamma_1^2 |\chi_m|^2 - \alpha^2 [Re(\mu_m)]^2} \quad (46)$$

for solution (43), and

$$\sigma \geq \sigma_2 = 2\sqrt{\gamma_1^2 |\chi_m|^2 - \alpha^2 [Re(\mu_m)]^2} \quad (47)$$

for solution (44). It can be proved [14] that the zero solution is unstable for $\sigma \in [\sigma_1, \sigma_2]$, solution (43) is always stable, and solution (44) is always unstable.

From Eqs (16), (26), (27) and (38), the approximate expression of the stable steady-state responses in the m -th principal parametric resonance is

$$u_0(x, t) = 2a_{m1} |\phi_m(x)| \cos\left(\frac{\omega}{2}t - \frac{1}{2}\theta_{m1} + \arccos \frac{Re\phi_m(x)}{|\phi_m(x)|}\right), \quad (48)$$

where

$$\theta_{m1} = \frac{1}{2} \arcsin\left(1 - \frac{\alpha^2 Re^2(\mu_m)}{\gamma_1^2 |\chi_m|^2}\right). \quad (49)$$

Hence the amplitude of the stable steady-state response is

$$A(x) = 2a_{m1} |\phi_m(x)|. \quad (50)$$

Principal parametric resonance in the system governed by Eq. (14) can be analyzed via the method of multiple scales in a similar way. The solvability condition is still expressed by Eqs (32) and (33) with coefficients given by Eq. (34), while the other coefficients are defined by the following equations rather than Eqs (35) and (36)

$$C_{nm} = \frac{3 \int_0^1 \left(\phi'_n \bar{\phi}'_m \phi''_m + \phi'_n \phi'_m \bar{\phi}''_m + \phi'_m \bar{\phi}'_m \phi''_n \right) \bar{\phi}_n dx}{2 \int_0^1 (i\omega_n \phi_n + \gamma_0 \phi'_n) \bar{\phi}_n dx}, \quad (51)$$

$$C_{mn} = \frac{3 \int_0^1 \left(\phi'_m \bar{\phi}'_n \phi''_n + \phi'_m \phi'_n \bar{\phi}''_n + \phi'_n \bar{\phi}'_n \phi''_m \right) \bar{\phi}_m dx}{2 \int_0^1 (i\omega_m \phi_m + \gamma_0 \phi'_m) \bar{\phi}_m dx},$$

$$\kappa_k = \frac{3 \left(\int_0^1 \bar{\phi}_k \bar{\phi}''_k \phi_k'^2 dx + \frac{3}{2} \int_0^1 \bar{\phi}_k \phi''_k \phi'_k \bar{\phi}'_k dx \right)}{4 \left(i\omega_k \int_0^1 \phi_k \bar{\phi}_k dx + \gamma_0 \int_0^1 \phi'_k \bar{\phi}_k dx \right)}. \quad (52)$$

4. Numerical illustrations of the analytical results

The effects of some parameters on the amplitude and the existence conditions can be inferred from Eqs (43), (44), (46), and (47). From Eq. (43), a_{n1} increases with σ and γ_1 , and decreases with k_1 and α . From Eq. (46), a_{n2} increases with σ and α , and decreases with k_1 and γ_1 . From Eq. (46), σ_1 increases with α , and decreases γ_1 . From Eq. (46), σ_1 increases with γ_1 , and decreases α . That is, γ_1 makes the instability interval of the straight equilibrium configuration smaller, and α makes it larger. However, to understand the effects of other parameters such as n , γ_0 , and k , numerical examples are necessary.

Consider an axially moving beam with $k_f = 0.8$. For given γ_0 and k , the modal parameters can be numerically solved from Eqs (24) and (25), and coefficients defined by Eqs (34)–(36), (51) and (52) can be calculated. For example, let $\gamma_0 = 2.0$, and $k = 0$. Then $\omega_1 = 5.3692$ and $\omega_2 = 30.1200$, and β_{jn} ($j = 1, 2, 3, 4$) and the coefficients μ_j ($j = m, n$), χ_m , C_{mn} , C_{nm} , and κ_j ($j = m, n$) are calculated for Eqs (24) and (25). The numerical result is

Table 1
The roots of Eq. (24) and μ_n , χ_n calculated from Eq. (34)

n	β_{1n}	β_{2n}	β_{3n}	β_{4n}	μ_n	χ_n
1	4.57	$-1.25+3.07i$	$-1.25-3.07i$	-2.07	148.57	$-0.15+1.03i$
2	7.73	$-1.25+6.36i$	$-1.25-6.36i$	-5.23	1145.52	$-0.12+0.97i$

Table 2
The coefficients respectively calculated from Eqs (35), (36) and (51), (52)

	C_{12}	C_{21}	κ_1	κ_2
Eq. (15)	170.509i	56.6002i	-40.2494i	-142.864i
Eq. (14)	69.4427i	23.0514i	-28.8418i	-88.5134i

Fig. 1. Amplitudes and stability of steady-state responses: Eq. (15).

Fig. 2. Amplitudes and stability of steady-state responses: Eq. (14).

shown in Tables 1 and 2. Similarly, let $\gamma_0 = 2.0$, and $k = 2.0$. Then $\omega_1 = 8.1570$, $\omega_2 = 32.9441$ and the coefficients can be obtained but are not listed.

Figures 1 and 2 illustrate the relationship between the amplitude at the beam center and the detuning parameter for first two principal parametric resonances for Eqs (14) and (15) respectively. In the both figures, the solid or dot lines stand for stable or unstable amplitudes respectively. Equations (14) and (15) yield the qualitatively same results. In both the first and the second principal resonance, only the trivial zero solution exists and is stable for $\sigma < \sigma_1$. At $\sigma = \sigma_1$, the trivial solution losses its stability and a stable nontrivial solution occurs. At $\sigma = \sigma_2$, the unstable trivial solution becomes stable again, and an unstable nontrivial solution bifurcates. The response amplitude of the first principal resonance is much larger than that of the second principal resonance, and the instability interval in the first principal resonance is larger than that in the second principal resonance. Therefore, the low order principal resonance is more significant.

The effects of the mean axial speed are shown in Figs 3 and 4 with $k_f = 0.8$, $k_1 = 20$, $\alpha = 0.0001$, $k = 2.0$, and $\gamma_1 = 0.2$. Eqs (14) and (15) predict the same tendencies. The instability interval increases with the mean speed.

Fig. 3. Effects of the mean axial speed: Eq. (15).

Fig. 4. Effects of the mean axial speed: Eq. (14).

Fig. 5. Effects of the rotational spring stiffness: Eq. (15).

The effects of the rotational spring stiffness are shown in Figs 5 and 6 with $k_f = 0.8$, $k_1 = 20$, $\alpha = 0.0001$, $\gamma_0 = 2.0$, and $\gamma_1 = 0.2$. Equations (14) and (15) predict the same tendencies again. The instability interval decreases with the spring stiffness.

The numerical examples indicate that results of Eqs (14) and (15) are qualitatively same, while there exist quantitative differences. Figure 7 is the superposition of Figs 1 and 2. The amplitude derived from Eq. (14) is smaller, and the instability intervals are the same.

5. Stable steady-state response via the differential quadrature method

Stable steady-state response can be determined by solving the governing equations numerically. In the following, the differential quadrature method [24,25,27,28] is applied to calculate stable steady-state responses. In numerical investigations here, let $\varepsilon = 1$ in Eqs (10) and (11).

Fig. 6. Effects of the rotational spring stiffness: Eq. (14).

Fig. 7. Comparison of results from Eqs (14) and (15).

Introduce N unequally spaced sampling points as

$$x_i = \frac{1}{2} \left[1 - \cos \frac{(i-1)\pi}{N-1} \right] \quad (i = 1, 2, \dots, N). \quad (53)$$

The quadrature rules for the derivatives of a function at the sampling points yield

$$\begin{aligned} v_{,x}(x_i, t) &= \sum_{j=1}^N A_{ij}^{(1)} v(x_j, t), \quad v_{,xx}(x_i, t) = \sum_{j=1}^N A_{ij}^{(2)} v(x_j, t), \\ v_{,xxxx}(x_i, t) &= \sum_{j=1}^N A_{ij}^{(4)} v(x_j, t), \quad v_{,xxxxx}(x_i, t) = \sum_{j=1}^N A_{ij}^{(5)} v(x_j, t), \end{aligned} \quad (54)$$

where the weighting coefficients are the expression [27]

$$A_{ij}^{(1)} = \frac{\prod_{k=1, k \neq i}^N (x_i - x_k)}{(x_i - x_j) \prod_{k=1, k \neq j}^N (x_j - x_k)} \quad (i, j = 1, 2, \dots, N; j \neq i) \quad (55)$$

and the recurrence relationship

$$A_{ij}^{(r)} = r \left[A_{ii}^{(r-1)} A_{ij}^{(1)} - \frac{A_{ij}^{(r-1)}}{x_i - x_j} \right] \quad (r = 2, 3, 4, 5; i, j = 1, 2, \dots, N; j \neq i), \quad (56)$$

$$A_{ii}^{(r)} = - \sum_{k=1, k \neq i}^N A_{ik}^{(r)} \quad (r = 1, 2, 3, 4, 5; i = 1, 2, \dots, N). \quad (57)$$

Consider the beam simply supported at both ends. Then the boundary conditions are

$$v(0, t) = 0, v_{,xx}(0, t) = 0, v(1, t) = 0, v_{,xx}(1, t) = 0. \quad (58)$$

Substitution of Eq. (54) into Eqs (10) and (58) and modification of the weighting coefficient matrices to implement the boundary conditions [34] yield

$$\begin{aligned} & \ddot{v}_j + \left\{ -k_2 \left(\sum_{k=1}^N A_{jk}^{(1)} v_k \right) \left\{ 2 \left[\sum_{k=1}^N A_{jk}^{(2)} v_k \right] \sum_{k=1}^N A_{jk}^{(1)} + \left[\sum_{k=1}^N A_{jk}^{(1)} v_k \right] \sum_{k=1}^N A_{jk}^{(2)} \right\} \right. \\ & \left. + \sum_{k=1}^N \left[2\gamma A_{jk}^{(1)} + \alpha A_{jk}^{(4)} \right] \right\} \dot{v}_k - \sum_{k=1}^N \left[\dot{\gamma} A_{jk}^{(1)} + (\gamma^2 - 1) A_{jk}^{(2)} + k_f^2 A_{jk}^{(4)} + \alpha \gamma A_{jk}^{(5)} \right] v_k \\ & = \gamma k_2 \left[\sum_{k=1}^N A_{jk}^{(1)} v_k \right] \left\{ 2 \left[\sum_{k=1}^N A_{jk}^{(2)} v_k \right]^2 + \left[\sum_{k=1}^N A_{jk}^{(1)} v_k \right] \left[\sum_{k=1}^N A_{jk}^{(3)} v_k \right] \right\} \\ & + \frac{3}{2} k_1^2 \left(\sum_{k=1}^N A_{jk}^{(2)} v_k \right) \left(\sum_{k=1}^N A_{jk}^{(1)} v_k \right)^2, \quad (j = 2, 3, 4, \dots, N-1), \quad v_0 = v_N = 0, \end{aligned} \quad (59)$$

where

$$v_j(t) = v(x_j, t) \quad (j = 1, 2, \dots, N). \quad (60)$$

Similarly, Eq. (11) under boundary condition (58) can be discretized into

$$\begin{aligned} & \ddot{v}_j + \left(\sum_{k=1}^N (2\gamma A_{jk}^{(1)} + \alpha A_{jk}^{(4)}) \right) \dot{v}_k + \sum_{k=1}^N \left[\dot{\gamma} A_{jk}^{(1)} + (\gamma^2 - 1) A_{jk}^{(2)} + v_f^2 A_{jk}^{(4)} + \alpha \gamma A_{jk}^{(5)} \right] v_k \\ & = \left(\sum_{k=1}^N A_{jk}^{(2)} v_k \right) \left\{ \sum_{g=1}^N I_g \left[\frac{1}{2} k_1^2 \left(\sum_{k=1}^N A_{gk}^{(1)} v_k \right)^2 + \gamma k_2 \left(\sum_{k=1}^N A_{gk}^{(2)} v_k \right) \left(\sum_{k=1}^N A_{gk}^{(1)} v_k \right) \right. \right. \\ & \left. \left. + k_2 \left(\sum_{k=1}^N A_{gk}^{(1)} v_k \right) \left(\sum_{k=1}^N A_{gk}^{(1)} \dot{v}_k \right) \right] \right\}, \quad (j = 2, 3, \dots, N-1), \quad v_0 = v_N = 0, \end{aligned} \quad (61)$$

where weighting coefficients I_g ($g = 1, 2, \dots, N$) for integrals are solved from

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & \dots & x_{N-1} & x_N \\ \vdots & \vdots & & \vdots & \vdots \\ x_1^{N-2} & x_2^{N-2} & \dots & x_{N-1}^{N-2} & x_N^{N-2} \\ x_1^{N-1} & x_1^{N-1} & \dots & x_{N-1}^{N-1} & x_N^{N-1} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_{N-1} \\ I_N \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ \vdots \\ 1/(N-1) \\ 1/N \end{pmatrix}. \quad (62)$$

For a set of given parameters $k_f, \gamma_0, \gamma_1, \omega, \alpha, k_1$, and k_2 , v_j can be numerically solved from Eq. (59) or (61). For an odd N , $v_{(N+1)/2}$ is the beam center displacement. The stable steady-state response has constant local maximums of $v_{(N+1)/2}$, which is the amplitude of the beam center A . Changing ω and calculating corresponding A yield the curve of the amplitude via the detuning parameter.

Consider an axially moving beam with $k_f = 0.8$ and $\gamma_0 = 2.0$. Then the first two natural frequencies of system (18) are $\omega_1 = 5.3692$ and $\omega_2 = 30.12$. Let $\gamma_1 = 0.2$, $\alpha = 0.0001$, and $k_1 = 20$. Based on the numerical solutions of Eq. (61), Fig. 8 shows the amplitudes of the stable steady-state responses changing with the detuning parameter. The numerical results demonstrate that the lower order principal resonance has the larger stable response amplitude and the larger instability interval of the straight equilibrium.

Figure 9 shows the effects of the mean axial speed based on the numerical solutions of Eq. (61). In Fig. 9, $k_f = 0.8$, $k_1 = 20$, $k_2 = 0$, $\alpha = 0.0001$, and $\gamma_1 = 0.2$. The larger mean axial speed leads to the larger stable response amplitude and the larger instability interval of the straight equilibrium.

Fig. 8. Amplitudes of stable steady-state responses.

Fig. 9. Effects of the mean axial speed.

Fig. 10. Effects of the axial speed variation amplitude.

Figure 10 shows the effects of the axial speed variation amplitude based on the numerical solutions of Eq. (61). In Fig. 10, $k_f = 0.8$, $k_1 = 20$, $k_2 = 0$, $\alpha = 0.0001$, and $\gamma_0 = 2.0$. Both the stable response amplitude and the equilibrium instability interval increase with the axial speed variation amplitude.

Figure 11 shows the effects of the dynamic viscosity based on the numerical solutions of Eq. (61). In Fig. 11, $k_f = 0.8$, $k_1 = 20$, $k_2 = 0$, $\gamma_1 = 0.2$, and $\gamma_0 = 2.0$. The larger dynamic viscosity leads to the smaller stable response amplitude and the smaller instability interval of the straight equilibrium.

Figure 12 shows the effects of the non-linear coefficient k_1 based on the numerical solutions of Eq. (61). In Fig. 12, $k_f = 0.8$, $k_2 = 0$, $\alpha = 0.0001$, $\gamma_0 = 2.0$, and $\gamma_1 = 0.2$. The stable response amplitude decreases with the non-linear coefficient k_1 , while the equilibrium instability interval is independent of k_1 .

Fig. 11. Effects of the dynamic viscosity.

Fig. 12. Effects of the non-linear coefficient k_1 . Fig. 13. Effects of the non-linear coefficient k_2 .

Figure 13 shows the effects of the non-linear coefficient k_2 based on the numerical solutions of Eq. (61). In Fig. 13, $k_f = 0.8$, $k_1 = 20$, $\alpha = 0.0001$, $\gamma_0 = 2.0$, and $\gamma_1 = 0.2$. In practical cases, k_2 is usual smaller than 1. The stable response amplitude and the equilibrium instability interval are not sensitive to k_2 .

Equation (59) yields the qualitatively same changing tendencies as Eq. (61), so that the numerical results are not shown. The quantitative comparisons between Eqs (59) and (61) are made in Fig. 14. In Fig. 14, $k_f = 0.8$, $k_1 = 20$, $k_2 = 0$, $\alpha = 0.0001$, $\gamma_0 = 2.0$, $\gamma_1 = 0.2$. The amplitude based on the numerical solution of Eq. (59) is smaller, and the instability intervals are the same.

Fig. 14. Comparison of results from Eqs (59) and (61).

Fig. 15. Comparison of analytical and numerical results: Eq. (10).

Fig. 16. Comparison of analytical and numerical results: Eq. (11).

6. Comparisons of approximate analytical results and the numerical results

The stable steady-state response can be determined via the method of multiple scales, as documented in Sec. 3, and calculated via the differential quadrature method, as presented in Sec. 5. Although the analytical studies considered a more general boundary conditions, the boundary conditions can easily reduced to the simple supports by inserting $K = 0$ into Eq. (8). The both results are quantitatively contrasted in the case with $k_f = 0.8$, $k_1 = 20$, $k_2 = 0$, $\alpha = 0.0001$, $\gamma_0 = 2.0$, $\gamma_1 = 0.2$. Figures 15 and 16 show the comparison. The amplitudes from both methods are almost coincided, especially near the exact-resonance ($\sigma = 0$) and in the first resonance. However, there are small errors in the equilibrium instability interval, especially for Eq. (10). On the whole, it can be concluded that the outcomes of the multi-scale method are verified by the differential quadrature method.

7. Conclusions

This paper is devoted to steady-state response in principal parametric resonance of axially accelerating viscoelastic beams. The method of multiple scales is applied to a non-linear partial-differential equation and a non-linear integro-partial-differential equation to determine steady-state responses. The differential quadrature schemes are developed to verify the approximate analytical results. The investigation leads to the following conclusions.

- (1) Analytical study demonstrates that the mode uninvolved in the resonance has no effect on the steady-state response.
- (2) Both analytical study and numerical calculations demonstrate that, when the axial speed variation frequency is close to 2 times of any linear natural frequency enough, or the detuning parameter belongs to an interval, the straight equilibrium configuration become unstable. The instability interval increases with the mean and the variation amplitude of the axial speed, decreases with the principal parametric resonance order and the dynamic viscosity, and is independent of non-linear coefficients. Analytical study also demonstrates that the instability interval decreases with the boundary constraint stiffness.
- (3) Both analytical study and numerical calculations demonstrate that, when the straight equilibrium configuration begins to lose its stability, a stable steady-state response emerges. The stable steady-state response amplitude increases with the detuning parameter and the mean and the variation amplitude of the axial speed, decreases with the principal parametric resonance order, the dynamic viscosity, and the non-linear coefficients.
- (4) Analytical study demonstrates that, when the unstable straight equilibrium configuration becomes stable again, an unstable steady-state response emerges. The unstable steady-state response amplitude increases with the detuning parameter, the dynamic viscosity, and the boundary constraint stiffness, decreases with the principal parametric resonance order, the variation amplitude of the axial speed, and the non-linear coefficient.
- (5) Both analytical study and numerical calculations demonstrates that two non-linear models yield the qualitatively same results. Quantitatively, both models predict the same instability interval, while the non-linear integro-partial-differential equation leads to the larger amplitude of steady-state responses.

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