Free vibration of rectangular plates with attached discrete sprung masses

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Abstract. A direct approach is used to derive the exact solution for the free vibration of thin rectangular plates with discrete sprung masses attached. The plate is simply supported along two opposite edges and elastically supported along the two other edges. The elastic support can represent a range of boundary conditions from free to clamped supports. Considering only the compatibility of the internal forces between the plate and the sprung masses, the equations of the coupled vibration of the plate-spring-mass system are derived. The exact expressions for mode and frequency equations of the coupled vibration of the plate and sprung masses are determined. The solutions converge steadily and monotonically to exact values. The correctness and accuracy of the solutions are demonstrated through comparison with published results. A parametric study is undertaken focusing on the plate with one or two sprung masses. The results can be used as a benchmark for further investigation.

The solution provided in the paper is general and includes several special cases, such as the plate with classical boundary conditions, the plate attached with discrete rigid masses, the plate supported by discrete springs and the plate restricted by rigid vertical point-supports.

Keywords: Coupled vibration, eigenfrequency, plate, sprung mass, exact solution

1. Introduction

There are several situations where the vibration of plates with attached discrete masses needs to be considered, one example being a structure supporting people [1]. In this situation the person (or crowd) can be represented by mass-spring-damper model(s). Another example occurs when the vibrations from a machine are reduced using a rubber membrane between it and the floor, thereby creating a sprung-mass system. This paper presents a theoretical study which provides an improved understanding of the vibration of couple spring-mass and plate systems.

There are three possible ways to study the above problem. First by converting the floor into a single degree-of-freedom (SDOF) system and the discrete sprung masses into a single sprung mass; thus forms a two degree-of-freedom system. The advantage of the method is that it greatly simplifies the solution of the problem. On the other hand, the model neglects higher modes of vibration of the system. Second using a FE package makes direct modelling. This method is simple and quick for particular solutions, but it may be tedious for parametric studies of the plate-spring-mass problem considering several variable parameters. Thirdly using analytical studies to derive expressions for the solution of the problem in order to conduct parametric studies thereby provides an improved understanding of the problem. The paper investigates the problem using the third route.

The general study to vibration characteristics of rectangular plates with arbitrarily located point-supports came from Fan and Cheung [2] using the spline finite strip. Kim and Dickinson [3] used the Lagrangian multiplier method

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Fig. 1. A rectangular plate having two opposite edges simply supported and other two opposite edges arbitrarily supported, with discrete sprung masses attached.

combining with the orthogonally generated polynomials to study the rectangular plates with point-supports. Zhou [4] and Cheung and Zhou [5] used the static beam functions as admissible functions to study tapered rectangular plates with point-supports and composite rectangular plates with point-supports respectively. Moreover, the 3-D vibration of composite plates with internal point-supports has been studied by Zhou et al. [6] using the finite layer method. Bergman et al. [7] derived the dynamic Green function for rectangular plates having two opposite edges simply supported and attached by substructures. Li [8] gave the exact solution of rectangular plates with two opposite edges simply supported and with line-concentrated masses or elastic line-supports. Avalos et al. [9,10] used the optimized Rayleigh-Ritz approach and sinusoidal eigenfunction expansion method respectively to study the free vibration of a circular plate carrying an elastically attached mass at its centre and a simply supported rectangular plate having an elastically attached mass at an arbitrary position. Wong [11] applied the Rayleigh-Ritz method to study the effect of distributed mass on plate vibration behaviour. Wu et al. [12] used the mode expansion method to investigate the effect of various concentrated elements. Jacquot [13] used the two-dimensional Fourier series to study the random vibration of rectangular plate attached with absorbers. Chiba and Sugimoto [14] studied the vibration characteristics of a cantilevered rectangular plate attached by a sprung mass in detail. Liew et al. [15] studied the free vibration of arbitrarily shaped Mindlin plates with internal point-supports. McMillan and Keane [16, 17] investigated the possibility of using attached masses to control the vibration of rectangular plates. Adhikari et al. [18,19] experimentally studied the effect of uncertainty quantification on structural dynamics. Although some investigators [10,12,13] used the two-dimensional Fourier sinusoidal series to obtain the exact solutions of the dynamic characteristics of plates attached with elastic/rigid masses, such a method is only applicable to plates with four edges simply supported. It should be mentioned that the existing studies mainly focused on plates with classical boundary conditions and with one sprung mass. No study on plates with elastic boundary conditions and with multi-sprung- masses has been found.

In the present study, the exact analytical solution for a thin rectangular plate attached to sprung masses is derived from the governing differential equations. The plate is simply supported along two opposite edges and elastically supported along the other edges. The eigenvalue equations are obtained by considering the compatibility of internal forces between the plate and the sprung masses at the attached positions. The order of the eigenvalue equations is equal to the number of the attached sprung masses. The eigenfrequencies are obtained using the root searching method. Several special cases can be derived from the given solutions. A parametric study is conducted to examine the effects of the relative mass and stiffness, and the locations of the sprung mass on the dynamic characteristics of the coupled system.

2. Governing differential equations

Consider a uniform thin rectangular plate with a length of a and a width of b attached to I sprung masses as shown in Fig. 1. It is assumed that the plate is simply supported along two opposite edges $x = 0$ and $x = a$. According to the theory of thin plate vibration when damping is neglected, the governing differential equation of the plate under the forces from the sprung masses is

$$D \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + \rho h \frac{\partial^2 w}{\partial t^2} = f(x, y, t) \quad (1)$$

where w is the transverse deflection of the plate, ρ is the mass density per unit volume of the plate, h and D are the thickness and the flexural rigidity of the plate, $f(x, y, t)$ is the action forces of the sprung masses on the plate. The simply supported boundary conditions of the plate are

$$w = 0, \quad \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} = 0 \text{ at } x = 0 \text{ and } a \quad (2)$$

In which, ν is the Poisson's ratio. The loading term $f(x, y, t)$ can be expressed as:

$$f(x, y, t) = \sum_{i=1}^I p_i(t) \delta(x - x_i) \delta(y - y_i) \quad (3)$$

where $p_i(t)$ is the force of the i th sprung mass on the plate, x_i and y_i are the coordinates of the i th sprung mass, $\delta(x - x_i)$ and $\delta(y - y_i)$ are Dirac delta functions. The force $p_i(t)$ has the following relation with the plate deflection $w(x_i, y_i, t)$ at point (x_i, y_i) and the displacement $z_i(t)$ of the i th mass as follows

$$p_i(t) = -k_i[w(x_i, y_i, t) - z_i(t)] \quad (4)$$

where z_i should satisfy the following equation of motion

$$m_i \frac{d^2 z_i(t)}{dt^2} = -p_i(t), \quad i = 1, 2, \dots, I \quad (5)$$

Equations (1)–(5) form the governing equations of free vibration of the plate-spring-mass system. Considering free vibration of the system, the solutions of Eqs (1)–(5) have the following form:

$$w = W(x, y)e^{j\omega t}; \quad f(x, y, t) = F(x, y)e^{j\omega t}; \quad p_i(t) = P_i e^{j\omega t}; \quad z_i(t) = Z_i e^{j\omega t} \quad (6)$$

where ω is the eigenfrequency of the plate-spring-mass system and $j = \sqrt{-1}$. Using the following dimensionless coordinates

$$\xi = x/a; \quad \eta = y/b \quad (7)$$

Equation (1) can be written as

$$\frac{1}{\beta^4} \frac{\partial^4 W}{\partial \xi^4} + \frac{2}{\beta^2} \frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 W}{\partial \eta^4} - \frac{\rho h}{D} b^4 \omega^2 W = \frac{b^4}{D} F(\xi, \eta) \quad (8)$$

where $\beta = a/b$ is the aspect ratio of the plate. Similarly, Eqs (3)–(5) can be expressed as:

$$F(\xi, \eta) = \frac{1}{ab} \sum_{i=1}^I P_i \delta(\xi - \xi_i) \delta(\eta - \eta_i) \quad (9)$$

$$P_i = -k_i[W(\xi_i, \eta_i) - Z_i]; \quad m_i \omega^2 Z_i = P_i \quad (10)$$

where $\xi_i = x_i/a$ and $\eta_i = y_i/b$. Equations (8)–(10) are the non-dimensional differential governing equations of free vibration of a thin plate attached to several sprung masses.

3. Solutions

For the plate with two opposite edges simply supported at $x = 0$ and a , its solution can be given in the form of

$$W(\xi, \eta) = \sum_{n=1}^{\infty} \sin(n\pi\xi) W_n(\eta) \quad (11)$$

The above equation exactly satisfies the simply supported boundary conditions at edges $\xi = 0$ and $\xi = 1$. Expand $F(\xi, \eta)$ into a Fourier sinusoidal series about ξ as follows

$$F(\xi, \eta) = 2 \sum_{n=1}^{\infty} \sin(n\pi\xi) F_n(\eta); \quad F_n(\eta) = \int_0^1 F(\xi, \eta) \sin(n\pi\xi) d\xi \quad (12)$$

It should be mentioned that the convergence error in the above equation has not been eliminated due to the Gibb's effect. Substituting Eqs (11) and (12) into Eq. (8) gives

$$\frac{d^4 W_n(\eta)}{d\eta^4} - 2 \frac{(n\pi)^2}{\beta^2} \frac{d^2 W_n(\eta)}{d\eta^2} + \left[\frac{(n\pi)^4}{\beta^4} - \alpha^4 \right] W_n(\eta) = 2 \frac{b^4}{D} F_n(\eta) \quad (13)$$

where $\alpha^4 = \rho h b^4 \omega^2 / D$. According to the theory of ordinary differential equations, the solution of Eq. (13) can be obtained. When $\alpha < n\pi/\beta$, one has

$$W_n(\eta) = A_n \sinh(\lambda_{n1}\eta) + B_n \cosh(\lambda_{n1}\eta) + C_n \sinh(\lambda_{n2}\eta) + D_n \cosh(\lambda_{n2}\eta) + \tilde{W}_n(\eta) \quad (14a)$$

where

$$\tilde{W}_n(\eta) = \frac{b^4}{D\alpha^2 \lambda_{n1} \lambda_{n2}} \int_0^\eta F_n(s) [\lambda_{n1} \sinh(\lambda_{n2}(\eta - s)) - \lambda_{n2} \sinh(\lambda_{n1}(\eta - s))] ds \quad (14b)$$

$$\lambda_{n1}^4 = \left[\frac{(n\pi)^2}{\beta^2} - \alpha^2 \right]^2; \quad \lambda_{n2}^4 = \left[\frac{(n\pi)^2}{\beta^2} + \alpha^2 \right]^2 \quad (15)$$

When $\alpha > n\pi/\beta$, one has

$$W_n(\eta) = A_n \sin(\lambda_{n1}\eta) + B_n \cos(\lambda_{n1}\eta) + C_n \sinh(\lambda_{n2}\eta) + D_n \cosh(\lambda_{n2}\eta) + \tilde{W}_n(\eta) \quad (16a)$$

where

$$\tilde{W}_n(\eta) = \frac{b^4}{D\alpha^2 \lambda_{n1} \lambda_{n2}} \int_0^\eta F_n(s) [\lambda_{n1} \sinh(\lambda_{n2}(\eta - s)) - \lambda_{n2} \sin(\lambda_{n1}(\eta - s))] ds \quad (16b)$$

In Eqs (14a) and (16a), the first four terms with unknown constants, A_n , B_n , C_n and D_n , are the homogeneous solutions while $\tilde{W}_n(\eta)$ is the special solution.

Assuming a positive integer L , when $n < L$, $\alpha > n\pi/\beta$ and when $n \geq L$, $\alpha < n\pi/\beta$, the mode shape $W(\xi, \eta)$ in Eq. (11) can be expressed in the form of

$$\begin{aligned} W(\xi, \eta) = & \sum_{n=1}^{L-1} \sin(n\pi\xi) \{ A_n \sin(\lambda_{n1}\eta) + B_n \cos(\lambda_{n1}\eta) + C_n \sinh(\lambda_{n2}\eta) + D_n \cosh(\lambda_{n2}\eta) + \\ & \frac{b^4}{D\alpha^2 \lambda_{n1} \lambda_{n2}} \int_0^\eta F_n(s) [\lambda_{n1} \sinh(\lambda_{n2}(\eta - s)) - \lambda_{n2} \sin(\lambda_{n1}(\eta - s))] ds \} + \\ & \sum_{n=L}^{\infty} \sin(n\pi\xi) \{ A_n \sinh(\lambda_{n1}\eta) + B_n \cosh(\lambda_{n1}\eta) + C_n \sinh(\lambda_{n2}\eta) + D_n \cosh(\lambda_{n2}\eta) + \\ & \frac{b^4}{D\alpha^2 \lambda_{n1} \lambda_{n2}} \int_0^\eta F_n(s) [\lambda_{n1} \sinh(\lambda_{n2}(\eta - s)) - \lambda_{n2} \sinh(\lambda_{n1}(\eta - s))] ds \} \end{aligned} \quad (17)$$

Substituting Eq. (9) into Eq. (12) leads to

$$F_n(\eta) = \frac{1}{ab} \sum_{i=1}^I P_i \sin(n\pi\xi_i) \delta(\eta - \eta_i) \quad (18)$$

Substituting Eq. (18) into Eq. (17) gives

$$\begin{aligned}
W(\xi, \eta) = & \sum_{n=1}^{L-1} \sin(n\pi\xi) \{A_n \sin(\lambda_{n1}\eta) + B_n \cos(\lambda_{n1}\eta) + C_n \sinh(\lambda_{n2}\eta) + D_n \cosh(\lambda_{n2}\eta) + \\
& \frac{b^2}{\beta D \alpha^2 \lambda_{n1} \lambda_{n2}} \sum_{i=1}^I P_i \sin(n\pi\xi_i) [\lambda_{n1} \sinh(\lambda_{n2}(\eta - \eta_i)) - \lambda_{n2} \sin(\lambda_{n1}(\eta - \eta_i))] H(\eta - \eta_i)\} + \\
& \sum_{n=L}^{\infty} \sin(n\pi\xi) \{A_n \sinh(\lambda_{n1}\eta) + B_n \cosh(\lambda_{n1}\eta) + C_n \sinh(\lambda_{n2}\eta) + D_n \cosh(\lambda_{n2}\eta) + \\
& \frac{b^2}{\beta D \alpha^2 \lambda_{n1} \lambda_{n2}} \sum_{i=1}^I P_i \sin(n\pi\xi_i) [\lambda_{n1} \sinh(\lambda_{n2}(\eta - \eta_i)) - \lambda_{n2} \sinh(\lambda_{n1}(\eta - \eta_i))] H(\eta - \eta_i)\}
\end{aligned} \quad (19)$$

where

$$H(\eta - \eta_i) = \begin{cases} 0 & \eta < \eta_i \\ 1 & \eta \geq \eta_i \end{cases} \text{ is the Heaviside function.}$$

4. Boundary conditions

Using the boundary conditions of the plate at the edges $\eta = 0$ and $\eta = 1$, the unknown constants A_n, B_n, C_n and D_n can be determined. Considering a general case: the plate is elastically restrained at the edges $\eta = 0$ and $\eta = 1$, the boundary conditions can be given by [20]:

$$\begin{aligned}
\frac{\partial^2 W}{\partial \eta^2} + \nu \frac{\partial^2 W}{\beta^2 \partial \xi^2} &= \frac{b k_{r0}}{D} \frac{\partial W}{\partial \eta}, \text{ at } \eta = 0, \\
\frac{\partial^2 W}{\partial \eta^2} + \nu \frac{\partial^2 W}{\beta^2 \partial \xi^2} &= -\frac{b k_{r1}}{D} \frac{\partial W}{\partial \eta}, \text{ at } \eta = 1; \\
\frac{\partial^3 W}{\partial \eta^3} + (2 - \nu) \frac{\partial^3 W}{\beta^2 \partial \eta \partial \xi^2} &= -\frac{b^3 k_{t0}}{D} W, \text{ at } \eta = 0; \\
\frac{\partial^3 W}{\partial \eta^3} + (2 - \nu) \frac{\partial^3 W}{\beta^2 \partial \eta \partial \xi^2} &= \frac{b^3 k_{t1}}{D} W, \text{ at } \eta = 1
\end{aligned} \quad (20)$$

where k_{r0} and k_{r1} are the stiffnesses of the rotational restraints at boundaries $y = 0$ and $y = b$ respectively. k_{t0} and k_{t1} are the stiffnesses of the translational restraints at boundaries $y = 0$ and $y = b$ respectively. By taking these stiffness coefficients to be infinite or zero, various classical boundary conditions can be obtained. For examples, if the plate is simply supported at $y = 0$ and $y = b$, then $k_{r0} = k_{r1} = 0$ and $k_{t0} = k_{t1} = \infty$. If the plate is clamped at $y = 0$ and $y = b$, then $k_{r0} = k_{r1} = \infty$ and $k_{t0} = k_{t1} = \infty$. And if the plate is free at $y = 0$ and $y = b$, then $k_{r0} = k_{r1} = 0$ and $k_{t0} = k_{t1} = 0$.

Substituting Eq. (19) into Eq. (20) gives the coefficients equations. When $n < L$

$$\begin{aligned}
-B_n \lambda_{n1}^2 + D_n \lambda_{n2}^2 - \frac{\nu(n\pi)^2}{\beta^2} (B_n + D_n) &= K_{r0} \{A_n \lambda_{n1} + C_n \lambda_{n2}\}, \\
-A_n \lambda_{n1}^2 \sin \lambda_{n1} - B_n \lambda_{n1}^2 \cos \lambda_{n1} + C_n \lambda_{n2}^2 \sinh \lambda_{n2} + D_n \lambda_{n2}^2 \cosh \lambda_{n2} + \\
\frac{b^2}{\beta D \alpha^2} \sum_{i=1}^I P_i \sin(n\pi\xi_i) [\lambda_{n2} \sinh(\lambda_{n2}(1 - \eta_i)) + \lambda_{n1} \sin(\lambda_{n1}(1 - \eta_i))] - \\
\frac{\nu(n\pi)^2}{\beta^2} \{A_n \sin \lambda_{n1} + B_n \cos \lambda_{n1} + C_n \sinh \lambda_{n2} + D_n \cosh \lambda_{n2} +
\end{aligned}$$

$$\begin{aligned}
& \frac{b^2}{\beta D \alpha^2 \lambda_{n1} \lambda_{n2}} \sum_{i=1}^I P_i \sin(n\pi \xi_i) [\lambda_{n1} \sinh(\lambda_{n2}(1 - \eta_i)) - \lambda_{n2} \sin(\lambda_{n1}(1 - \eta_i))] \} = \\
& -K_{r1} \{ A_n \lambda_{n1} \cos \lambda_{n1} - B_n \lambda_{n1} \sin \lambda_{n1} + C_n \lambda_{n2} \cosh \lambda_{n2} + D_n \lambda_{n2} \sinh \lambda_{n2} + \\
& \frac{b^2}{\beta D \alpha^2} \sum_{i=1}^I P_i \sin(n\pi \xi_i) [\cosh(\lambda_{n2}(1 - \eta_i)) - \cos(\lambda_{n1}(1 - \eta_i))] \}, \\
& -A_n \lambda_{n1}^3 + C_n \lambda_{n2}^3 - \frac{(2 - \nu)(n\pi)^2}{\beta^2} (A_n \lambda_{n1} + C_n \lambda_{n2}) = -K_{t0} \{ B_n + D_n \}, \\
& -A_n \lambda_{n1}^3 \cos \lambda_{n1} + B_n \lambda_{n1}^3 \sin \lambda_{n1} + C_n \lambda_{n2}^3 \cosh \lambda_{n2} + D_n \lambda_{n2}^3 \sinh \lambda_{n2} + \\
& \frac{b^2}{\beta D \alpha^2} \sum_{i=1}^I P_i \sin(n\pi \xi_i) [\lambda_{n2}^2 \cosh(\lambda_{n2}(1 - \eta_i)) + \lambda_{n1}^2 \cos(\lambda_{n1}(1 - \eta_i))] - \\
& \frac{(2 - \nu)(n\pi)^2}{\beta^2} \{ A_n \lambda_{n1} \cos \lambda_{n1} - B_n \lambda_{n1} \sin \lambda_{n1} + C_n \lambda_{n2} \cosh \lambda_{n2} + D_n \lambda_{n2} \sinh \lambda_{n2} + \\
& \frac{b^2}{\beta D \alpha^2} \sum_{i=1}^I P_i \sin(n\pi \xi_i) [\cosh(\lambda_{n2}(1 - \eta_i)) - \cos(\lambda_{n1}(1 - \eta_i))] \} = \\
& K_{t1} \{ A_n \sin \lambda_{n1} + B_n \cos \lambda_{n1} + C_n \sinh \lambda_{n2} + D_n \cosh \lambda_{n2} + \\
& \frac{b^2}{\beta D \alpha^2 \lambda_{n1} \lambda_{n2}} \sum_{i=1}^I P_i \sin(n\pi \xi_i) [\lambda_{n1} \sinh(\lambda_{n2}(1 - \eta_i)) - \lambda_{n2} \sin(\lambda_{n1}(1 - \eta_i))] \} \tag{21a}
\end{aligned}$$

when $n \geq L$

$$\begin{aligned}
& B_n \lambda_{n1}^2 + D_n \lambda_{n2}^2 - \frac{\nu(n\pi)^2}{\beta^2} (B_n + D_n) = K_{r0} \{ A_n \lambda_{n1} + C_n \lambda_{n2} \}, \\
& A_n \lambda_{n1}^2 \sinh \lambda_{n1} + B_n \lambda_{n1}^2 \cosh \lambda_{n1} + C_n \lambda_{n2}^2 \sinh \lambda_{n2} + D_n \lambda_{n2}^2 \cosh \lambda_{n2} + \\
& \frac{b^2}{\beta D \alpha^2} \sum_{i=1}^I P_i \sin(n\pi \xi_i) [\lambda_{n2} \sinh(\lambda_{n2}(1 - \eta_i)) - \lambda_{n1} \sinh(\lambda_{n1}(1 - \eta_i))] - \\
& \frac{\nu(n\pi)^2}{\beta^2} \{ A_n \sinh \lambda_{n1} + B_n \cosh \lambda_{n1} + C_n \sinh \lambda_{n2} + D_n \cosh \lambda_{n2} + \\
& \frac{b^2}{\beta D \alpha^2 \lambda_{n1} \lambda_{n2}} \sum_{i=1}^I P_i \sin(n\pi \xi_i) [\lambda_{n1} \sinh(\lambda_{n2}(1 - \eta_i)) - \lambda_{n2} \sinh(\lambda_{n1}(1 - \eta_i))] \} = \\
& -K_{r1} \{ A_n \lambda_{n1} \cosh \lambda_{n1} + B_n \lambda_{n1} \sinh \lambda_{n1} + C_n \lambda_{n2} \cosh \lambda_{n2} + D_n \lambda_{n2} \sinh \lambda_{n2} + \\
& \frac{b^2}{\beta D \alpha^2} \sum_{i=1}^I P_i \sin(n\pi \xi_i) [\cosh(\lambda_{n2}(1 - \eta_i)) - \cosh(\lambda_{n1}(1 - \eta_i))] \}, \\
& A_n \lambda_{n1}^3 + C_n \lambda_{n2}^3 - \frac{(2 - \nu)(n\pi)^2}{\beta^2} (A_n \lambda_{n1} + C_n \lambda_{n2}) = -K_{t0} \{ B_n + D_n \},
\end{aligned}$$

$$\begin{aligned}
& A_n \lambda_{n1}^3 \cosh \lambda_{n1} + B_n \lambda_{n1}^3 \sinh \lambda_{n1} + C_n \lambda_{n2}^3 \cosh \lambda_{n2} + D_n \lambda_{n2}^3 \sinh \lambda_{n2} + \\
& \frac{b^2}{\beta D \alpha^2} \sum_{i=1}^I P_i \sin(n\pi \xi_i) [\lambda_{n2}^2 \cosh(\lambda_{n2}(1 - \eta_i)) - \lambda_{n1}^2 \cosh(\lambda_{n1}(1 - \eta_i))] - \\
& \frac{(2 - \nu)(n\pi)^2}{\beta^2} \{A_n \lambda_{n1} \cosh \lambda_{n1} + B_n \lambda_{n1} \sinh \lambda_{n1} + C_n \lambda_{n2} \cosh \lambda_{n2} + D_n \lambda_{n2} \sinh \lambda_{n2} + \\
& \frac{b^2}{\beta D \alpha^2} \sum_{i=1}^I P_i \sin(n\pi \xi_i) [\cosh(\lambda_{n2}(1 - \eta_i)) - \cosh(\lambda_{n1}(1 - \eta_i))]\} = \\
& K_{t1} \{A_n \sinh \lambda_{n1} + B_n \cosh \lambda_{n1} + C_n \sinh \lambda_{n2} + D_n \cosh \lambda_{n2} + \\
& \frac{b^2}{\beta D \alpha^2 \lambda_{n1} \lambda_{n2}} \sum_{i=1}^I P_i \sin(n\pi \xi_i) [\lambda_{n1} \sinh(\lambda_{n2}(1 - \eta_i)) - \lambda_{n2} \sinh(\lambda_{n1}(1 - \eta_i))]\}
\end{aligned} \tag{21b}$$

in which,

$$K_{r0} = \frac{bk_{r0}}{D}, \quad K_{r1} = \frac{bk_{r1}}{D}, \quad K_{t0} = \frac{b^3 k_{t0}}{D}, \quad K_{t1} = \frac{b^3 k_{t1}}{D} \tag{22}$$

From Eqs (21a) and (21b), the coefficients A_n , B_n , C_n and D_n can be solved and expressed as follows

$$\begin{aligned}
A_n &= \frac{b^2}{\beta D \alpha^2} \sum_{i=1}^I P_i \sin(n\pi \xi_i) q_{ni}^1, \quad B_n = \frac{b^2}{\beta D \alpha^2} \sum_{i=1}^I P_i \sin(n\pi \xi_i) q_{ni}^2, \\
C_n &= \frac{b^2}{\beta D \alpha^2} \sum_{i=1}^I P_i \sin(n\pi \xi_i) q_{ni}^3, \quad D_n = \frac{b^2}{\beta D \alpha^2} \sum_{i=1}^I P_i \sin(n\pi \xi_i) q_{ni}^4
\end{aligned} \tag{23}$$

q_{ni}^s ($s = 1, 2, 3, 4$) are given in the Appendix.

5. Eigenfrequency equation

Substituting Eq. (23) into Eq. (19), the following displacement function is obtained

$$\begin{aligned}
W(\xi, \eta) &= \frac{b^2}{\beta D \alpha^2} \sum_{i=1}^I P_i \left\{ \sum_{n=1}^{L-1} \sin(n\pi \xi_i) \sin(n\pi \xi) \{q_{ni}^1 \sin(\lambda_{n1} \eta) + q_{ni}^2 \cos(\lambda_{n1} \eta) + \right. \\
& q_{ni}^3 \sinh(\lambda_{n2} \eta) + q_{ni}^4 \cosh(\lambda_{n2} \eta) + \left[\frac{1}{\lambda_{n2}} \sinh(\lambda_{n2}(\eta - \eta_i)) - \frac{1}{\lambda_{n1}} \sin(\lambda_{n1}(\eta - \eta_i)) \right] H(\eta - \eta_i) \} + \\
& \sum_{n=L}^{\infty} \sin(n\pi \xi_i) \sin(n\pi \xi) \{q_{ni}^1 \sinh(\lambda_{n1} \eta) + q_{ni}^2 \cosh(\lambda_{n1} \eta) + q_{ni}^3 \sinh(\lambda_{n2} \eta) + \\
& \left. q_{ni}^4 \cosh(\lambda_{n2} \eta) + \left[\frac{1}{\lambda_{n2}} \sinh(\lambda_{n2}(\eta - \eta_i)) - \frac{1}{\lambda_{n1}} \sinh(\lambda_{n1}(\eta - \eta_i)) \right] H(\eta - \eta_i) \} \right\}
\end{aligned} \tag{24}$$

From Eq. (10), one has

$$(1 - \frac{k_i}{m_i \omega^2}) P_i + k_i W(\xi_i, \eta_i) = 0, \quad i = 1, 2, \dots, I \tag{25}$$

Substituting Eq. (24) into the above equation, a group of homogeneous equations about P_i ($i = 1, 2, \dots, I$) is obtained as follows

$$\begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1I} \\ B_{21} & B_{22} & \cdots & B_{2I} \\ \vdots & \vdots & \ddots & \vdots \\ B_{I1} & B_{I2} & \cdots & B_{II} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_I \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \quad (26)$$

in which,

$$\begin{aligned} B_{ij} = & \sum_{n=1}^{L-1} \sin(n\pi\xi_i) \sin(n\pi\xi_j) \{q_{ni}^1 \sin(\lambda_{n1}\eta_j) + q_{ni}^2 \cos(\lambda_{n1}\eta_j) + q_{ni}^3 \sinh(\lambda_{n2}\eta_j) + \\ & q_{ni}^4 \cosh(\lambda_{n2}\eta_j) + \left[\frac{1}{\lambda_{n2}} \sinh(\lambda_{n2}(\eta_j - \eta_i)) - \frac{1}{\lambda_{n1}} \sin(\lambda_{n1}(\eta_j - \eta_i)) \right] H(\eta_j - \eta_i)\} + \\ & \sum_{n=L}^{\infty} \sin(n\pi\xi_i) \sin(n\pi\xi_j) \{q_{ni}^1 \sinh(\lambda_{n1}\eta_j) + q_{ni}^2 \cosh(\lambda_{n1}\eta_j) + q_{ni}^3 \sinh(\lambda_{n2}\eta_j) + \\ & q_{ni}^4 \cosh(\lambda_{n2}\eta_j) + \left[\frac{1}{\lambda_{n2}} \sinh(\lambda_{n2}(\eta_j - \eta_i)) - \frac{1}{\lambda_{n1}} \sinh(\lambda_{n1}(\eta_j - \eta_i)) \right] H(\eta_j - \eta_i)\}, \quad i \neq j, \\ B_{ii} = & \sum_{n=1}^{L-1} \sin^2(n\pi\xi_i) \{q_{ni}^1 \sin(\lambda_{n1}\eta_i) + q_{ni}^2 \cos(\lambda_{n1}\eta_i) + q_{ni}^3 \sinh(\lambda_{n2}\eta_i) + q_{ni}^4 \cosh(\lambda_{n2}\eta_i) + \\ & \sum_{n=L}^{\infty} \sin^2(n\pi\xi_i) \{q_{ni}^1 \sinh(\lambda_{n1}\eta_i) + q_{ni}^2 \cosh(\lambda_{n1}\eta_i) + q_{ni}^3 \sinh(\lambda_{n2}\eta_i) + \\ & q_{ni}^4 \cosh(\lambda_{n2}\eta_i) + \frac{\alpha^2}{\tau_j} - \frac{1}{\alpha^2 \gamma_j} \end{aligned} \quad (27)$$

in which,

$$\tau_j = \frac{k_j b^2}{\beta D}; \quad \gamma_j = \frac{m_j}{\rho h a b} \quad (28)$$

where τ_j and γ_j are referred to as the stiffness ratio and mass ratio. In order to obtain non-zero solutions about P_i ($i = 1, 2, \dots, I$), the coefficient determinant of Eq. (25) should be zero, i.e.

$$\begin{vmatrix} B_{11} & B_{12} & \cdots & B_{1I} \\ B_{21} & B_{22} & \cdots & B_{2I} \\ \vdots & \vdots & \ddots & \vdots \\ B_{I1} & B_{I2} & \cdots & B_{II} \end{vmatrix} = 0 \quad (29)$$

It can be seen from Eq. (29) that the order of the eigenvalue equations is equal to the number of attached sprung masses. The equation is non-linear in respect to eigenfrequency ω which can be determined using the root searching method. Substituting the known values of ω into Eq. (26) yields the solution of coefficient vector $\{P\}$. Then, substituting these results into Eq. (24) gives the mode shapes of the plate.

From Eq. (24) that if $W(\xi_j, \eta_j) = 0$ for a mode, then $P_j = 0$, thus the j th sprung mass has no effect on that mode. Similarly, if $W(\xi_i, \eta_i) = 0$ ($i = 1, 2, \dots, I$) for a mode, then $P_i = 0$ ($i = 1, 2, \dots, I$), which indicates that the mode is independent of all the sprung masses. Moreover, it is well known that if a bare plate has duplicate eigenfrequencies, an arbitrary linear combination of the two modes is still the mode of the same eigenfrequency [21]. Therefore, one can always find such a coefficient that the combination mode is equal to zero at an arbitrarily given point, i.e. the node of the combination mode can be arranged at an arbitrary location of the plate. When such a plate is attached to one

Table 1
The convergence study of the first five coupled eigenfrequencies for a SSSS square plate with one sprung mass in respect to the number of terms (n) used

ξ_1, η_1	τ_1, γ_1	n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	
0.5, 0.5	100.0, 0.1	10	15.81	36.93	102.99	178.81	258.18	
		20	15.81	36.89	102.98	178.81	258.18	
		30	15.81	36.88	102.97	178.81	258.17	
		40	15.81	36.88	102.97	178.80	258.17	
	500, 0.2	10	14.23	51.68	117.57	183.44	264.26	
		20	14.23	51.51	117.34	183.35	264.15	
		30	14.23	51.48	117.29	183.34	264.13	
		40	14.23	51.47	117.28	183.33	264.12	
	2500, 0.5	10	10.98	55.87	139.09	197.20	282.77	
		20	10.98	55.58	138.30	196.46	281.77	
		30	10.98	55.53	138.16	196.33	281.66	
		40	10.98	55.51	138.11	196.29	281.54	
	0.6, 0.7	100, 0.1	10	16.84	31.58	55.02	79.93	99.42
			20	16.84	31.55	55.00	79.92	99.42
			30	16.84	31.55	55.00	79.92	99.42
			40	16.84	31.55	55.00	79.92	99.42
500, 0.2		10	15.51	35.64	67.27	84.69	103.05	
		20	15.51	35.59	67.15	84.60	102.98	
		30	15.51	35.58	67.13	84.58	102.96	
		40	15.51	35.58	67.12	84.58	102.96	
2500, 0.5		10	12.33	34.22	71.31	89.97	111.22	
		20	12.32	34.16	71.19	89.82	110.88	
		30	12.32	34.15	71.17	89.79	110.82	
		40	12.32	34.15	71.16	89.78	110.80	

sprung mass, the separation of the duplicate eigenfrequencies will inevitably occur. In other words, one of the two eigenfrequencies will hold the line ($P_1 = 0$), while the other varies with the parameters of the system ($P_1 \neq 0$), such as the attached position of the sprung mass, the mass ratio and the stiffness ratio. Taking a fully simply supported square plate as an example, the (1, 3) mode $\sin(\pi x/a) \sin(3\pi y/b)$ and the (3,1) mode $\sin(3\pi x/a) \sin(\pi y/b)$ are both the double symmetric modes and have the same eigenfrequency. Whether a sprung mass is attached on any position of the plate, one always obtains an invariable eigenfrequency $\alpha^2 = 10\pi^2$.

The invariable eigenfrequencies cannot be predicted by the present root searching method because the coefficient vector $\{P\}$ equals zero. However, each invariable mode can be judged by observing the modal shapes of the bare plate and the attached positions of the sprung masses. For an invariable mode, all the sprung masses are on the nodes of the modal shape of the bare plate.

The solution provided in the paper is general and includes several special cases when taking the stiffness ratio τ_j and/or the mass ratios γ_j to their extreme values, such as

1. When $\tau_j = \infty$, a rigid mass is attached at (x_j, y_j) on the plate;
2. When $\gamma_j = \infty$, an elastic point-support is provided at (x_j, y_j) on the plate and
3. When $\tau_j = \infty$ and $\gamma_j = \infty$, a rigid vertical support is given at (x_j, y_j) on the plate.

6. Numerical studies

To demonstrate the accuracy and applicability of the present approach, numerical calculations have been conducted. Only the results concerned with the coupled vibration (where the sprung masses have effect on the plate vibration) are presented and discussed. The Poisson's ratio is taken to be $\nu = 0.3$ and 10^{10} is used to simulate a rigid support in the calculations.

6.1. Convergence study

The first problem considered is a simply supported (SSSS) square plate with one sprung mass. Three groups of parameters of the sprung mass are used in the computations respectively: $\tau_1 = 100$, $\gamma_1 = 0.1$; $\tau_1 = 500$,

Table 2
The convergence study of the first five coupled eigenfrequencies for square plates with two sprung masses in respect to the number of terms (n) used

B.C.	n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
$\xi_1 = \eta_1 = 0.4, \xi_2 = 0.7, \eta_2 = 0.6, \tau_1 = \tau_2 = 200,$ $\gamma_1 = 0.1, \gamma_2 = 0.2$						
SSSS	10	13.91	26.29	46.89	51.27	62.90
	20	13.91	26.25	46.84	51.22	62.83
	30	13.91	26.25	46.83	51.22	62.82
	40	13.91	26.25	46.83	51.21	62.81
SF ₁ SF*	10	8.02	16.79	25.04	34.77	42.29
	20	8.02	16.79	25.01	34.74	42.27
	30	8.02	16.79	25.01	34.74	42.27
	40	8.02	16.79	25.01	34.74	42.27
$\xi_1 = \eta_1 = 0.25, \xi_2 = \eta_2 = 0.5, \tau_1 = 100, \tau_2 = 50,$ $\gamma_1 = \gamma_2 = 0.1$						
SSSS	10	14.49	24.87	31.64	54.04	81.81
	20	14.49	24.86	31.63	54.03	81.80
	30	14.49	24.85	31.62	54.03	81.80
	40	14.49	24.85	31.62	54.03	81.80
SF ₁ SF*	10	8.40	16.43	22.12	29.71	41.93
	20	8.40	16.43	22.10	29.69	41.92
	30	8.40	16.43	22.10	29.69	41.92
	40	8.40	16.43	22.10	29.69	41.92

*Two opposite edges are simply supported and the two other edges are free.

$\gamma_1 = 0.2$ and $\tau_1 = 2500$, $\gamma_1 = 0.5$. Two different locations of the sprung mass are considered: $\xi_1 = 0.5, \eta_1 = 0.5$ and $\xi_1 = 0.6, \eta_1 = 0.7$. The convergence corresponding to different terms of the series for the first five coupled eigenfrequencies $\Omega_i = \alpha_i^2 = \omega_i b^2 \sqrt{\rho h / D}$ ($i = 1, 2, \dots, 5$) is shown in Table 1.

It can be seen from Table 1 that the calculations converge quickly for all of the studied cases. With the increase of the number of terms, all the eigenfrequencies monotonically and steadily decrease. Therefore, the results obtained are the upper bounds of the exact values, similar to those obtained from the use of the Rayleigh-Ritz method. Taking only 20 terms, the maximum difference relative to those using 40 terms is less than 0.2% for the studied cases. The CPU time to use 40 terms to search the first five eigenvalues is less than two seconds on a Pentium 4 microcomputer. It is observed that the computation converges slightly slower when the sprung mass is placed at the centre of the plate than when it is away from the centre of the plate. When the mass and stiffness ratios increase, more terms are needed to maintain the same accuracy. However, the differences between the convergence rates for the studied cases are small.

Table 2 compares the convergence of the eigenvalues of square plates attached with two sprung masses. Two kinds of boundary conditions and two locations of the sprung masses are considered. Compared with the results in Table 1, it is seen from Table 2 that a better convergence is achieved. Therefore, in the following calculations, 40 terms are taken for all the cases.

6.2. Comparative study

A comparative study is given in Table 3 for a simply supported rectangular plate with one sprung mass. The solutions from the eigenfunction expansion (EE) method presented by Avalos et al. [10] and Wu et al. [12] and the solutions from the finite element (FE) method presented by Wu et al. [12] are taken as the reference. Moreover, the solutions from the Lagrangian multiplier (LM) method presented by Kim and Dickinson [3] are taken for comparison for the case of a rigid vertical point-support.

Table 3 shows that the present solutions agree well with those obtained from the finite element solutions and with most of those from the eigenfunction expansion solutions. However, almost all the answers from the present method are smaller or equal to those quoted in literature.

Table 3
Comparison of the first five coupled eigenfrequencies with published data (SSSS rectangular plates with one sprung mass)

Parameters	Methods	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
$\tau_1 = 3.375, \gamma_1 = 0.5,$ $\beta = 2/3, \xi_1 = 0.75, \eta_1 = 0.75$	Present	2.588	32.129	61.740	98.730	111.048
	FE [12]	2.588	32.937	63.743	99.844	114.314
$\tau_1 = 0.0625, \gamma_1 = 0.25,$ $\beta = 2, \xi_1 = 0.75, \eta_1 = 0.75$	Present	0.500	12.340	19.742	32.077	41.947
	FE [12]	0.500	12.655	20.697	33.812	42.318
	EE [12]	0.500	12.339	19.742	32.077	41.947
	EE [10]	0.500	12.340	19.742	32.077	41.947
$\tau_1 = 0.125, \gamma_1 = 0.5,$ $\beta = 2, \xi_1 = 0.75, \eta_1 = 0.75$	Present	0.499	12.342	19.746	32.078	41.949
	FE [12]	0.499	12.658	20.700	33.813	42.319
	EE [12]	0.499	12.342	19.745	32.078	41.948
	EE [10]	0.499	12.342	19.746	32.078	41.949
$\tau_1 = 5, \gamma_1 = 0.1,$ $\beta = 1, \xi_1 = 0.75, \eta_1 = 0.5$	Present	6.923	20.022	49.554	98.798	128.383
	EE [10]	6.930	20.022	49.555	98.798	
$\tau_1 = 5, \gamma_1 = 1,$ $\beta = 1, \xi_1 = 0.5, \eta_1 = 0.5$	Present	2.173	20.242	98.898	177.709	256.688
	EE [10]	2.174	20.243	98.899	177.709	
$\tau_1 = \infty, \gamma_1 = 1,$ $\beta = 1, \xi_1 = 0.5, \eta_1 = 0.5$	Present	8.474	56.310	148.089	208.663	298.017
	EE [10]	8.493	57.175	150.570	212.787	
$\tau_1 = \infty, \gamma_1 = 0.5,$ $\beta = 1, \xi_1 = 0.75, \eta_1 = 0.75$	Present	14.779	29.898	64.646	92.758	115.406
	EE [10]	14.892	30.437	65.466	93.333	
$\tau_1 = \infty, \gamma_1 = 0.5,$ $\beta = 2, \xi_1 = 0.75, \eta_1 = 0.5$	Present	7.343	15.426	29.105	54.658	76.608
	EE [10]	7.428	15.506	29.347	68.003	93.760
$\tau_1 = \infty, \gamma_1 = \infty,$ $\beta = 1, \xi_1 = 0.5, \eta_1 = 0.5$	Present	52.631	146.700	207.679	297.446	405.937
	LM [3]	53.170	148.201			

6.3. Parametric study

Consider a square plate with one sprung mass. The plate has two opposite edges simply supported and the two other edges clamped (SCSC). Table 4 gives the first six eigenfrequencies of the coupled vibration with respect to the mass ratio γ_1 and stiffness ratio τ_1 . Three different locations of the sprung mass are considered, i.e. ($\xi_1 = 0.5; \eta_1 = 0.5$), ($\xi_1 = 0.5; \eta_1 = 0.25$) and ($\xi_1 = 0.25; \eta_1 = 0.25$). It can be seen from Table 4 that increasing the mass ratio γ_1 will decrease the eigenfrequencies, while increasing the stiffness ratio will increase the eigenfrequencies. Moreover, if a given finite mass is placed at the centre of the plate, it will lead to the smallest fundamental eigenfrequency; while an elastic point-support located at the centre of the plate provides the largest fundamental eigenfrequency.

Consider a simply supported square plate attached with one sprung mass at its centre. Figures 2–4 give the first four eigenfrequencies of coupled vibration with respect to the stiffness ratio τ_1 when the mass ratio γ_1 is equal to 0.5, 5 and infinity. The parameter τ_1 is plotted along a logarithmic axis. It is shown that with the increase of the stiffness of the sprung mass, all of the eigenfrequencies monotonically increase. It is seen from Figs 2 and 3 that the variations of the eigenfrequencies increase with increase of τ_1 for moderate spring stiffness. The stiffness of the sprung mass, which results in the steep variation of the eigenfrequencies, increases with the order of the eigenfrequencies. It can be seen that the first eigenfrequency is sensitive in the range of $5 < \tau_1 < 100$; the second eigenfrequency is sensitive within $50 < \tau_1 < 500$, the third within $100 < \tau_1 < 10000$ and the fourth within $500 < \tau_1 < 10000$. Thus when the stiffness of the sprung mass is small, the coupled vibration between the plate and the sprung mass is rather weak because the natural frequency of the sprung mass is much lower than the natural frequencies of the bare plate. However, when the stiffness of the sprung mass is very large, the sprung mass acts as a rigid mass attached on the plate. Figure 4 shows that the smaller the stiffness of the internal support, the smaller the effect of the elastic point-support on the eigenfrequencies of the plate and the large internal stiffness acts as a rigid point support. It should be mentioned that all of the coupled modes in Figs 2–4 are double symmetric because the sprung mass is just placed at the centre of the plate. Correspondingly, all of the non symmetric-symmetric modes belong to the invariable modes.

Table 5 studies the effect of boundary conditions on the coupled eigenfrequencies of the square plate. The plate is simply supported along all edges. However, rotational springs with the same stiffness are added to two opposite

Table 4
The first six eigenfrequencies of coupled vibration for a SCSC square plate with one sprung mass

γ_1, τ_1	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
$\xi_1 = 0.5, \eta_1 = 0.5$						
0.1, 100	20.728	41.679	104.474	130.814	200.882	259.462
0.5, 100	10.555	36.687	104.320	130.730	200.861	259.452
0.5, 1000	14.198	59.083	112.901	145.768	209.107	266.006
0.1, ∞	23.415	83.600	119.593	177.645	233.957	286.155
0.5, ∞	14.794	69.702	117.209	169.783	228.676	282.962
$\infty, 100$	35.763	104.284	130.710	200.856	259.449	307.965
$\infty, 1000$	54.068	112.328	144.591	208.772	265.838	313.127
∞, ∞	64.003	116.541	167.428	227.307	282.160	328.076
$\xi_1 = 0.5, \eta_1 = 0.25$						
0.1, 100	23.775	34.920	72.556	103.091	130.403	141.976
0.5, 100	11.892	31.468	72.109	103.028	130.350	141.899
0.5, 1000	18.222	39.587	82.565	106.993	133.588	159.052
0.1, ∞	26.640	56.464	93.804	113.806	135.033	184.900
0.5, ∞	19.384	43.339	88.249	110.396	134.584	180.147
$\infty, 100$	31.066	72.014	103.014	130.337	141.881	200.624
$\infty, 1000$	36.169	81.282	106.623	133.485	157.987	205.464
∞, ∞	38.629	86.539	109.641	134.471	178.680	207.124
$\xi_1 = 0.25, \eta_1 = 0.25$						
0.1, 100	24.927	32.633	56.316	70.963	96.785	102.825
0.5, 100	12.197	30.193	55.993	70.727	96.620	102.768
0.5, 1000	19.722	34.476	58.985	75.675	100.454	110.916
0.1, ∞	27.649	48.201	63.207	82.271	101.093	120.601
0.5, ∞	21.221	36.662	60.133	78.208	100.864	117.205
$\infty, 100$	29.983	55.931	70.678	96.582	102.755	129.761
$\infty, 1000$	32.286	58.472	75.014	100.355	110.100	132.514
∞, ∞	33.333	59.438	77.240	100.794	116.199	134.110

Fig. 2. The first four coupled eigenfrequencies of a SSSS square plate with one sprung mass at its centre ($\xi_1 = \eta_1 = 0.5$), with respect to the stiffness ratio τ_1 when the mass ratio γ_1 is equal to 0.5.

edges. In the present investigation, two cases are considered: the plate with one sprung mass and the plate attached with two sprung masses. It can be seen from Table 5 that increasing the stiffness of the rotational springs does not always lead to increases of the coupled eigenfrequencies.

Fig. 3. The first four coupled eigenfrequencies of a SSSS square plate with one sprung mass at its centre ($\xi_1 = \eta_1 = 0.5$), with respect to the stiffness ratio τ_1 when the mass ratio γ_1 is equal to 5.

Fig. 4. The first four coupled eigenfrequencies of a SSSS square plate with an elastic point-support ($\gamma_1 = \infty$) at its centre ($\xi_1 = \eta_1 = 0.5$), with respect to the stiffness ratio τ_1 .

Figures 5–7 give the first two coupled modes of square plates attached with two sprung masses. Three kinds of classical boundary conditions, SSSS, SCSC and SFSF (where SFSF means two opposite edges simply supported and the other two edges are free) are considered. The locations and parameters of the sprung masses are $\xi_1 = 0.4$, $\eta_1 = 0.4$, $\xi_2 = 0.6$, $\eta_2 = 0.4$, $\tau_1 = \tau_2 = 100.0$, $\gamma_1 = \gamma_2 = 0.1$. It is seen from Figs 5–7 that the mode shapes of the coupled system are similar to those of the bare plate although the effect of the sprung masses on the eigenfrequencies is significant. This means that the smaller masses have no significant effects on the mode shapes of the plate. Moreover, it is seen that the effect of the attached masses on the mode shapes of the SFSF plate is larger than that on the SSSS and SCSC plates. For each mode the sprung masses are away from the modal nodes of the bare plate, the larger the effect of the sprung masses on the mode shape and the eigenfrequency of the coupled system.

Table 5

The first six eigenfrequencies of coupled vibration for a simply supported square plate with two opposite edges rotationally restrained, with one and two sprung masses respectively

$K_{r0} = K_{r1}$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
$\xi_1 = 0.3, \eta_1 = 0.4, \tau_1 = 100.0, \gamma_1 = 0.1$						
0.0	16.839	31.553	55.004	79.924	99.418	128.868
1.0	17.121	31.803	49.970	55.259	80.345	98.900
10.0	18.184	32.855	52.972	56.649	82.635	99.493
100.0	19.099	33.760	55.714	58.869	85.725	100.261
1000.0	20.037	33.963	56.149	59.457	86.445	100.452
∞	22.099	36.181	58.778	70.792	95.679	102.460
$\xi_1 = 0.2, \eta_1 = 0.3, \xi_2 = 0.5, \eta_2 = 0.6, \tau_1 = \tau_2 = 100.0, \gamma_1 = \gamma_2 = 0.1$						
0.0	15.495	26.619	37.080	50.806	53.999	81.211
1.0	15.804	26.651	37.138	51.321	54.534	81.645
10.0	17.025	26.794	37.364	53.172	57.817	83.985
100.0	18.134	26.936	37.554	54.658	61.928	87.099
1000.0	19.076	26.986	37.630	54.944	62.770	87.818
∞	20.646	27.665	41.492	57.074	72.013	96.541

Fig. 5. The first two coupled modes of a SSSS square plate with two sprung masses ($\xi_1 = 0.4, \eta_1 = 0.4, \xi_2 = 0.6, \eta_2 = 0.4$). (a) the mode corresponding to $\Omega_1 = 14.407$, (b) the mode corresponding to $\Omega_2 = 28.267$.

Fig. 6. The first two coupled modes of a SFSF square plate with two sprung masses ($\xi_1 = 0.4, \eta_1 = 0.4, \xi_2 = 0.6, \eta_2 = 0.4$). (a) the mode corresponding to $\Omega_1 = 8.2505$, (b) the mode corresponding to $\Omega_2 = 16.736$.

Fig. 7. The first two coupled modes of a SCSC square plate with two sprung masses ($\xi_1 = 0.4$, $\eta_1 = 0.4$, $\xi_2 = 0.6$, $\eta_2 = 0.4$). (a) the mode corresponding to $\Omega_1 = 19.027$, (b) the mode corresponding to $\Omega_2 = 28.609$.

7. Conclusions

This paper presents an exact solution of free vibration of a thin rectangular plate attached with sprung masses. The plate is simply supported at two opposite edges and elastically supported at the other edges. The governing differential equations of the free vibration of the plate-spring-mass system are directly solved without introducing further assumptions.

Numerical examples of the solution and parametric studies are provided for comparing the results with published data and for investigating the convergence of the method. As expected, the exact solutions would provide more accurate results than any approximate methods. Thus the solutions provided can be used as benchmark.

A parametric study is conducted to examine the coupled vibration of the sprung mass and plate system.

The solution derived in this paper is general and includes some special cases such as classical boundary conditions, rigid masses, elastic point-supports and rigid vertical supports.

The present analysis is based on a linear model that is a simplification of practical problems. When nonlinear effects are added to the model, suitable approximate calculation should be taken. In that case, the coexistence of solutions and vibration synchronization may play an important role.

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Appendix

q_{ni}^s ($s = 1, 2, 3, 4$) in Eq. (23) can be written as

$$\begin{Bmatrix} q_{ni}^1 \\ q_{ni}^2 \\ q_{ni}^3 \\ q_{ni}^4 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ b_1^i \\ 0 \\ b_2^i \end{Bmatrix}$$

in which, when $n < L$

$$\begin{aligned} a_{11} &= K_{r0}\lambda_{n1}, & a_{12} &= \lambda_{n1}^2 + \frac{\nu(n\pi)^2}{\beta^2}, & a_{13} &= K_{r0}\lambda_{n2}, & a_{14} &= -\lambda_{n2}^2 + \frac{\nu(n\pi)^2}{\beta^2}, \\ a_{21} &= K_{r1}\lambda_{n1} \cos \lambda_{n1} - \left(\lambda_{n1}^2 + \frac{\nu(n\pi)^2}{\beta^2} \right) \sin \lambda_{n1}, \end{aligned}$$

$$\begin{aligned}
a_{22} &= -K_{r1}\lambda_{n1}\sin\lambda_{n1} - \left(\lambda_{n1}^2 + \frac{\nu(n\pi)^2}{\beta^2}\right)\cos\lambda_{n1}, \\
a_{23} &= K_{r1}\lambda_{n2}\cosh\lambda_{n2} + \left(\lambda_{n2}^2 - \frac{\nu(n\pi)^2}{\beta^2}\right)\sinh\lambda_{n2}, \\
a_{24} &= K_{r1}\lambda_{n2}\sinh\lambda_{n2} + \left(\lambda_{n2}^2 - \frac{\nu(n\pi)^2}{\beta^2}\right)\cosh\lambda_{n2}, \\
a_{31} &= -\lambda_{n1}^3 - \frac{(2-\nu)(n\pi)^2}{\beta^2}\lambda_{n1}, \quad a_{32} = K_{t0}, \quad a_{33} = \lambda_{n2}^3 - \frac{(2-\nu)(n\pi)^2}{\beta^2}\lambda_{n2}, \quad a_{34} = K_{t0}, \\
a_{41} &= K_{t1}\sin\lambda_{n1} + \left(\lambda_{n1}^3 + \frac{(2-\nu)(n\pi)^2}{\beta^2}\lambda_{n1}\right)\cos\lambda_{n1}, \\
a_{42} &= K_{t1}\cos\lambda_{n1} - \left(\lambda_{n1}^3 + \frac{(2-\nu)(n\pi)^2}{\beta^2}\lambda_{n1}\right)\sin\lambda_{n1}, \\
a_{43} &= K_{t1}\sinh\lambda_{n2} - \left(\lambda_{n2}^3 - \frac{(2-\nu)(n\pi)^2}{\beta^2}\lambda_{n2}\right)\cosh\lambda_{n2}, \\
a_{44} &= K_{t1}\cosh\lambda_{n2} - \left(\lambda_{n2}^3 - \frac{(2-\nu)(n\pi)^2}{\beta^2}\lambda_{n2}\right)\sinh\lambda_{n2}, \\
b_1^i &= \left(\frac{\nu(n\pi)^2}{\beta^2\lambda_{n2}} - \lambda_{n2}\right)\sinh(\lambda_{n2}(1-\eta_i)) - \left(\frac{\nu(n\pi)^2}{\beta^2\lambda_{n1}} + \lambda_{n1}\right)\sin(\lambda_{n1}(1-\eta_i)) - \\
&K_{r1}[\cosh(\lambda_{n2}(1-\eta_i)) - \cos(\lambda_{n1}(1-\eta_i))], \\
b_2^i &= \left(\lambda_{n2}^2 - \frac{(2-\nu)(n\pi)^2}{\beta^2}\right)\cosh(\lambda_{n2}(1-\eta_i)) + \left(\lambda_{n1}^2 + \frac{(2-\nu)(n\pi)^2}{\beta^2}\right)\cos(\lambda_{n1}(1-\eta_i)) - \\
&K_{t1}\left[\frac{1}{\lambda_{n2}}\sinh(\lambda_{n2}(1-\eta_i)) - \frac{1}{\lambda_{n1}}\sin(\lambda_{n1}(1-\eta_i))\right]
\end{aligned}$$

when $n \geq L$

$$\begin{aligned}
a_{11} &= K_{r0}\lambda_{n1}, \quad a_{12} = \frac{\nu(n\pi)^2}{\beta^2} - \lambda_{n1}^2, \quad a_{13} = K_{r0}\lambda_{n2}, \quad a_{14} = \frac{\nu(n\pi)^2}{\beta^2} - \lambda_{n2}^2, \\
a_{21} &= K_{r1}\lambda_{n1}\cosh\lambda_{n1} + \left(\lambda_{n1}^2 - \frac{\nu(n\pi)^2}{\beta^2}\right)\sinh\lambda_{n1}, \\
a_{22} &= K_{r1}\lambda_{n1}\sinh\lambda_{n1} + \left(\lambda_{n1}^2 - \frac{\nu(n\pi)^2}{\beta^2}\right)\cosh\lambda_{n1}, \\
a_{23} &= K_{r1}\lambda_{n2}\cosh\lambda_{n2} + \left(\lambda_{n2}^2 - \frac{\nu(n\pi)^2}{\beta^2}\right)\sinh\lambda_{n2}, \\
a_{24} &= K_{r1}\lambda_{n2}\sinh\lambda_{n2} + \left(\lambda_{n2}^2 - \frac{\nu(n\pi)^2}{\beta^2}\right)\cosh\lambda_{n2}, \\
a_{31} &= \lambda_{n1}^3 - \frac{(2-\nu)(n\pi)^2}{\beta^2}\lambda_{n1}, \quad a_{32} = K_{t0}, \quad a_{33} = \lambda_{n2}^3 - \frac{(2-\nu)(n\pi)^2}{\beta^2}\lambda_{n2}, \quad a_{34} = K_{t0},
\end{aligned}$$

$$\begin{aligned}
a_{41} &= K_{t1} \sinh \lambda_{n1} - \left(\lambda_{n1}^3 - \frac{(2-\nu)(n\pi)^2}{\beta^2} \lambda_{n1} \right) \cosh \lambda_{n1}, \\
a_{42} &= K_{t1} \cosh \lambda_{n1} - \left(\lambda_{n1}^3 - \frac{(2-\nu)(n\pi)^2}{\beta^2} \lambda_{n1} \right) \sinh \lambda_{n1}, \\
a_{43} &= K_{t1} \sinh \lambda_{n2} - \left(\lambda_{n2}^3 - \frac{(2-\nu)(n\pi)^2}{\beta^2} \lambda_{n2} \right) \cosh \lambda_{n2}, \\
a_{44} &= K_{t1} \cosh \lambda_{n2} - \left(\lambda_{n2}^3 - \frac{(2-\nu)(n\pi)^2}{\beta^2} \lambda_{n2} \right) \sinh \lambda_{n2}, \\
b_1^i &= \left(\lambda_{n1} - \frac{\nu(n\pi)^2}{\beta^2 \lambda_{n1}} \right) \sinh \left(\lambda_{n1}(1 - \eta_i) \right) - \left(\lambda_{n2} - \frac{\nu(n\pi)^2}{\beta^2 \lambda_{n2}} \right) \sinh(\lambda_{n2}(1 - \eta_i)) - \\
&K_{r1} [\cosh(\lambda_{n2}(1 - \eta_i)) - \cosh(\lambda_{n1}(1 - \eta_i))], \\
b_2^i &= \left(\lambda_{n2}^2 - \frac{(2-\nu)(n\pi)^2}{\beta^2} \right) \cosh(\lambda_{n2}(1 - \eta_i)) - \left(\lambda_{n1}^2 - \frac{(2-\nu)(n\pi)^2}{\beta^2} \right) \cosh(\lambda_{n1}(1 - \eta_i)) - \\
&K_{t1} \left[\frac{1}{\lambda_{n2}} \sinh(\lambda_{n2}(1 - \eta_i)) - \frac{1}{\lambda_{n1}} \sinh(\lambda_{n1}(1 - \eta_i)) \right]
\end{aligned}$$

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