Vibration analysis of thin plates resting on Pasternak foundations by element free Galerkin method

Ehsan Bahmyari^{a,*}, Mohammad Mahdi Banatehrani^b, Mohammad Ahmadi^c and Marzieh Bahmyari^d

^a*Department of Marine Technology, Amirkabir University of Technology, Tehran, Iran*

^b*Institute for Aerospace Studies, University of Toronto, Toronto, ON, Canada*

^c*Department of Mechanical Engineering, Shahid Chamran University of Ahwaz, Khuzestan, Iran*

^d*Department of Mathematics, Persian Gulf University, Bushehr, Iran*

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Abstract. The element free Galerkin method is used to analyze free vibration of thin plates resting on Pasternak elastic foundations with all possible types of classical boundary conditions. Convergence of solution is studied by increasing number of nodes for different boundary conditions and foundation parameters. Upon comparison with available results in literature, it was found that the method converges very fast and has very good accuracy even with small number of nodes. Applicability of the method was shown by solving numerical examples with all possible combinations of boundary conditions and different values of foundation parameters.

Keywords: Element free Galerkin method, Pasternak foundation, vibration of plate

1. Introduction

Plate is an essential part of structural members in many civil, aeronautical, mechanical and marine structures. Vibration behavior of plates on elastic foundations is of interest for the design of many engineering problems such as pavement of roads, footing of buildings and bases of machines. Winkler type elastic foundation is the simplest model to describe the mechanical behavior of elastic supports. In this model interaction between lateral springs are ignored. Two parameters elastic foundation models have been developed to consider this interaction. They are characterized by two independent elastic constants which are derived by extension of the Winkler's model.

Vibration analysis of plates resting on two parameter elastic foundations was subject of several researches by various approaches. Xiang et al. [1] have studied analytically vibration of rectangular Mindlin plates with simply supported boundary conditions on Pasternak foundation. Omurtag et al. [2] used Finite element method to study the free vibration of thin plates resting on Pasternak foundation. Lam et al. [3] used the Green's functions to derive canonical exact solutions of elastic bending, buckling and vibration for Levy-plates resting on two parameters elastic foundations. Matsunaga [4] developed a two-dimensional higher-order theory to study vibration and stability of thick elastic plates resting on elastic foundation. Shen et al. [5] studied the free and forced vibration of Reissner-Mindlin

*Corresponding author: Ehsan Bahmyari, Department of Marine Technology, Amirkabir university of Technology Hafez Ave, No. 424, Tehran, Iran. Tel.: +98 216 454 3120; E-mail: Ehsan_Bahmyari@aut.ac.ir.

plates with four free edges resting on a Pasternak elastic foundation by employing the Rayleigh-Ritz method. Zhou et al. [6] used Rayleigh-Ritz method to study the three dimensional vibration of rectangular thick plates on elastic foundations. Ferreira et al. [7] employed the radial basis function collocation method to analysis static deformation and free vibration of plates on Pasternak foundations.

Meshless method is an approximate solution to many boundary value problems which approximations are entirely based on nodal parameters. Variety of meshless methods have been introduced, namely the smoothed particle hydrodynamics (SPH) [8], the meshless local Petrov Galerkin method (MLPG) [9], the reproducing kernel particle method (RKPM) [10], the element free Galerkin method (EFG) [11] and the radial basis function approach [12–15]. Element free Galerkin method (EFG) is considered as one of meshless methods which is used for solution of bending, buckling and vibration of plates and shells. Yan et al. [16] have used this method for vibration analysis of isotropic rectangular plate with interior elastic point support and elastically restrained edges. Chen et al. [17] and Dai et al. [18] have used EFG method for vibration analysis of laminated composite plates.

The previous publications have concentrated their studies on limited types of boundary conditions. For example, only one type of boundary condition has been used in References [1,4,5] and also two types of boundary condition in Reference [2], three types of boundary condition in References [6,7] and six types of boundary condition in Reference [3] have been analyzed.

It is the main objective of this paper to use EFG method for vibration analysis of thin plates with all possible types of boundary conditions resting on two parameters elastic foundation. The plate is discretized by a set of regularly distributed nodes. On the basis of classical plate theory (CTP) basic equation of vibration is derived. Moving least square (MLS) approximation is employed to produce displacement shape functions and Penalty method is used for imposing essential boundary conditions. Convergence, accuracy and applicability of the method were demonstrated by vibration solutions of thin plates with general boundary conditions. These important results can be served as benchmark results for researchers to verify their numerical methods and also for engineers to use such plates in their structures in the future.

2. Moving least square approximation (MLS)

Moving least square approximation was developed by Lancaster and Salkauskas [19] for surface construction. This method has been used widely for generation of shape functions in the meshfree methods. According to the Moving Least Square method, the unknown function $u(\mathbf{x})$ is approximated by $u^h(x)$ as follows:

$$u^h(\mathbf{x}) = \sum_{j=1}^m p_j(\mathbf{x}) a_j(\mathbf{x}) = \mathbf{p}^T(\mathbf{x}) \mathbf{a}(\mathbf{x}) \quad (1)$$

where $\mathbf{p}(\mathbf{x})$ is a vector of monomial basis functions, and $\mathbf{a}(\mathbf{x})$ is a vector of unknown coefficients which depend on location x . Also, m is the numbers of terms in the basis. Commonly used bases are the linear and quadratic basis functions. Quadratic basis in two-dimensional domain has following form:

$$\mathbf{p}^T = [1, x, y, x^2, xy, y^2] \quad (2)$$

By minimizing weighted square of residual function, the unknown coefficients $a_j(x)$ in Eq. (1) can be obtained as follows:

$$J = \sum_{I=1}^n w(\mathbf{x} - \mathbf{x}_I) [\mathbf{p}^T(\mathbf{x}_I) \mathbf{a}(\mathbf{x}) - u_I]^2 \quad (3)$$

where u_I is nodal parameter of field variable at node I and n is the number of nodes in the neighborhood of $\mathbf{x}$ which called domain of influence. The unknown coefficients vector $\mathbf{a}(\mathbf{x})$ can be determined by minimizing the functional of the weighted residual as follows:

$$\frac{\partial J}{\partial \mathbf{a}} = 0 \quad (4)$$

Table 1
First and sixth frequency parameters, $\bar{\omega} = \omega a^2 \sqrt{\frac{\rho h}{D}}$, of a square thin plate resting on Winkler's type elastic foundation

K_1	Mode number	Method	Natural frequency parameter					
			SSSS	SCSC	SFSF	SSSC	SCSF	SSSF
100	First mode	Pres.(5×5)	22.1997	39.7748	13.9200	27.7797	16.6801	15.4248
		Pres.(7×7)	22.1388	31.0169	13.8917	25.9713	16.1896	15.3891
		Pres.(9×9)	22.1380	30.8754	13.8955	25.7790	16.1742	15.3900
		Pres.(11×11)	22.1385	30.8351	13.8943	25.7450	16.1697	15.3889
		Pres.(13×13)	22.1363	30.7795	13.8919	25.7246	16.1655	15.3867
		Pres.(15×15)	22.1342	30.7364	13.8901	25.7121	16.1628	15.3850
		Pres.(17×17)	22.1326	30.7141	13.8888	25.7041	16.1611	15.3838
		Pres.(19×19)	22.1315	30.6992	13.8879	25.6983	16.1598	15.3829
		Pres.(23×23)	22.1299	30.6786	13.8866	25.6906	16.1581	15.3817
		Ref [3]	22.13	30.63	13.88	25.67	16.15	15.38
	Sixth mode	Pres.(5×5)	221.2307	2.6824×10^3	81.4949	559.4443	159.4388	158.5789
		Pres.(7×7)	103.3783	145.4672	71.6906	121.7760	95.7161	95.2384
		Pres.(9×9)	101.3152	132.1131	71.6137	116.2383	92.9194	92.6162
		Pres.(11×11)	100.0937	130.9692	71.5525	114.8928	92.0405	91.7153
		Pres.(13×13)	99.7609	130.9690	71.5187	114.6726	91.7934	91.4683
		Pres.(15×15)	99.6474	130.8205	71.5015	114.5373	91.6701	91.3523
		Pres.(17×17)	99.5727	130.6771	71.4892	114.4154	91.5865	91.2701
		Pres.(19×19)	99.5162	130.5475	71.4791	114.3138	91.5233	91.2069
		Pres.(23×23)	99.4325	130.3091	71.4642	114.1546	91.4292	91.1130
		Ref [3]	99.20	129.48	71.44	113.67	91.16	90.85
1000	First mode	Pres.(5×5)	37.3263	51.5988	33.0712	41.2286	34.3728	33.7327
		Pres.(7×7)	37.2827	43.1561	33.0591	39.6853	34.0886	33.7155
		Pres.(9×9)	37.2820	43.0487	33.0607	39.5528	34.0809	33.7158
		Pres.(11×11)	37.2823	43.0193	33.0602	39.5304	34.0787	33.7153
		Pres.(13×13)	37.2809	42.9792	33.0592	39.5171	34.0767	33.7143
		Pres.(15×15)	37.2797	42.9483	33.0584	39.5089	34.0754	33.7135
		Pres.(17×17)	37.2787	42.9324	33.0579	39.5037	34.0746	33.7130
		Pres.(19×19)	37.2781	42.9217	33.0575	39.4999	34.0740	33.7126
		Pres.(23×23)	37.2771	42.9070	33.0570	39.4949	34.0732	33.7120
		Ref [3]	37.28	42.87	31.62	39.49	34.07	33.71
	Sixth mode	Pres.(5×5)	238.9642	3.0594×10^3	86.8640	630.8066	162.3892	161.5506
		Pres.(7×7)	107.6658	148.6465	77.7132	125.4638	100.3083	99.8551
		Pres.(9×9)	105.6708	135.4854	77.6402	120.0558	97.6396	97.3515
		Pres.(11×11)	104.4904	134.3589	77.5831	118.7426	96.8028	96.4937
		Pres.(13×13)	104.1706	134.3583	77.5518	118.5287	96.5678	96.2588
		Pres.(15×15)	104.0618	134.2132	77.5360	118.3977	96.4506	96.1486
		Pres.(17×17)	103.9903	134.0734	77.5246	118.2798	96.3711	96.0705
		Pres.(19×19)	103.9362	133.9471	77.5153	118.1815	96.3110	96.0105
		Pres.(23×23)	103.8561	133.7148	77.5016	118.0275	96.2216	95.9213
		Ref [3]	103.64	132.91	77.49	117.56	95.97	95.67

which yields following system of linear equations:

$$\mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\mathbf{U}_s \quad (5)$$

where,

$$\mathbf{A}(\mathbf{x}) = \sum_{I=1}^n w(\mathbf{x} - \mathbf{x}_I) \mathbf{p}(\mathbf{x}_I) \mathbf{p}^T(\mathbf{x}_I) \quad (6)$$

$$\mathbf{B}(\mathbf{x}) = [w(\mathbf{x} - \mathbf{x}_1) \mathbf{p}(\mathbf{x}_1), \dots, w(\mathbf{x} - \mathbf{x}_n) \mathbf{p}(\mathbf{x}_n)] \quad (7)$$

$$\mathbf{U}_s = \{u_1 \ u_2 \ \dots \ u_n\}^T \quad (8)$$

By substituting Eq. (5) into Eq. (1), MLS approximant can be written as:

$$u^h(\mathbf{x}) = \sum_{I=1}^n \phi_I(\mathbf{x}) u_I = \Phi(\mathbf{x}) \mathbf{U}_s \quad (9)$$

Fig. 1. Domain of influence.

Fig. 2. Thin plate resting on two parameter elastic foundation and co-ordinate system.

Fig. 3. Prescribed boundary conditions.

where $\Phi(\mathbf{x})$ is matrix of shape functions and is defined as follows:

$$\Phi(\mathbf{x}) = [\phi_1(\mathbf{x}) \ \phi_2(\mathbf{x}) \ \dots \ \phi_n(\mathbf{x})] = \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x}) \quad (10)$$

In order to calculate partial derivatives of $\Phi(\mathbf{x})$, Eq. (10) is rewritten as follows [20]:

$$\Phi(\mathbf{x}) = \gamma^T(\mathbf{x})\mathbf{B}(\mathbf{x}) \quad (11)$$

where

$$\gamma(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x})\mathbf{p}(\mathbf{x}) \quad (12)$$

or,

$$\mathbf{A}(\mathbf{x})\gamma(\mathbf{x}) = \mathbf{p}(\mathbf{x}) \quad (13)$$

The partial derivatives of $\gamma(\mathbf{x})$ can be expressed as follows:

$$\mathbf{A}\gamma_{,x} = \mathbf{p}_{,x} - \mathbf{A}_{,x}\gamma \quad (14)$$

$$\mathbf{A}\gamma_{,y} = \mathbf{p}_{,y} - \mathbf{A}_{,y}\gamma \quad (15)$$

$$\mathbf{A}\gamma_{,xx} = \mathbf{p}_{,xx} - (\mathbf{A}_{,xx}\gamma + 2\mathbf{A}_{,x}\gamma_{,x}) \quad (16)$$

$$\mathbf{A}\gamma_{,xy} = \mathbf{p}_{,xy} - (\mathbf{A}_{,xy}\gamma + \mathbf{A}_{,x}\gamma_{,y} + \mathbf{A}_{,y}\gamma_{,x}) \quad (17)$$

$$\mathbf{A}\gamma_{,yy} = \mathbf{p}_{,yy} - (\mathbf{A}_{,yy}\gamma + 2\mathbf{A}_{,y}\gamma_{,y}) \quad (18)$$

Table 2
Dimensionless parameter of natural frequencies, $\bar{\omega} = \frac{\omega a^2}{\pi^2} \sqrt{\frac{\rho h}{D}}$, of a square SSSS plate resting on Winkler's type elastic foundation

K ₁	Method	Mode number					
		1st	2nd	3rd	4th	5th	6th
100	Pres.(5×5)	2.2493	6.8840	6.8840	15.6987	16.2849	22.4154
	Pres.(7×7)	2.2431	5.2509	5.2509	8.4292	10.4681	10.4744
	Pres.(9×9)	2.2430	5.1231	5.1231	8.0986	10.2630	10.2654
	Pres.(11×11)	2.2431	5.1153	5.1153	8.0797	10.1380	10.1416
	Pres.(13×13)	2.2429	5.1125	5.1125	8.0752	10.1074	10.1079
	Pres.(15×15)	2.2427	5.1107	5.1107	8.0730	10.0961	10.0964
	Pres.(17×17)	2.2425	5.1092	5.1092	8.0714	10.0886	10.0888
	Pres.(19×19)	2.2424	5.1079	5.1079	8.0700	10.0830	10.0831
	Pres.(23×23)	2.2422	5.1059	5.1059	8.0678	10.0746	10.0746
	Ref. [1]	2.2413	5.0971	5.0971	8.0523		
	Ref. [6]	2.2413	5.0973	5.0973	8.0527		
	Ref. [7]	2.2414	5.0967	5.0967	8.0542		
500	Pres.(5×5)	3.0278	7.2613	7.2613	15.8295	17.1184	23.2311
	Pres.(7×7)	3.0228	5.6306	5.6306	8.6774	10.6635	10.6697
	Pres.(9×9)	3.0227	5.5092	5.5092	8.3483	10.4615	10.4638
	Pres.(11×11)	3.0228	5.5020	5.5020	8.3298	10.3384	10.3420
	Pres.(13×13)	3.0226	5.4993	5.4993	8.3254	10.3084	10.3089
	Pres.(15×15)	3.0225	5.4976	5.4976	8.3233	10.2973	10.2976
	Pres.(17×17)	3.0223	5.4962	5.4962	8.3217	10.2900	10.2902
	Pres.(19×19)	3.0223	5.4950	5.4950	8.3203	10.2844	10.2845
	Pres.(23×23)	3.0221	5.4932	5.4932	8.3182	10.2762	10.2762
	Ref. [1]	3.0215	5.4850	5.4850	8.3032		
	Ref. [6]	3.0214	5.4850	5.4850	8.3035		
	Ref. [7]	3.0216	5.4846	5.4846	8.3051		

The partial derivatives of $\Phi(\mathbf{x})$ would be as follows:

$$\Phi_{I,x} = \gamma_{I,x}^T \mathbf{B}_I + \gamma^T \mathbf{B}_{I,x} \quad (19)$$

$$\Phi_{I,y} = \gamma_{I,y}^T \mathbf{B}_I + \gamma^T \mathbf{B}_{I,y} \quad (20)$$

$$\Phi_{I,xx} = \gamma_{I,xx}^T \mathbf{B}_I + 2\gamma_{I,x}^T \mathbf{B}_{I,x} + \gamma^T \mathbf{B}_{I,xx} \quad (21)$$

$$\Phi_{I,xy} = \gamma_{I,xy}^T \mathbf{B}_I + \gamma_{I,x}^T \mathbf{B}_{I,y} + \gamma_{I,y}^T \mathbf{B}_{I,x} + \gamma^T \mathbf{B}_{I,xy} \quad (22)$$

$$\Phi_{I,yy} = \gamma_{I,yy}^T \mathbf{B}_I + 2\gamma_{I,y}^T \mathbf{B}_{I,y} + \gamma^T \mathbf{B}_{I,yy} \quad (23)$$

Weight function plays an important role in the formulation of MLS method. This function should be non-zero in the domain of influence and zero outside of the domain. The precise character of this function seems to be unimportant although it is almost mandatory that it be positive and increase monotonically as $\|\mathbf{x} - \mathbf{x}_I\|$ decreases, furthermore it is desirable that weight function be smooth [21]. Several weight functions have been proposed by researchers. In this work quartic spline weight function is chosen,

$$w(\mathbf{x} - \mathbf{x}_I) \equiv w(r) = \begin{cases} 1 - 6r^2 + 8r^3 - 3r^4, & r \leq 1 \\ 0, & r > 1 \end{cases} \quad (24)$$

where,

$$r = \frac{\|\mathbf{x} - \mathbf{x}_I\|}{d_I} \quad (25)$$

where d_I determines the size of the influence domain at node I . The most commonly used domains are circles and rectangles (Fig. 1). For circular domain, d_I is the radius of circle and for rectangular domain, d_I is equal to length of rectangle in x , and y direction which are denoted by d_{Ix} and d_{Iy} , respectively. For the latter case, weight function can be written as follows:

$$w(\mathbf{x} - \mathbf{x}_I) = w(r_x) \cdot w(r_y) \quad (26)$$

Fig. 4. Comparison of natural frequencies for Winkler and Pasternak's type of elastic foundation ($K_1 = 100$, $h/a = 0.01$ and number of nodes 19×19).

where r_x and r_y are given by

$$r_x = \frac{\|x - x_I\|}{d_{Ix}} \quad (27)$$

$$r_y = \frac{\|y - y_I\|}{d_{Iy}} \quad (28)$$

where

$$d_{Ix} = d_{\max} \cdot c_{Ix} \quad (29)$$

$$d_{Iy} = d_{\max} \cdot c_{Iy} \quad (30)$$

where $d_{\max}$ is scaling parameter, c_{Ix} and c_{Iy} are distance around node I which are determined in such a way that enough nodes be in the domain in order that matrix $\mathbf{A}$ in Eq. (13) to be invertible at every point in the domain [21].

3. Governing equations

3.1. Strain and kinetic energies of thin plate

A thin plate of thickness h resting on a two-parametric elastic foundation with Cartesian coordinate system origin at center is shown in Fig. 2. Displacements functions in x , y and z directions are defined by u , v and w , respectively. Based on CPT, displacement field can be expressed as follows [17]:

$$\mathbf{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \left\{ -z \frac{\partial}{\partial x} \quad -z \frac{\partial}{\partial y} \quad 1 \right\}^T w = \mathbf{L}_u w, \quad (31)$$

Table 3
First and sixth frequency parameters, $\bar{\omega} = \omega a^2 \sqrt{\frac{\rho h}{D}}$, of a square thin plate resting on Pasternak's type elastic foundation

K ₁	K ₂	Mode number	Method	Natural frequency parameter					
				SSSS	SCSC	SFSF	SSSC	SCSF	SSSF
0	0	First mode	Pres.(5×5)	19.8187	38.2349	9.6837	25.8562	13.3355	11.7442
			Pres.(7×7)	19.7520	29.3599	9.6430	23.9680	12.7323	11.6976
			Pres.(9×9)	19.7511	29.2113	9.6485	23.7607	12.7129	11.6989
			Pres.(11×11)	19.7517	29.1688	9.6468	23.7239	12.7072	11.6973
			Pres.(13×13)	19.7492	29.1100	9.6433	23.7018	12.7018	11.6945
			Pres.(15×15)	19.7468	29.0644	9.6407	23.6882	12.6984	11.6923
			Pres.(17×17)	19.7451	29.0409	9.6388	23.6795	12.6962	11.6907
			Pres.(19×19)	19.7438	29.0251	9.6375	23.6732	12.6946	11.6895
			Pres.(23×23)	19.7421	29.0033	9.6356	23.6649	12.6923	11.6879
			Ref [3]	19.74	28.95	9.63	23.65	12.69	11.68
		Sixth mode	Pres.(5×5)	219.1715	2.6372×10 ³	80.8759	550.9448	159.1064	158.2405
			Pres.(7×7)	102.8908	145.1095	70.9898	121.3593	95.1922	94.7116
			Pres.(9×9)	100.8196	131.7330	70.9124	115.8063	92.3800	92.0751
			Pres.(11×11)	99.5932	130.5871	70.8508	114.4571	91.4960	91.1689
			Pres.(13×13)	99.2588	130.5870	70.8166	114.2361	91.2475	90.9204
			Pres.(15×15)	99.1447	130.4380	70.7993	114.1003	91.1234	90.8037
			Pres.(17×17)	99.0697	130.2942	70.7868	113.9779	91.0393	90.7211
			Pres.(19×19)	99.0129	130.1643	70.7767	113.8760	90.9757	90.6575
			Pres.(23×23)	98.9288	129.9252	70.7616	113.7161	90.8811	90.5630
			Ref [3]	98.69	129.09	70.74	113.23	90.61	90.29
0	100	First mode	Pres.(5×5)	48.6709	65.9214	32.9237	54.0423	38.8593	37.1864
			Pres.(7×7)	48.6194	55.1007	32.9062	51.5800	38.0427	37.1545
			Pres.(9×9)	48.6183	54.8322	32.9083	51.4035	37.9990	37.1548
			Pres.(11×11)	48.6188	54.8028	32.9080	51.3710	37.9905	37.1545
			Pres.(13×13)	48.6177	54.7798	32.9071	51.3577	37.9869	37.1536
			Pres.(15×15)	48.6167	54.7552	32.9063	51.3501	37.9850	37.1529
			Pres.(17×17)	48.6160	54.7431	32.9057	51.3452	37.9836	37.1524
			Pres.(19×19)	48.6154	54.7349	32.9053	51.3414	37.9825	37.1520
			Pres.(23×23)	48.6146	54.7225	32.9047	51.3359	37.9809	37.1515
			Ref [3]	48.62	54.68	32.90	51.32	37.98	37.15
		Sixth mode	Pres.(5×5)	298.1788	3.9594×10 ³	126.8551	810.8308	210.1817	191.3902
			Pres.(7×7)	143.6404	178.7094	115.5538	158.8996	135.7528	135.2073
			Pres.(9×9)	141.7370	167.9889	115.4758	154.3975	133.3230	132.9989
			Pres.(11×11)	140.6843	166.8371	115.3670	153.1456	132.5973	132.2724
			Pres.(13×13)	140.4231	166.8450	115.3114	152.9621	132.4086	132.0906
			Pres.(15×15)	140.3387	166.7491	115.2892	152.8701	132.3183	132.0076
			Pres.(17×17)	140.2832	166.6527	115.2765	152.7868	132.2575	131.9487
			Pres.(19×19)	140.2413	166.5637	115.2671	152.7155	132.2117	131.9033
			Pres.(23×23)	140.1794	166.3948	115.2541	152.6009	132.1440	131.8361
			Ref [3]	140.04	165.75	115.25	152.24	131.98	131.67
100	100	First mode	Pres.(5×5)	49.6881	66.8246	34.4088	54.9899	40.1342	38.5076
			Pres.(7×7)	49.6370	56.0014	34.3920	52.5408	39.3351	38.4766
			Pres.(9×9)	49.6359	55.7365	34.3941	52.3671	39.2927	38.4768
			Pres.(11×11)	49.6364	55.7075	34.3938	52.3351	39.2845	38.4765
			Pres.(13×13)	49.6353	55.6849	34.3928	52.3220	39.2810	38.4757
			Pres.(15×15)	49.6344	55.6607	34.3921	52.3146	39.2791	38.4751
			Pres.(17×17)	49.6336	55.6488	34.3915	52.3098	39.2778	38.4746
			Pres.(19×19)	49.6331	55.6407	34.3911	52.3060	39.2767	38.4742
			Pres.(23×23)	49.6323	55.6285	34.3905	52.3006	39.2752	38.4737
			Ref [3]	49.63	55.59	34.39	52.29	39.27	38.47
		Sixth mode	Pres.(5×5)	299.6952	3.9897×10 ³	127.2539	816.6297	210.5833	191.6662
			Pres.(7×7)	143.9899	178.9988	115.9859	159.2181	136.1206	135.5767
			Pres.(9×9)	142.0899	168.2871	115.9078	154.7217	133.6973	133.3742
			Pres.(11×11)	141.0391	167.1364	115.7994	153.4715	132.9735	132.6496
			Pres.(13×13)	140.7784	167.1442	115.7439	153.2884	132.7854	132.4683

Table 3, continued

K ₁	K ₂	Mode number	Method	Natural frequency parameter				
				SSSS	SCSC	SFSF	SSSC	SCSF
			Pres.(15×15)	140.6942	167.0484	115.7218	153.1965	132.6953
			Pres.(17×17)	49.6336	166.9522	115.7091	153.1135	132.6348
			Pres.(19×19)	140.5970	166.8633	115.6998	153.0423	132.5891
			Pres.(23×23)	140.5353	166.6948	115.6868	152.9279	132.5215
			Ref [3]	140.39	166.05	115.69	152.57	132.36

The pseudo-strains of the plate are denoted as

$$\boldsymbol{\varepsilon}_p = \left\{ -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - 2\frac{\partial^2}{\partial x \partial y} \right\}^T w = \mathbf{L}_{ps} w, \quad (32)$$

The pseudo-stresses of the plate are given by

$$\boldsymbol{\sigma}_p = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}, \quad (33)$$

The relationship between the pseudo-strains and pseudo-stress is expressed as

$$\boldsymbol{\sigma}_p = \mathbf{D} \boldsymbol{\varepsilon}_p, \quad (34)$$

where $\mathbf{D}$ is stiffness matrix and is given as,

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \quad (35)$$

For an isotropic plate, $\mathbf{D}$ would be as follows:

$$\mathbf{D} = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \quad (36)$$

where E is Young's module and ν is Poisson's ratio. The strain and kinetic energies of the plate can be written as:

$$U_p = \frac{1}{2} \int_S \boldsymbol{\varepsilon}_p^T \boldsymbol{\sigma}_p dS \quad (37)$$

$$T_p = \frac{1}{2} \int_V \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} dV, \quad (38)$$

3.2. Strain energy due to elastic foundation

The surface pressure vector induce by foundation is defined as follows;

$$\mathbf{q} = \{ 0 \ 0 \ k_f w - G_f \nabla^2 w \}^T \quad (39)$$

where k_f is the Winkler foundation stiffness while G_f is the shear stiffness of the elastic foundation. The strain energy due to the Pasternak foundation is given as follows,

$$U_F = \frac{1}{2} \int_S \mathbf{u}^T \mathbf{q} dS = \frac{1}{2} \int_S \boldsymbol{\varepsilon}_F^T \boldsymbol{\sigma}_F dS \quad (40)$$

Table 4

Dimensionless parameter of natural frequencies, $\bar{\omega} = \frac{\omega a^2}{\pi^2} \sqrt{\frac{\rho h}{D}}$, of a square SSSS plate resting on Pasternak's type elastic foundation ($K_2 = 10$)

h/a	K_1	Method	Mode number					
			1st	2nd	3rd	4th	5th	6th
0.01	100	Pres.(5×5)	2.6624	7.3457	7.3457	16.0736	17.1893	23.3326
		Pres.(7×7)	2.6567	5.7185	5.7185	8.9137	10.9498	10.9558
		Pres.(9×9)	2.6566	5.5958	5.5958	8.5849	10.7480	10.7502
		Pres.(11×11)	2.6566	5.5886	5.5886	8.5665	10.6261	10.6296
		Pres.(13×13)	2.6564	5.5860	5.5860	8.5622	10.5966	10.5971
		Pres.(15×15)	2.6563	5.5843	5.5843	8.5601	10.5858	10.5861
		Pres.(17×17)	2.6561	5.5829	5.5829	8.5585	10.5786	10.5788
		Pres.(19×19)	2.6560	5.5817	5.5817	8.5572	10.5732	10.5733
		Pres.(23×23)	2.6559	5.5799	5.5799	8.5551	10.5651	10.5652
		Ref. [1]	2.6551	5.5718	5.5718	8.5405		
		Ref. [6]	2.6551	5.5717	5.5717	8.5406		
		Ref. [7]	2.6559	5.5718	5.5718	8.5384		
	500	Pres.(5×5)	3.3461	7.7004	7.7004	16.2013	17.9808	24.1174
		Pres.(7×7)	3.3412	6.0691	6.0691	9.1488	11.1368	11.1426
		Pres.(9×9)	3.3411	5.9513	5.9513	8.8208	10.9377	10.9399
		Pres.(11×11)	3.3412	5.9445	5.9445	8.8028	10.8174	10.8209
		Pres.(13×13)	3.3410	5.9420	5.9420	8.7986	10.7885	10.7889
		Pres.(15×15)	3.3409	5.9404	5.9404	8.7965	10.7778	10.7781
		Pres.(17×17)	3.3408	5.9391	5.9391	8.7950	10.7708	10.7710
		Pres.(19×19)	3.3407	5.9380	5.9380	8.7937	10.7655	10.7656
		Pres.(23×23)	3.3406	5.9363	5.9363	8.7917	10.7576	10.7576
		Ref. [1]	3.3400	5.9287	5.9287	8.7775		
		Ref. [6]	3.3398	5.9285	5.9285	8.7775		
		Ref. [7]	3.3406	5.9285	5.9285	8.7754		
	200	Pres.(5×5)	2.8258	7.2782	7.2782	15.3872	16.8089	22.4779
		Pres.(7×7)	2.8204	5.6932	5.6932	8.6942	10.5696	10.5753
		Pres.(9×9)	2.8203	5.5745	5.5745	8.3761	10.3805	10.3827
		Pres.(11×11)	2.8204	5.5675	5.5675	8.3583	10.2646	10.2680
		Pres.(13×13)	2.8202	5.5649	5.5649	8.3542	10.2365	10.2369
		Pres.(15×15)	2.8200	5.5633	5.5633	8.3522	10.2261	10.2264
		Pres.(17×17)	2.8199	5.5620	5.5620	8.3507	10.2193	10.2195
		Pres.(19×19)	2.8198	5.5608	5.5608	8.3494	10.2141	10.2142
		Pres.(23×23)	2.8197	5.5591	5.5591	8.3475	10.2064	10.2065
		Ref. [1]	2.7842	5.3043	5.3043	7.7287		
		Ref. [6]	2.7756	5.2954	5.2954	7.7279		
		Ref. [7]	2.7902	5.3452	5.3452	7.8255		
	1000	Pres.(5×5)	4.0088	7.9497	7.9497	15.6297	18.2908	23.9416
		Pres.(7×7)	4.0044	6.3524	6.3524	9.1412	10.9246	10.9300
		Pres.(9×9)	4.0043	6.2421	6.2421	8.8244	10.7406	10.7428
		Pres.(11×11)	4.0043	6.2358	6.2358	8.8073	10.6279	10.6312
		Pres.(13×13)	4.0042	6.2335	6.2335	8.8033	10.6007	10.6011
		Pres.(15×15)	4.0041	6.2321	6.2321	8.8014	10.5907	10.5909
		Pres.(17×17)	4.0040	6.2309	6.2309	8.8000	10.5841	10.5842
		Pres.(19×19)	4.0039	6.2298	6.2298	8.7988	10.5791	10.5792
		Pres.(23×23)	4.0038	6.2283	6.2283	8.7969	10.5717	10.5717
		Ref. [1]	3.9805	6.0078	6.0078	8.2214		
		Ref. [6]	3.9566	5.9757	5.9757	8.1954		
		Ref. [7]	3.9844	6.0430	6.0430	8.3112		

Where ε_F and σ_F are regarded as strain and stress vectors associated with Pasternak foundation and are defines as follows,

$$\varepsilon_F = \left\{ 1 \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right\}^T w = \mathbf{L}_f s w \quad (41)$$

$$\sigma_F = \mathbf{D}_F \varepsilon_F \quad (42)$$

Table 5

Dimensionless parameter of natural frequencies, $\bar{\omega} = \frac{\omega a^2}{\pi^2} \sqrt{\frac{\rho h}{D}}$, of a square CCCC plate resting on Pasternak's type elastic foundation ($K_1 = 1390.2$, $K_2 = 16683$, $h/a = 0.015$ and $\nu = 0.15$)

Method	Mode number					
	1st	2nd	3rd	4th	5th	6th
Pres.(5×5)	11.2731	487.5740	487.5740	691.8086	749.8748	773.9894
Pres.(7×7)	8.4543	14.2932	14.2932	22.5495	25.5613	39.8698
Pres.(9×9)	8.1614	12.9504	12.9504	17.2984	19.5261	19.8530
Pres.(11×11)	8.1367	12.7800	12.7800	16.8966	19.2998	19.4499
Pres.(13×13)	8.1281	12.7478	12.7478	16.7403	19.2333	19.2945
Pres.(15×15)	8.1254	12.7336	12.7336	16.7012	19.2142	19.2536
Pres.(17×17)	8.1240	12.7261	12.7261	16.6797	19.2071	19.2358
Pres.(19×19)	8.1230	12.7214	12.7214	16.6700	19.2001	19.2236
Pres.(23×23)	8.1214	12.7152	12.7152	16.6606	19.1867	19.2058
Ref. [2]	8.1375	12.898	12.898	16.932		
Ref. [6]	8.1675	12.823	12.823	16.833		
Ref. [7]	8.1669	12.821	12.821	16.842

Fig. 5. Comparison of natural frequencies for Winkler and Pasternak's type of elastic foundation ($K_1 = 500$, $h/a = 0.01$ and number of nodes 19×19).

where D_F is given by,

$$D_F = \begin{bmatrix} k_f & 0 & 0 \\ 0 & G_f & 0 \\ 0 & 0 & G_f \end{bmatrix} \quad (43)$$

3.3. Approximation of field variables

By using the MLS method the following approximation of deflection function in z direction is obtained:

$$w^h(\mathbf{x}) = \sum_{I=1}^n \phi_I(\mathbf{x}) w_I, \quad (44)$$

Table 6

Dimensionless parameter of natural frequencies, $\bar{\omega} = \frac{\omega a^2}{\pi^2} \sqrt{\frac{\rho h}{D}}$, of a square plate on Pasternak's type elastic foundation ($K_2 = 10$, $h/a = 0.01$ and number of nodes 19×19)

B.C.	K_1	Mode number					
		1st	2nd	3rd	4th	5th	6th
SCSC	100	3.4527	6.0949	7.5255	10.1074	10.9209	13.6416
	500	4.0033	6.4228	7.7934	10.3084	11.1072	13.7912
	1000	4.5999	6.8105	8.1159	10.5543	11.3357	13.9759
SSCC	100	3.2945	6.6644	6.6846	9.9326	12.1658	12.1805
	500	3.8677	6.9656	6.9849	10.1370	12.3332	12.3478
	1000	4.4823	7.3246	7.3430	10.3870	12.5394	12.5537
SFSS	100	1.9313	3.5429	4.7833	6.5913	6.9730	9.7393
	500	2.7992	4.0814	5.1947	6.8956	7.2613	9.9478
	1000	3.6011	4.6679	5.6671	7.2580	7.6063	10.2023
FFSS	100	1.2994	2.6769	2.7733	4.5871	5.9674	6.1585
	500	2.4072	3.3572	3.4346	5.0146	6.3019	6.4831
	1000	3.3057	4.0500	4.1144	5.5024	6.6966	6.8674
FFSC	100	1.4006	2.7949	3.2738	5.0515	6.1053	7.1681
	500	2.4633	3.4521	3.8501	5.4427	6.4326	7.4489
	1000	3.3467	4.1290	4.4671	5.8952	6.8197	7.7856
FSFS	100	1.7330	2.4271	4.5545	4.5559	5.3624	7.8203
	500	2.6663	3.1617	4.9848	4.9861	5.7323	8.0784
	1000	3.4988	3.8895	5.4753	5.4765	6.1636	8.3899
FSFC	100	2.1435	2.7784	4.8181	5.5511	6.2747	8.5774
	500	2.9497	3.4387	5.2267	5.9093	6.5937	8.8133
	1000	3.7193	4.1179	5.6964	6.3286	6.9719	9.0997
FSFF	100	1.1427	1.7063	2.4654	3.3679	3.6260	5.7051
	500	2.3264	2.6490	3.1912	3.9303	4.1536	6.0541
	1000	3.2473	3.4856	3.9135	4.5364	4.7311	6.4640
SFSC	100	2.0138	4.0416	4.8327	6.9808	8.0060	9.7713
	500	2.8568	4.5211	5.2402	7.2688	8.2583	9.9791
	1000	3.6460	5.0568	5.7088	7.6135	8.5632	10.2328
CCCC	100	4.1067	7.9246	7.9246	11.4696	13.8906	13.9520
	500	4.5793	8.1795	8.1795	11.6471	14.0375	14.0983
	1000	5.1090	8.4873	8.4873	11.8652	14.2190	14.2790
CFCC	100	2.9256	4.6645	6.9083	8.4202	8.7004	12.4004
	500	3.5588	5.0856	7.1993	8.6605	8.9332	12.5648
	1000	4.2187	5.5672	7.5472	8.9517	9.2158	12.7672
CFSC	100	2.3923	4.3066	5.7953	7.7673	8.1903	11.2920
	500	3.1351	4.7594	6.1393	8.0271	8.4371	11.4723
	1000	3.8680	5.2710	6.5438	8.3405	8.7358	11.6937
CFCF	100	2.7181	3.2639	5.1602	6.6934	7.3379	8.8761
	500	3.3903	3.8417	5.5437	6.9933	7.6124	9.1043
	1000	4.0775	4.4599	5.9886	7.3510	7.9422	9.3817
FFCC	100	1.4975	3.2617	3.4074	5.4896	7.0872	7.3222
	500	2.5197	3.8398	3.9643	5.8516	7.3710	7.5972
	1000	3.3884	4.4582	4.5659	6.2747	7.7112	7.9276
FFCF	100	1.2460	1.8048	3.0164	3.7057	3.8825	6.2183
	500	2.3788	2.7135	3.6337	4.2234	4.3794	6.5400
	1000	3.2851	3.5349	4.2820	4.7925	4.9306	6.9211
CCSC	100	3.7221	6.9323	7.6980	10.7266	12.3447	13.7673
	500	4.2379	7.2223	7.9601	10.9162	12.5097	13.9155
	1000	4.8054	7.5691	8.2761	11.1487	12.7131	14.0985
CSSS	100	2.9899	5.8036	6.4784	9.2667	10.7290	12.0336
	500	3.6118	6.1470	6.7878	9.4856	10.9185	12.2028
	1000	4.2635	6.5511	7.1558	9.7522	11.1509	12.4112

Table 6, continued

B.C.	K_1	Mode number					
		1st	2nd	3rd	4th	5th	6th
FFFF	100	1.0132	1.4870	1.4870	2.2656	3.1826	3.4509
	500	2.2656	2.5134	2.5134	3.0395	3.7727	4.0016
	1000	3.2041	3.3837	3.3837	3.7908	4.4005	4.5983

3.4. Imposing essential boundary conditions

The essential boundary conditions in the element free Galerkin method cannot be easily and directly imposed because moving least square approximation does not satisfy the Kronecker delta function property, [22]. Various techniques have been proposed for imposing the essential boundary conditions in the EFG method, such as Lagrange multipliers [21], penalty method [23], coupling with FEM [24], and coupling with boundary element method [25]. In this paper, a Penalty method is used for enforcing the essential boundary condition in the EFG method.

In the absence of body forces and assumed prescribed tractions, Lagrangian of free vibration of a thin plate resting on elastic foundation can be written as follows

$$L = T_P - U_p - U_F + \int_{\Gamma_u} \frac{1}{2} (\tilde{\mathbf{u}} - \bar{\mathbf{u}})^T \boldsymbol{\alpha} (\tilde{\mathbf{u}} - \bar{\mathbf{u}}) d\Gamma, \quad (45)$$

where $\boldsymbol{\alpha}$ is a diagonal matrix of penalty coefficients, $\bar{\mathbf{u}}$ is prescribed displacement and defines as [26],

$$\tilde{\mathbf{u}} = \mathbf{L}_b w, \quad (46)$$

where $\mathbf{L}_b$ is a vector of differential operators. Proper choice of penalty coefficients is very important on accuracy of solution. From literature review [26], usually large numbers of order $1 \times 10^{4-13} \times \max$ (diagonals in stiffness matrix) can be chosen. A plate with general boundary condition is considered (Fig. 3). For clamped boundary condition $\mathbf{L}_b$ is defined as,

$$\mathbf{L}_b = \left\{ 1 \frac{\partial}{\partial n} \right\}^T, \quad (47)$$

where n denotes outward normal direction on boundary of the plate. For simply supported boundary condition $\mathbf{L}_b$ is defined as,

$$\mathbf{L}_b = \{1 \ 0\}^T, \quad (48)$$

3.5. Derivation of stiffness and mass matrices for free vibration analysis

The dynamical equation of a plate resting on elastic foundation can be derived according to the Hamilton's variational principle,

$$\delta \int_{t_1}^{t_2} L dt = 0, \quad (49)$$

where L is the Lagrangian function of system. By substituting Eqs (37), (38) and (40) into Eq. (45) and replacing results in Eq. (49), following variational form is found:

$$\int_S \delta \boldsymbol{\varepsilon}_P^T \boldsymbol{\sigma}_P dS + \int_S \delta \boldsymbol{\varepsilon}_F^T \boldsymbol{\sigma}_F dS + \int_V \rho \delta \mathbf{u}^T \ddot{\mathbf{u}} dV - \delta \int_{\Gamma_u} \frac{1}{2} (\tilde{\mathbf{u}} - \bar{\mathbf{u}})^T \boldsymbol{\alpha} (\tilde{\mathbf{u}} - \bar{\mathbf{u}}) d\Gamma = 0, \quad (50)$$

Table 7

Dimensionless parameter of natural frequencies, $\bar{\omega} = \frac{\omega a^2}{\pi^2} \sqrt{\frac{\rho h}{D}}$, of a square plate on Pasternak's type elastic foundation ($K_2 = 50$, $h/a = 0.01$ and number of nodes 19×19)

B.C	K_1	Mode number					
		1st	2nd	3rd	4th	5th	6th
SSSS	100	3.8935	7.1703	7.1703	10.2777	12.3409	12.3410
	500	4.3892	7.4510	7.4510	10.4754	12.5060	12.5062
	1000	4.9393	7.7877	7.7877	10.7175	12.7094	12.7096
SCSC	100	4.5622	7.6189	8.9035	11.6903	12.6604	15.1832
	500	4.9919	7.8837	9.1311	11.8646	12.8214	15.3177
	1000	5.4819	8.2027	9.4078	12.0788	13.0199	15.4842
SSCC	100	4.4560	8.1437	8.1547	11.5434	13.8154	13.8260
	500	4.8950	8.3920	8.4026	11.7198	13.9631	13.9736
	1000	5.3938	8.6923	8.7026	11.9366	14.1456	14.1559
SFSS	100	2.9739	5.1403	6.3433	8.3280	8.9734	11.5147
	500	3.5987	5.5252	6.6590	8.5709	9.1993	11.6915
	1000	4.2524	5.9716	7.0337	8.8651	9.4740	11.9088
FFSS	100	1.9380	4.2715	4.2980	6.3562	8.0792	8.1668
	500	2.8039	4.7277	4.7517	6.6713	8.3293	8.4143
	1000	3.6048	5.2424	5.2641	7.0454	8.6317	8.7138
FFSC	100	2.0550	4.3485	4.8120	6.7846	8.1631	9.1225
	500	2.8860	4.7974	5.2211	7.0807	8.4107	9.3447
	1000	3.6690	5.3054	5.6914	7.4342	8.7103	9.6153
FSFS	100	2.6590	3.7868	6.0872	6.5713	7.0341	9.7677
	500	3.3431	4.2949	6.4156	6.8765	7.3201	9.9755
	1000	4.0384	4.0384	6.8037	7.2400	7.6626	10.2294
FSFC	100	3.0310	4.0790	6.7819	6.9614	7.8353	10.4368
	500	3.6460	4.5545	7.0781	7.2503	8.0930	10.6316
	1000	4.2925	5.0868	7.4317	7.5959	8.4040	10.8701
FSFF	100	1.5287	2.8786	4.0043	5.0427	5.7190	7.7912
	500	2.5384	3.5203	4.4878	5.4345	6.0673	8.0503
	1000	3.4024	4.1863	5.0271	5.8877	6.4763	8.3629
SFSC	100	3.0571	5.6181	6.3909	8.6962	9.9272	11.5463
	500	3.6676	5.9722	6.7044	8.9291	10.1318	11.7226
	1000	4.3109	6.3874	7.0767	9.2119	10.3819	11.9394
CCCC	100	5.1501	9.2762	9.2762	12.9650	15.4286	15.4739
	500	5.5344	9.4949	9.4949	13.1223	15.5610	15.6059
	1000	5.9801	9.7613	9.7613	13.3163	15.7249	15.7694
CFCC	100	3.8338	6.1366	8.2287	10.2369	10.2900	14.1867
	500	4.3364	6.4624	8.4745	10.4355	10.4875	14.3306
	1000	4.8925	6.8479	8.7720	10.6784	10.7292	14.5084
CFSC	100	3.3938	5.8436	7.2391	9.4024	10.0895	12.9407
	500	3.9527	6.1849	7.5172	9.6182	10.2909	13.0982
	1000	4.5559	6.5866	7.8511	9.8813	10.5372	13.2926
CFCF	100	3.5109	4.4624	7.0497	7.9770	8.7751	11.2077
	500	4.0537	4.9009	7.3351	8.2303	9.0060	11.3893
	1000	4.6438	5.3991	7.6768	8.5363	9.2865	11.6122
FFCC	100	2.1659	4.8504	4.8892	7.1927	9.0988	9.2193
	500	2.9661	5.2566	5.2924	7.4726	9.3215	9.4393
	1000	3.7323	5.7239	5.7569	7.8083	9.5927	9.7072
FFCF	100	1.6709	2.9646	4.5505	5.5258	5.7739	8.2012
	500	2.6265	3.5909	4.9812	5.8856	6.1190	8.4477
	1000	3.4686	4.2458	5.4721	6.3065	6.5248	8.7461
CCSC	100	4.8151	8.3789	9.0677	12.2733	13.9808	15.3046
	500	5.2240	8.6204	9.2913	12.4393	14.1268	15.4380
	1000	5.6941	8.9130	9.5634	12.6438	14.3072	15.6032

Table 7, continued

B.C	K_1	Mode number					
		1st	2nd	3rd	4th	5th	6th
CSSS	100	4.1831	7.3679	7.9683	10.9262	12.4853	13.6884
	500	4.6480	7.6414	8.2218	11.1124	12.6486	13.8374
	1000	5.1707	7.9701	8.5281	11.3409	12.8497	14.0215
FFFF	100	1.0132	2.5610	2.5610	3.7055	5.3937	5.4901
	500	2.2656	3.2657	3.2657	4.2233	5.7617	5.8520
	1000	3.2041	3.9746	3.9746	4.7925	6.1910	6.2752

Fig. 6. Comparison of natural frequencies for Winkler and Pasternak's type of elastic foundation ($K_1 = 1000$, $h/a = 0.01$ and number of nodes 19×19).

After substituting Eqs (31), (32), (34), (41), (42) and (46) into Eq. (50), it can be rewritten as:

$$\int_S \delta(\mathbf{L}_{ps}w)^T \mathbf{D}(\mathbf{L}_{ps}w) dS + \int_S \delta(\mathbf{L}_{fs}w)^T \mathbf{D}_F(\mathbf{L}_{fs}w) dS + \int_V \rho \delta(\mathbf{L}_u w)^T \mathbf{L}_u \ddot{w} dV - \frac{1}{2} \int_{\Gamma_u} \delta(\mathbf{L}_b w - \bar{\mathbf{u}})^T \boldsymbol{\alpha}(\mathbf{L}_b w - \bar{\mathbf{u}}) d\Gamma = 0, \quad (51)$$

By substituting the approximated deflection function $w^h(\mathbf{x})$ Eq. (44) into Eq. (51), the discrete dynamical equations for free vibration analysis of plates resting on elastic foundation is deduced as:

$$\mathbf{M}\ddot{\mathbf{U}} + (\mathbf{K}_P + \mathbf{K}_F + \tilde{\mathbf{K}})\mathbf{U} = 0, \quad (52)$$

where $\mathbf{U}$ is the vector of deflection of all nodes, and is defined by

$$\mathbf{U} = \{w_1, w_2, \dots, w_{n_t}\}^T, \quad (53)$$

Where n_t is the total number of nodes in the entire domain of the plate. The notations of $\mathbf{K}_P$, $\mathbf{K}_F$, $\tilde{\mathbf{K}}$ and $\mathbf{M}$ denote the global plate stiffness matrix, the global foundation stiffness matrix, the global penalty matrix and global mass matrix which are given by:

$$\mathbf{K}_P = \int_S \mathbf{B}_I^T \mathbf{D} \mathbf{B}_J dS, \quad (54)$$

Table 8

Dimensionless parameter of natural frequencies, $\bar{\omega} = \frac{\omega a^2}{\pi^2} \sqrt{\frac{\rho h}{D}}$, of a square plate on Pasternak's type elastic foundation ($K_2 = 100$, $h/a = 0.01$ and number of nodes 19×19)

B.C	K_1	Mode number					
		1st	2nd	3rd	4th	5th	6th
SSSS	100	5.0289	8.7599	8.7599	12.0887	14.2454	14.2455
	500	5.4217	8.9911	8.9911	12.2573	14.3886	14.3887
	1000	5.8760	9.2720	9.2720	12.4648	14.5658	14.5659
SCSC	100	5.6376	9.1698	10.3581	13.4012	14.5434	16.9068
	500	5.9907	9.3910	10.5544	13.5535	14.6838	17.0277
	1000	6.4047	9.6603	10.7947	13.7414	14.8574	17.1776
SSCC	100	5.5585	9.6676	9.6740	13.2766	15.6292	15.6370
	500	5.9163	9.8776	9.8839	13.4303	15.7599	15.7676
	1000	6.3352	10.1340	10.1401	13.6199	15.9217	15.9294
SFSS	100	3.8983	6.5779	7.8644	10.0715	10.9149	13.4029
	500	4.3934	6.8829	8.1212	10.2732	11.1013	13.5551
	1000	4.9432	7.2461	8.4312	10.5199	11.3300	13.7430
FFSS	100	2.5106	5.5992	5.6107	7.9947	10.0262	10.0714
	500	3.2263	5.9545	5.9653	8.2474	10.2288	10.2731
	1000	3.9423	6.3709	6.3810	8.5528	10.4766	10.5198
FFSC	100	2.6297	5.6653	6.1227	8.4042	10.0874	11.0035
	500	3.3199	6.0167	6.4493	8.6449	10.2888	11.1884
	1000	4.0192	6.4290	6.8355	8.9368	10.5352	11.4154
FSFS	100	3.4845	4.9657	7.5744	8.3694	8.6680	11.7228
	500	4.0309	5.3632	7.8407	8.6111	8.9017	11.8966
	1000	4.6239	5.8220	8.1614	8.9041	9.1853	12.1102
FSFC	100	3.8419	5.2369	8.3833	8.5583	9.4074	12.3379
	500	4.3436	5.6152	8.6247	8.7948	9.6231	12.5031
	1000	4.8989	6.0550	8.9172	9.0819	9.8861	12.7066
FSFF	100	1.8997	3.8433	5.2736	6.5140	7.4703	9.7105
	500	2.7776	4.3447	5.6494	6.8218	7.7402	9.9196
	1000	3.5844	4.8999	6.0867	7.1881	8.0649	10.1749
SFSC	100	3.9796	7.0432	7.9106	10.4253	11.8284	13.4341
	500	4.4657	7.3288	8.1660	10.6203	12.0006	13.5859
	1000	5.0075	7.6710	8.4743	10.8592	12.2124	13.7734
CCCC	100	6.1900	10.7124	10.7124	14.6078	17.1481	17.1820
	500	6.5132	10.9023	10.9023	14.7476	17.2673	17.3010
	1000	6.8959	11.1351	11.1351	14.9205	17.4152	17.4486
CFCC	100	4.6985	7.5146	9.6048	11.8496	12.1599	16.1052
	500	5.1168	7.7829	9.8161	12.0216	12.3275	16.2320
	1000	5.5959	8.1059	10.0741	12.2331	12.5338	16.3893
CFSC	100	4.3011	7.2528	8.6962	11.0821	11.9783	14.7354
	500	4.7545	7.5305	8.9291	11.2657	12.1484	14.8740
	1000	5.2667	7.8638	9.2119	11.4912	12.3577	15.0454
CFCF	100	4.2778	5.5757	8.7914	9.3159	10.2668	13.0550
	500	4.7335	5.9324	9.0218	9.5336	10.4648	13.2112
	1000	5.2477	6.3503	9.3018	9.7990	10.7071	13.4039
FFCC	100	2.7438	6.1706	6.1862	8.7973	11.0065	11.0719
	500	3.4109	6.4947	6.5096	9.0275	11.1914	11.2557
	1000	4.0948	6.8784	6.8925	9.3074	11.4183	11.4813
FFCF	100	2.0523	3.9254	5.8134	6.9810	7.5194	10.0809
	500	2.8841	4.4176	6.1564	7.2691	7.7876	10.2825
	1000	3.6676	4.9646	6.5600	7.6139	8.1103	10.5290
CCSC	100	5.8824	9.8819	10.5168	13.9569	15.7839	17.0247
	500	6.2216	10.0874	10.7102	14.1031	15.9133	17.1448
	1000	6.6212	10.3386	10.9471	14.2838	16.0736	17.2937

Table 8, continued

B.C	K ₁	Mode number					
		1st	2nd	3rd	4th	5th	6th
CSSS	100	5.2997	8.9436	9.4994	12.6946	14.3813	15.5064
	500	5.6739	9.1702	9.7131	12.8552	14.5233	15.6382
	1000	6.1094	9.4458	9.9737	13.0532	14.6988	15.8013
FFFF	100	1.0132	3.4292	3.4292	4.9146	7.1491	7.1959
	500	2.2656	3.9831	3.9831	5.3159	7.4307	7.4757
	1000	3.2041	4.5823	4.5823	5.7785	7.7683	7.8114

$$\mathbf{K}_F = \int_S \mathbf{H}_I^T \mathbf{D}_F \mathbf{H}_J dS, \quad (55)$$

$$\mathbf{M} = \int_V \rho \mathbf{N}_I^T \mathbf{N}_J dV, \quad (56)$$

$$\tilde{\mathbf{K}} = \int_{\Gamma_u} \Psi_I^T \alpha \Psi_J d\Gamma, \quad (57)$$

where

$$\mathbf{B} = \mathbf{L}_{ps} \phi_I = \begin{Bmatrix} -\phi_{I,xx} \\ -\phi_{I,yy} \\ -2\phi_{I,xy} \end{Bmatrix}, \quad (58)$$

$$\mathbf{H} = \mathbf{L}_{fs} \phi_I = \begin{Bmatrix} \phi_I \\ \phi_{I,x} \\ \phi_{I,y} \end{Bmatrix} \quad (59)$$

$$\mathbf{N} = \mathbf{L}_u \phi_I = \begin{Bmatrix} -z\phi_{I,x} \\ -z\phi_{I,y} \\ \phi_I \end{Bmatrix}, \quad (60)$$

Ψ_I depends on boundary condition. For clamped boundary condition, Ψ_I is defined as follows,

$$\Psi_I = \begin{Bmatrix} \phi_I \\ \phi_{I,n} \end{Bmatrix}, \quad (61)$$

For simply support boundary condition it is defined as follows,

$$\Psi_I = \begin{Bmatrix} \phi_I \\ 0 \end{Bmatrix}, \quad (62)$$

Assuming a harmonic vibration form for the plate, the deflection vector $\mathbf{U}$ can be expressed as:

$$\mathbf{U} = \bar{\mathbf{U}} e^{i\omega t}, \quad (63)$$

where $\bar{\mathbf{U}}$ is the amplitude of the vibration and ω is the circular frequency. Substituting Eq. (63) into Eq. (52), the following eigenvalue equation is obtained:

$$(\mathbf{K} - \omega^2 \mathbf{M}) \bar{\mathbf{U}} = 0, \quad (64)$$

where ω^2 is the eigenvalue and represents the square of circular frequency of transverse vibration and $\bar{\mathbf{U}}$ is the eigenvector that represents the vector of amplitude of transverse vibration.

4. Numerical results and discussion

In this paper, quadratic basis is adopted for basis function $\mathbf{P}(\mathbf{x})$, scaling parameter $d_{\max}$ is chosen as 3.9 [27] and a 4×4 gauss integration scheme is used to perform the integration in computing the system matrixes. Also the Poisson ratio ν is assumed 0.3. To show accuracy and applicability of the present method, a computer code in MATLAB was developed and some examples of square plates were solved. Different boundary conditions, foundation parameters and number of nodes were considered.

To ease comparison of results, the following dimensionless parameters are defined:

1- Natural frequency of plate resting on elastic foundation,

$$\bar{\omega} = \frac{\omega a^2}{\pi^2} \sqrt{\frac{\rho h}{D}}, \quad (65)$$

where D is the bending rigidity of plate and defined as follows:

$$D = \frac{Eh^3}{12(1-\nu^2)}, \quad (66)$$

2- Foundation parameters,

$$K_1 = \frac{k_f a^4}{D}, \quad (67)$$

$$K_2 = \frac{G_f a^2}{D}, \quad (68)$$

4.1. Free vibration of a square plate resting on Winkler's type elastic foundation

The convergency and accuracy of the element free Galerkin method in vibration analysis of thin plates resting on Winkler elastic foundation is studied for different foundation parameters, various types of boundary conditions and different numbers of modes of vibration. In Table 1 the convergency and accuracy of the first and sixth natural frequencies are shown for different types of boundary conditions and two values of Winkler foundation stiffness. As it is seen, the results are convergent and are in very good agreement with the exact solutions of Ref [3]. Also, it is obvious from Table 1 that when the plate edge boundary condition is clamped or the vibrational results of the sixth mode of vibration would be obtained, more nodes should be used in order to achieve accurate results. In Table 2, the convergency of the first six modes of vibration is studied for a square SSSS plates resting on Winkler elastic foundation and as it is seen the convergency is monotonic. Also the results of the first four modes of the frequency are compared with the numerical solutions obtained by Ref [1,6,7] and good consistency is shown. It is seen that, using 5×5 number of nodes gives very accurate solution for the fundamental frequency and by increasing the number of mode of vibration, more nodes are needed in order to obtain precise results.

4.2. Free vibration of a square plate resting on Pasternak's type elastic foundation

In the end vibration of square thin plates resting on Pasternak's type elastic foundation is investigated. To show convergency and accuracy of the method for this type of foundation, in Table 3 the first and sixth natural frequencies of square plates with different foundation parameters and various types of boundary conditions are calculated and compared with exact solutions of Ref [3]. As it is shown, the results are convergent and a very good agreement with Ref [3] is seen. In Table 4, the first six natural frequencies for a SSSS plate with different elastic foundation parameters and thickness ratios are calculated and compared with available results. As can be seen very good agreements is achieved in all cases. In Table 5 the first six natural frequencies for a CCCC plate are calculated and compared with available literature and good agreements is obtained.

In order to compare the Winkler's type of elastic foundation with Pasternak's, in Figs 4–6 the first nine natural frequencies of plates with these two foundations are shown. As can be seen, by increasing the mode number and the shear stiffness parameter of elastic foundation K_2 , differences between these two foundations increase.

The first six natural frequencies of a square plate resting on Pasternak foundation are presented in Tables 6–8 for all possible types of classical boundary conditions and various values of foundation parameters.

5. Conclusions

In this paper Element Free Galerkin method has been used for free vibration analysis of thin plates resting on two-parameter type elastic foundation with all possible types of boundary condition. Accuracy of the solution was examined for different foundation parameters, thickness to span ratio and boundary conditions. It was found that the method has very good agreement with available literature regardless of different parameters even with small number of nodes. Applicability of the method was demonstrated with solution of square plates with general boundary conditions, and foundation parameters. The numerical results provide valuable information for engineers in design applications and may also serve as benchmarks for further reference.

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