Buckling and vibration of non-homogeneous rectangular plates subjected to linearly varying in-plane force

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Abstract. The present work analyses the buckling and vibration behaviour of non-homogeneous rectangular plates of uniform thickness on the basis of classical plate theory when the two opposite edges are simply supported and are subjected to linearly varying in-plane force. For non-homogeneity of the plate material it is assumed that young’s modulus and density of the plate material vary exponentially along axial direction. The governing partial differential equation of motion of such plates has been reduced to an ordinary differential equation using the sine function for mode shapes between the simply supported edges. This resulting equation has been solved numerically employing differential quadrature method for three different combinations of clamped, simply supported and free boundary conditions at the other two edges. The effect of various parameters has been studied on the natural frequencies for the first three modes of vibration. Critical buckling loads have been computed. Three dimensional mode shapes have been presented. Comparison has been made with the known results.

Keywords: Non-homogeneous, rectangular, buckling, differential quadrature

1. Introduction

Plates of various geometries are the important components in many engineering applications. Of these, rectangular plates with different combinations of boundary conditions are commonly encountered in aerospace, mechanical, nuclear and off-shore structures. In many practical situations, particularly in the ship buildings and automotive industry these plates may be subjected to in-plane dynamic loads of different types, which may induce buckling, a phenomenon which is highly undesirable. In this regard, efforts have been made by researchers to analyses the effect of uniformly/non-uniformly distributed in-plane loads on the vibration characteristics of rectangular plates and prominent ones are reported in references [1–17]. Out of these, Leissa and Kang [6] employed the power series method to obtain the exact solutions for vibration and buckling of rectangular plates having two opposite edges simply supported and these are subjected to linearly varying in-plane stresses while the other two are clamped. Hu et al. [8] investigated the buckling behaviour of a symmetrically laminated composite rectangular plate under parabolic variation of axial loads using Rayleigh-Ritz method. Devarakonda and Bert [14] used Galerkin method to study the buckling of rectangular plate with non-linearly distributed compressive loading on two opposite sides. Kang and Leissa [10] obtained the exact solution for buckling of rectangular plates having linearly varying in-plane

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loading on two opposite simply supported edges, for all combinations of clamped, simply supported or free at the other two edges. Wang et al. [11] obtained the numerical results for the buckling and vibration of isotropic rectangular plate subjected to linearly varying in-plane stresses along two opposite simply supported edges while the other two are clamped using differential quadrature method. Jana and Bhaskar [12] used Galerkin’s approach to present the analytical solutions for buckling of rectangular plates under non-uniform biaxial compression. Wang et al. [13] analyzed the buckling of thin rectangular plates with cosine-distributed compressive loads on two opposite sides using differential quadrature method. Recently, Civalek et al. [15] used discrete singular convolution approach for buckling analysis of rectangular Kirchhoff plates subjected to compressive loads on two-opposite edges. Tang and Wang [16] studied the buckling of symmetrically laminated rectangular plates under parabolic edge compression using Rayleigh-Ritz method. Very recently, Eftekhar and Jafari [17] employed the mixed finite element and differential quadrature method for free and forced vibration and buckling analysis of rectangular plates.

Beside the above considerations, in many practical applications, particularly in aerospace industry, modern missile technology and microelectronics, plate type structural elements have to work under high temperature environment which causes non-homogeneity in the material i.e. mechanical properties of the material vary with space variables. However, many structural components possess initial non-homogeneity due to the inclusion of foreign materials or imperfection or being composite materials. Plywood, timber and fibre-reinforced plastic etc. form an important class of non-homogeneous materials. These materials are of considerable interest to design engineers in various technological situations [18–21]. Thus, their design requires an accurate analysis for their vibration characteristics. Up till now, several studies have been devoted to the dynamic behaviour of non-homogeneous plates of various geometries and reported in references [22–30], to mention a few. In these references various model such as linear, quadratic, exponential etc. for the Young modulus and density of the plate material have been considered. Recently, in two papers, namely Lal and Dhanpati [31] and Kumar and Lal [32] studied the combined effect of constant biaxial/uniaxial in-plane forces and non-homogeneity of the plate material on the transverse vibrations of rectangular plates of unidirectional/bidirectional varying thickness using quintic spline technique and Rayleigh-Ritz method, respectively.

The present paper analyzes the effect of linearly varying in-plane force together with non-homogeneity of the plate material on the transverse vibration of rectangular plates on the basis of classical plate theory. The two opposite edges \( y = 0 \) and \( y = b \) are taken simply supported and these are subjected to linearly varying in-plane force. Non-homogeneity of the plate material is assumed to arise due to exponential variation in Young’s modulus and density of the plate along axial direction. The Poisson ratio \( \nu \) is assumed to remain constant. The governing partial differential equation of motion for such plates has been reduced to an ordinary differential equation using the sine function for mode shapes between the simply supported edges. This resulting equation has been solved numerically employing differential quadrature method for three different combinations of clamped, simply supported and free boundary conditions at the other two edges. The effect of non-homogeneity parameter, density parameter, aspect ratio, in-plane force parameter and loading parameter on the natural frequencies has been illustrated for the first three modes of vibration. Three dimensional mode shapes have been presented for specified plates.

2. Mathematical formulation

Consider a non-homogeneous isotropic rectangular plate of length \( a \), breadth \( b \), thickness \( h \) and density \( \rho \). The plate is referred to a system of rectangular Cartesian co-ordinates \((x, y, \ z)\), the middle surface being \( z = 0 \) and origin is at the one of the corners of the plate. The \( x \)- and \( y \)-axes are taken along the edges of the plate, the axis of \( z \) is perpendicular to the \( xy \)-plane and a linearly varying compressive in-plane force \( N_y \) is applied along the two opposite simply supported edges \( y = 0 \) and \( y = b \) as shown in Fig. 1. Following Leissa and Kang [6] with the incorporation of non-homogeneity and assuming \( q = N_{sx} = N_x = 0 \), the differential equation governing the transverse vibration of such plates is given by

\[
D \left( \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} \right) + 2 \frac{\partial D}{\partial x} \frac{\partial^3 w}{\partial x^3} + \frac{\partial^2 D}{\partial x^2} \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 D}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 2 \frac{\partial D}{\partial x} \frac{\partial^3 w}{\partial x \partial y^2} + 2D \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^2 w}{\partial t^2} - N_y \frac{\partial^2 w}{\partial y^2} = 0,
\]

(1)
where \( D = Eh^3/12(1-\nu^2) \), \( N_y = -N_0(1-\gamma x/a) \), \( w(x, y, t) \) is the transverse deflection, \( t \) the time, \( D \) the flexural rigidity, \( E \) Young’s modulus, \( \nu \) Poisson ratio, \( N_0 \) the intensity of compressive force at the edge \( x = 0 \) and \( \gamma \) is the loading parameter.

For different values of loading parameter \( \gamma \), one can obtain various particular cases e.g. \( \gamma = 0 \) gives the case of uniformly distributed compressive force while for \( \gamma = 1 \), the compressive force varies linearly from \(-N_0\) at \( x = 0 \) to zero at \( x = a \). Except these values of \( \gamma \) in the range \( 0 < \gamma \leq 2 \), various combinations of bending and compression are obtained as shown in Fig. 2.

For a harmonic solution, the deflection \( w \) is assumed to be

\[
w(x, y, t) = \bar{w}(x) \sin(p\pi y/b) e^{i\omega t},
\]

where \( p \) is a positive integer and \( \omega \) is the frequency in rad/s.

Further, for elastically non-homogeneous material, it is assumed that the Young’s modulus \( E \) and density \( \rho \) are the functions of space variable \( x \) only.

Introducing the non-dimensional variables \( X = x/a, Y = y/b, \bar{h} = h/a, W = \bar{w}/a \), Eq. (1) reduces to

\[
E\bar{h}^3W'''' + 2E'\bar{h}^3W'''' + (E''''\bar{h}^3 - 2\lambda^2 E\bar{h}^3)W'' - 2E'\bar{h}^3\lambda^2W' + (E\bar{h}^3\lambda^4 - \nu E''''\bar{h}^3 - 12\bar{h}^2 \rho \omega^2 a^2(1-\nu^2) - N_0(1-\gamma X)(1-\nu^2)12\lambda^2)W = 0
\]

(3)
where \( \lambda^2 = \pi^2 \rho^2 a^2 / b^2 \) and prime denotes differentiation with respect to \( X \). Following references [24,29] for exponential variation in Young’s modulus \( E \) and density \( \rho \) of the plate material in \( X \) direction as follows:

\[
E = E_0 e^{\mu X}, \quad \rho = \rho_0 e^{\beta X}
\]

where \( E_0 \) and \( \rho_0 \) are the Young’s modulus and density of the plate material at \( X = 0 \), respectively, \( \mu \) is non-homogeneity parameter and \( \beta \) is the density parameter.

Equation (3) now, reduces to

\[
A_0 W^{iv} + A_1 W''' + A_2 W'' + A_3 W' + A_4 W = 0 \tag{5}
\]

where

\[
A_0 = 1, \quad A_1 = 2\mu, \quad A_2 = \mu^2 - 2\lambda^2, \quad A_3 = -2\mu\lambda^2
\]

\[
A_4 = \lambda^4 - \nu \mu^2 \lambda^2 - \Omega^2 e^{(\beta - \mu)X} - \lambda^2 N_0^2 (1 - \gamma X) e^{-\mu X}
\]

\[
\Omega^2 = 12\rho_0 (1 - \nu^2) a^2 \lambda^2 / E_0 h^3, \quad N_0^2 = 12N_0 (1 - \nu^2) / a Eh^3
\]

The solution of Eq. (5) together with the boundary conditions at the edges \( X = 0 \) and \( X = 1 \) constitutes a two-point boundary value problem. Due to the presence of variable coefficients in Eq. (5) its closed form solution is not possible. Hence, an approximate solution is obtained by employing the differential quadrature method.


Let \( X_1, X_2, \ldots, X_N \) be the \( N \) grid points in the applicability range \([0, 1]\) of the plate. According to the DQM, the \( n^{th} \) order derivative of \( W(X) \) with respect to \( X \) can be expressed discretely at the points \( X_i \) as

\[
\frac{d^n W(X_i)}{dX^n} = \sum_{j=1}^{N} c_{ij}^{(n)} W(X_j), \quad i = 1, 2, \ldots, N \tag{6}
\]

where \( c_{ij}^{(n)} \) are the weighting coefficients at discrete point \( X_i \) and given by

\[
c_{ij}^{(1)} = \frac{M^{(1)}(X_i)}{(X_i - X_j) M^{(1)}(X_j)}, \quad i, j = 1, 2, \ldots, N; i \neq j \tag{7}
\]

\[
M^{(1)}(X_i) = \prod_{j=1, j \neq i}^{N} (X_i - X_j), \tag{8}
\]

\[
c_{ij}^{(n)} = n \left( c_{ii}^{(n-1)} c_{ij}^{(1)} - c_{ij}^{(n-1)} c_{ij}^{(1)} / (X_i - X_j) \right) \quad \text{for } i, j = 1, 2, \ldots, N; j \neq i \text{ and } n = 2, 3, 4 \tag{9}
\]

\[
c_{ii}^{(n)} = -\sum_{j=1, j \neq i}^{N} c_{ij}^{(n)} \quad i = 1, 2, \ldots, N \text{ and } n = 1, 2, 3, 4 \tag{10}
\]

Discretizing Eq. (5) at the grid points \( X_i, i = 3, 4, \ldots, N - 2 \), it reduces to,

\[
A_0 W^{iv}(X_i) + A_1 W'''(X_i) + A_2 W''(X_i) + A_3 W'(X_i) + A_4 W(X_i) = 0 \tag{11}
\]
Substitution for $W(X)$ and its derivatives into Eq. (11) at the $i^{th}$ grid point, gives

$$\sum_{j=1}^{N} \left( A_0 c_{ij}^{(4)} + A_1 c_{ij}^{(3)} + A_2 c_{ij}^{(2)} + A_3 c_{ij}^{(1)} \right) W(X_j) + A_4 W(X_i) = 0$$

(12)

For $i = 3, 4, \ldots, (N - 2)$, one obtains a set of $(N - 4)$ equations in terms of unknowns $W_j (≡ W(X_j))$, $j = 1, 2, \ldots, N$, which can be written in the matrix form as

$$[A][W^*] = [0],$$

(13)

where $A$ and $W^*$ are matrices of order $(N - 4) \times N$ and $(N \times 1)$ respectively.

Here, the $(N - 2)$ internal grid points chosen for collocation are the zeroes of shifted Chebyshev polynomial of order $(N - 2)$ with orthogonality range $[0, 1]$ given by

$$X_{k+1} = \frac{1}{2} \left[ 1 + \cos \left( \frac{2k - 1}{N - 2} \pi \right) \right], \quad k = 1, 2, \ldots, N - 2$$

(14)

4. Boundary conditions and frequency equations

The three different combinations of boundary condition namely, C-C, C-S, and C-F have been considered in which first symbol represents the condition at the edge $X = 0$ and the second symbol at the edge $X = 1$ and C, S, F stand for clamped, simply supported and free edge, respectively. The relations that should be satisfied at clamped, simply supported and free edges are

$$W = \frac{dW}{dX} = 0; \quad W = \frac{d^2W}{dX^2} - \nu \lambda^2 W = 0; \quad \text{and}$$

$$W = \frac{d^3W}{dX^3} - \nu \lambda^2 W = \frac{d^3W}{dX^3} - (2 - \nu) \lambda^2 \frac{dW}{dX} = 0,$$

(15)

respectively.

Applying the boundary conditions for C-C plate, one can obtains a set of four homogeneous equations, which can be written as

$$[B_{CC}][W^*] = [0],$$

(17)

Equation (13) together with the Eq. (17) gives a complete set of $N$ equations in $N$ unknowns that can be denoted as

$$\begin{bmatrix} A \\ B_{CC} \end{bmatrix} [W^*] = [0],$$

(18)

For a non-trivial solution of Eq. (18), the frequency determinant must vanish and hence,

$$\left| \begin{bmatrix} A \\ B_{CC} \end{bmatrix} \right| = 0$$

(19)

Similarly for C-S and C-F plates, frequency determinants can be written as

$$\left| \begin{bmatrix} A \\ B_{CS} \end{bmatrix} \right| = 0$$

(20)

and

$$\left| \begin{bmatrix} A \\ B_{CF} \end{bmatrix} \right| = 0,$$

(21)

respectively.
5. Numerical results and discussions

Frequency Eqs (19)–(21) have been solved to obtain the numerical values of frequency parameter $\Omega$ for various values of plate and other parameters taken here. Following references [6,28–32], numerical results have been computed to investigate the effect of in-plane force parameter $N_0^{*} (= -70, -50, -30, 0, 30, 50, 70)$, loading parameter $\gamma (= 0.0, 0.5, 1.0, 1.5, 2.0)$, aspect ratio $a/b (= 0.5, 1.0, 1.5, 2.0)$, non-homogeneity parameter $\mu (= -0.5, -0.3, -0.1, 0.0, 0.1, 0.3, 0.5)$ and density parameter $\beta (= -0.5, -0.3, -0.1, 0.0, 0.1, 0.3, 0.5)$, on natural frequencies for the first three modes of vibration for $p = 1$.

To choose an appropriate number of grid points $N$, convergence studies have been carried out for various set of plate parameters. The normalized frequency parameter $\Omega/\Omega^{*}$ for the first three modes of vibration for a specified plate i.e. $a/b = 1$, $N_0^{*} = 20$, $\gamma = 1$, $\square$, first mode; $\bigtriangleup$, second mode; $+$, third mode. $\Omega^{*} –$ the DQ result using 20 grid points.

Fig. 3. Convergence of normalized frequency parameter $\Omega/\Omega^{*}$ (a) C-C (b) C-S (c) C-F, with grid refinement for $a/b = 1$, $N_0^{*} = 20$, $\gamma = 1$. $\square$, first mode; $\bigtriangleup$, second mode; $+$, third mode. $\Omega^{*} –$ the DQ result using 20 grid points.

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To choose an appropriate number of grid points $N$, convergence studies have been carried out for various set of plate parameters. The normalized frequency parameter $\Omega/\Omega^{*}$ for the first three modes of vibration for a specified plate i.e. $a/b = 1$, $N_0^{*} = 20$, $\gamma = 1$, $\mu = \beta = -0.5$ is shown in Fig. 3. For this data the maximum deviations were observed. The frequency parameter $\Omega$ converges with the increasing number of grid points and the nature of convergence is oscillatory for all the three boundary conditions. The value of $N$ has been fixed as 17, since there was no further improvement in the values of $\Omega$ even at the fourth place of decimal for all three plates.

The results have been reported to six significant digits in Tables 1 and 2 and Figs 4–12. It is observed that the frequency parameter $\Omega$ decreases in the order of boundary conditions C-C, C-S, and C-F for the same set of values of other parameters.

Figure 4(a) shows the plots for frequency parameter $\Omega$ versus in-plane force parameter $N_0^{*}$ for aspect ratio $a/b = 1$, non-homogeneity parameter $\mu = 0.5$, density parameter $\beta = 0.5$ and loading parameter $\gamma = 0, 1, 2$ for the first mode of vibration. It is observed that the frequency parameter $\Omega$ decrease with the increasing values of in-plane force parameter $N_0^{*}$ for $\gamma = 0$ and $\gamma = 1$ for all the three plates. The rate of decrease of $\Omega$ with increasing values of $N_0^{*}$ for a C-F plate is higher than that for a C-S and C-C plates when $\gamma = 0$ and it is higher for C-S plate than that for C-F and C-C when $\gamma = 1$. But, for $\gamma = 2$ the frequency parameter $\Omega$ increases with the increasing values of $N_0^{*}$ for C-S and C-F plates and remains almost same for C-C plate. The rate of increase of frequency parameter $\Omega$ with $N_0^{*}$ is in the order of boundary conditions C-F > C-S. In case of second and third modes of vibration the rate of decrease of $\Omega$ with decreases with the increase in the number of modes in the order of boundary conditions C-F > C-S > C-C, Figs 4(b) and (c).
Fig. 4. Frequency parameter $\Omega$ for (a) first mode (b) second mode (c) third mode. , C-C; ---, C-S; ------, C-F; $a/b = 1, \mu = \beta = 0.5$; □, $\gamma = 0$; +, $\gamma = 1$; △, $\gamma = 2$.

Fig. 5. Frequency parameter $\Omega$ for (a) first mode (b) second mode (c) third mode. , C-C; ---, C-S; ------, C-F; $N_0^* = 30, a/b = 1$; □, $\mu = 0.5, \beta = 0.5$; ■, $\mu = 0.5, \beta = -0.5$; △, $\mu = -0.5, \beta = 0.5$; ▲, $\mu = -0.5, \beta = -0.5$. 
A similar inference can be drawn from Figs 5(b) and (c) when the plate is vibrating in the second and third modes of vibration. However, in case of second and third modes of vibration, the rate of increase of frequency parameter is more pronounced for smaller values of loading parameter $\gamma$. The rate of increase is in the order of boundary conditions C-F > C-S > C-C. A similar inference can be drawn from Figs 5(b) and (c) when the plate is vibrating in the second and third modes, respectively except that the effect of loading parameter $\gamma$ and the rate of increase of $\Omega$ with $\gamma$ decrease with the increase in number of modes.

Table 1

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Figure 5(a) shows the behaviour of frequency parameter $\Omega$ with the increasing values of loading parameter $\gamma$ for a fixed value of in-plane force parameter $N_0 = 30$, aspect ratio $a/b = 1.0$, two values of non-homogeneity parameter $\mu = \pm 0.5$ and density parameter $\beta = \pm 0.5$ for the first mode of vibration. It is found that the effect of non-homogeneity parameter $\mu$ is more pronounced for smaller values of loading parameter $\gamma$ ($< 1$) while for density parameter $\beta$ it is more pronounced for larger values of $\gamma$ ($> 1$) keeping other parameters fixed, for all the three plates. The frequency parameter $\Omega$ increases with the increasing values of loading parameter $\gamma$. The rate of increase of frequency parameter $\Omega$ with loading parameter $\gamma$ is in the order of boundary conditions C-F > C-S > C-C. A similar inference can be drawn from Figs 5(b) and (c) when the plate is vibrating in the second and third modes, respectively except that the effect of loading parameter $\gamma$ and the rate of increase of $\Omega$ with $\gamma$ decrease with the increase in number of modes.

Figure 6(a) depicts the behaviour of frequency parameter $\Omega$ with increasing values of aspect ratio $a/b$ for in-plane force parameter $N_0 = 30$, loading parameter $\gamma = 1.0$, two values of non-homogeneity parameter $\mu = \pm 0.5$ and density parameter $\beta = \pm 0.5$ for the first mode of vibration. It is observed that the frequency parameter $\Omega$ increases with the increasing values of aspect ratio $a/b$ for the same set of values of plate parameters for all the three plates. The rate of increase of frequency parameter $\Omega$ with increasing values of aspect ratio $a/b$ for C-F plate is higher than that for C-S and C-C plates when $\mu = 0.5$ and for C-S plate is higher than that for C-C and C-F plates when $\mu = -0.5$ for both the values of $\beta = \pm 0.5$. This rate is higher for $\mu = 0.5$ as compared to $\mu = -0.5$ for all the three plates. However, in case of second and third modes of vibration i.e. Figures 6(b) and (c), the rate of increase $\Omega$ with $a/b$ for C-F plate is higher than that for C-S and C-C plates whatever be the values of other parameter.

The effect of non-homogeneity parameter $\mu$ on the frequency parameter $\Omega$ for in-plane force parameter $N_0 = 30$, aspect ratio $a/b = 1.0$, two values of density parameter $\beta = \pm 0.5$ and loading parameter $\gamma = 0$, has been shown.
Fig. 6. Frequency parameter $\Omega$ for (a) first mode (b) second mode (c) third mode. _____, C-C; - - - - -, C-S; ......, C-F; $N_0^* = 30$, $\gamma = 1$; □, $\mu = 0.5$, $\beta = 0.5$; ■, $\mu = 0.5$, $\beta = -0.5$; △, $\mu = -0.5$, $\beta = 0.5$; ▲, $\mu = -0.5$, $\beta = -0.5$.

Fig. 7. Frequency parameter $\Omega$ for (a) first mode (b) second mode (c) third mode. _____, C-C; - - - - -, C-S; ......, C-F; $N_0^* = 30$, $a/b = 1$; □, $\beta = 0.5$, $\gamma = 0$; ■, $\beta = 0.5$, $\gamma = 0$; △, $\beta = -0.5$, $\gamma = 1$; ▲, $\beta = 0.5$, $\gamma = 1$. 
Here, no frequencies for C-F plate are obtained as the values of critical buckling loads increase when the parameter $\gamma$ increases with the increasing values of density parameter $\beta$ in Fig. 7. From the Fig. 7(a), when the plate is vibrating in the first mode of vibration it is found that the frequency $\Omega$ for both values of $\beta = \pm 0.5$ and for all the three plates. The rate of increase of frequency parameter $\Omega$ with increasing value of non-homogeneity parameter $\mu$ is in the order of boundary conditions C-C > C-S > C-F. It is higher for $\beta = -0.5$ as compared to $\beta = 0.5$ for all the three plates. For the density parameter $\beta = \pm 0.5$ and loading parameter $\gamma = 0$, the frequency parameter $\Omega$ increases with the increasing values of non-homogeneity parameter $\mu$ for both the C-C and C-S plates. Here, no frequencies for C-F plate are obtained as the values of critical buckling loads $N_{cr}^* (= N_0^*)$ are less than 30. In case of second and third modes of vibration, the behaviour of frequency parameter $\Omega$ is observed to increases continuously with the increasing values of non-homogeneity parameter $\mu$ for the both values of density parameter $\beta = \pm 0.5$ and loading parameter $\gamma = 0, 1$. The rate of increase of frequency parameter $\Omega$ with $\mu$ for $\beta = 0.5$ is smaller than that for $\beta = -0.5$ for all the three plates shown in Figs 7(b) and (c).

Figure 8(a) shows the behaviour of frequency parameter $\Omega$ with increasing values of density parameter $\beta$ for in-plane force parameter $N^*_c$ = 30, aspect ratio $a/b = 1.0$, two values of density parameter $\mu = \pm 0.5$ and loading parameter $\gamma = 0, 1$ for the first mode of vibration. It is found that the frequency parameter $\Omega$ decreases with the increasing values of density parameter $\beta$ for the both values of $\mu = \pm 0.5$ and $\gamma = 1$ for all the three plates. The rate of decrease of frequency parameter $\Omega$ with increasing value of density parameter $\beta$ for C-C plate is higher than that for C-S and C-F plates. For non-homogeneity parameter $\mu = \pm 0.5$ and loading parameter $\gamma = 0$, the frequency parameter $\Omega$ decreases with increasing values of density parameter $\beta$ for both the C-C and C-S plates and no frequencies are found for C-F plate due to occurrence of critical buckling loads $N_{cr}^* (= N_0^*) < 30$. Figures 8(b) and (c) showing the behaviour of frequency parameter $\Omega$ with $\beta$ for second and third modes of vibration, it is found that the frequency parameter $\Omega$ decreases with the increasing values of density parameter $\beta$ for both values of non-homogeneity parameter $\mu = \pm 0.5$ and loading parameter $\gamma = 0, 1$. The rate of decrease of frequency parameter $\Omega$ with $\beta$ is higher for $\mu = 0.5$ as compared to $\mu = -0.5$ for all the three plates.

By allowing the frequency approaches to zero, the values of lowest critical buckling loads $N_{cr}^*$ for different values of aspect ratio $a/b = 0.5, 1.0$, non-homogeneity parameter $\mu = -0.5, 0.0, 0.5$ and loading parameter $\gamma = 0.0, 0.5, 1.0, 1.5, 2.0$ are reported in Tables 1 and 2. The analysis shows that the value of $N_{cr}^*$ does not depend upon the density parameter $\beta$. From Table 1, it is clear that the values of critical buckling loads increase with the increasing value of loading parameter $\gamma$ and non-homogeneity parameter $\mu$ for all the three plates. For C-C and C-S plates the values of critical buckling loads decrease with the increasing values of aspect ratio $a/b$. However, in case of C-F plate the values of critical buckling loads increase when $\gamma \leq 1$ and decrease when $\gamma > 1$ for the same.
Fig. 8. Frequency parameter $\Omega$ for (a) first mode (b) second mode (c) third mode. C-C; - - - - , C-S; ......, C-F; $N_0^* = 30$, $a/b = 1$; □, $\mu = -0.5$, $\gamma = 0$; ■, $\mu = 0.5$, $\gamma = 0$; △, $\mu = -0.5$, $\gamma = 1$; ▲, $\mu = 0.5$, $\gamma = 1$.

Fig. 9. Critical buckling loads $N_{cr}^*$ for (a) $\gamma = 0$ (b) $\gamma = 1$ (c) $\gamma = 2$. C-C; - - - - , C-S; ......, C-F; $\beta = 0.5$; □, $\mu = 0.5$; △, $\mu = -0.5$. 
non-homogeneity parameter $\mu$. For $\gamma = 1.5, 2$, (Fig. 2), as the plate undergoes to compressive and tensile forces simultaneously, one obtains two values of $N_{cr}^*$ (one positive and other negative). The negative values are reported in Table 2. It is observed that for $\gamma = 1.5$, the positive values of $N_{cr}^*$ are smaller in magnitude as compared to the negative values of $N_{cr}^*$ for C-C and C-S plates while for C-F plate the behaviour is just the reverse. In case of $\gamma = 2$, the positive values of $N_{cr}^*$ are greater in magnitude as compared to negative values of $N_{cr}^*$ for all three plates. The graphs for critical buckling loads $N_{cr}^*$ for $\gamma = 0, 1, 2$ are shown in Fig. 9.

For the specified plate i.e. $\mu = \beta = 0.5, a/b = 1, \gamma = 1, N_0^* = 30$, three dimensional mode shapes for all the three plates are shown in Figs 10–12.

A comparison for the convergence study of the frequency parameter $\Omega$ with $N$ for an unloaded ($N_0^* = 0$), homogeneous ($\mu = \beta = 0$), C-C square plate ($a/b = 1$) with quintic spline technique Lal et al. [33], differential quadrature method Wang et al. [11], power series method Leissa and Kang [6] for $p = 1$ has been shown in Table 3. The result shows that the present approach has faster rate of convergence. A comparison of critical buckling
loads \( N_{cr}^* \) for homogeneous \((\mu = 0, \beta = 0)\) C-C plate for the values of loading parameter \( \gamma = 0, 1, 2 \), aspect ratio \( b/a = 0.4, 0.5, 0.6 \) and \( p = 1 \) with Lal and Dhanpati [31] obtained by quintic spline technique, differential quadrature method Wang et al. [11], and power series method Leissa and Kang [6] has been presented in Table 4. An excellent agreement of the results shows the versatility of the technique.

On the suggestion of one of the learned reviewers, the practical situation arising due to \( \mu \neq 0 \) and \( \beta = 0 \) has been analyzed. Physically, this consideration gives rise a type of non-homogeneity arising due to the change in Young’s modulus of the material only. It is found that the values of frequency parameter \( \Omega \) increases with the increasing values of non-homogeneity parameter \( \mu \) for all the three boundary conditions for the same set of values of other parameters. It also increases with the increasing values of aspect ratio \( a/b \). However, the values of frequency parameter \( \Omega \) decreases with the increasing values of loading parameter \( \gamma \) for all the three boundary conditions.
Fig. 12. Vibration modes of C-F plate (a) first mode (b) second mode (c) third mode; for $a/b = 1, \mu = \beta = 0.5, \gamma = 1, N_0^* = 30$.

whatever be the set of values of other parameter. Similarly, it is noticed that the frequency parameter $\Omega$ decreases with the increasing value of in-plane force parameter $N_0^*$, keeping other parameters fixed for all the three plates. The percentage change in the values of lowest frequency parameter $\Omega$ increases with the increasing values of loading parameter $\gamma$ as well as in-plane force parameter $N_0^*$, whatever be the set of values of plate parameters in the order of boundary conditions C-C > C-S > C-F.

6. Conclusions

Differential quadrature method has been used to study the combine effect of linearly varying in-plane force and non-homogeneity of the plate material on the transverse vibration of thin rectangular plates of uniform thickness. The non-homogeneity is assumed to arise due to exponential variation in young modulus and density of the plate material along axial direction. It is observed that the values of frequency parameter $\Omega$ increases with the increasing values of in-plane force parameter $N_0^*$ as the plate become more and more stiff towards the edge $x = 0$ to $x = a$, keeping other plate parameters fixed. However, the values of frequency parameter $\Omega$ decreases as the plate becomes more and more dense towards the edge $x = a$ for fixed values of other plate parameters and this further decreases with the increasing values of in-plane force parameter $N_0^*$. The frequency parameter $\Omega$ also increases with the
increasing value of aspect ratio \(a/b\) and decreases with the increasing value of loading parameter \(\gamma\) for the same set of values of other parameters. A case of pure in-plane bending has been arisen for homogeneous \((\mu = \beta = 0)\) C-C plate when \(\gamma = 2\). The value of critical buckling loads \(N_{cr}^*\) increases with the increasing value of loading parameter \(\gamma\). It is found that for \(\gamma > 1.5\) there exist two values of critical buckling loads \(N_{cr}^*\) (one positive and another negative), as the plate undergoes to compressive and tensile forces simultaneously for all the three boundary conditions. The percentage variation in the value of lowest frequency parameter \(\Omega\) are \(-11.8\) to \(13.3\), \(-10.9\) to \(12.1\), \(-10.9\) to \(13.0\) for C-C, C-S and C-F boundary conditions, respectively, when the non-homogeneity arises due to the change in only \(\mu\) (i.e. \(\beta = 0\)) from \(-0.5\) to \(0.5\) with respect to \(\mu = 0\) for \(\gamma = 0\), \(N_{cr}^* = 0\), \(a/b = 0.5\). This effect increases with the increasing values of aspect ratio \(a/b\) in the order of boundary conditions C-C < C-S < C-F. These variations are \(13.1\) to \(-11.9\), \(15.0\) to \(-13.4\), \(21.9\) to \(-18.1\) for C-C, C-S and C-F boundary conditions, respectively, when the non-homogeneity arises due to the change in only density parameter \(\beta\) (i.e. \(\mu = 0\)) from \(-0.5\) to \(0.5\) with respect to \(\beta = 0\) for the increasing values of aspect ratio \(a/b\). This effect remains almost same for all the three boundary condition. In case of critical buckling loads \(N_{cr}^*\), the percentage variation are \(-22.2\) to \(28.2\), \(-22.4\) to \(28.6\) and \(-27.8\) to \(38.9\) for C-C, C-S and C-F plates, respectively, when \(\mu\) changes from \(-0.5\) to \(0.5\) with respect to \(\mu = 0\) for \(\gamma = 0\) and \(a/b = 1\). These variations remain unchanged due to variation in density parameter \(\beta\). The corresponding changes become \(-19.8\) to \(24.4\), \(-19.9\) to \(24.6\) and \(-23.3\) to \(30.4\) for \(\gamma = 1\). Almost similar percentage variations were obtained for second and third modes of vibration. The present analysis will be of great use to the design engineers in obtaining the desired frequency by varying one or more plate parameters considered here.

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References


