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Buckling and vibration of non-homogeneous rectangular plates subjected to linearly varying in-plane force

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Abstract. The present work analyses the buckling and vibration behaviour of non-homogeneous rectangular plates of uniform thickness on the basis of classical plate theory when the two opposite edges are simply supported and are subjected to linearly varying in-plane force. For non-homogeneity of the plate material it is assumed that young's modulus and density of the plate material vary exponentially along axial direction. The governing partial differential equation of motion of such plates has been reduced to an ordinary differential equation using the sine function for mode shapes between the simply supported edges. This resulting equation has been solved numerically employing differential quadrature method for three different combinations of clamped, simply supported and free boundary conditions at the other two edges. The effect of various parameters has been studied on the natural frequencies for the first three modes of vibration. Critical buckling loads have been computed. Three dimensional mode shapes have been presented. Comparison has been made with the known results.

Keywords: Non-homogeneous, rectangular, buckling, differential quadrature

1. Introduction

Plates of various geometries are the important components in many engineering applications. Of these, rectangular plates with different combinations of boundary conditions are commonly encountered in aerospace, mechanical, nuclear and off-shore structures. In many practical situations, particularly in the ship buildings and automotive industry these plates may be subjected to in-plane dynamic loads of different types, which may induce buckling, a phenomenon which is highly undesirable. In this regard, efforts have been made by researchers to analyses the effect of uniformly/non-uniformly distributed in-plane loads on the vibration characteristics of rectangular plates and prominent ones are reported in references [1–17]. Out of these, Leissa and Kang [6] employed the power series method to obtain the exact solutions for vibration and buckling of rectangular plates having two opposite edges simply supported and these are subjected to linearly varying in-plane stresses while the other two are clamped. Hu et al. [8] investigated the buckling behaviour of a symmetrically laminated composite rectangular plate under parabolic variation of axial loads using Rayleigh-Ritz method. Devarakonda and Bert [14] used Galerkin method to study the buckling of rectangular plate with non-linearly distributed compressive loading on two opposite sides. Kang and Leissa [10] obtained the exact solution for buckling of rectangular plates having linearly varying in-plane

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loading on two opposite simply supported edges, for all combinations of clamped, simply supported or free at the other two edges. Wang et al. [11] obtained the numerical results for the buckling and vibration of isotropic rectangular plate subjected to linearly varying in-plane stresses along two opposite simply supported edges while the other two are clamped using differential quadrature method. Jana and Bhaskar [12] used Galerkin's approach to present the analytical solutions for buckling of rectangular plates under non-uniform biaxial compression. Wang et al. [13] analyzed the buckling of thin rectangular plates with cosine-distributed compressive loads on two opposite sides using differential quadrature method. Recently, Civalek et al. [15] used discrete singular convolution approach for buckling analysis of rectangular Kirchhoff plates subjected to compressive loads on two-opposite edges. Tang and Wang [16] studied the buckling of symmetrically laminated rectangular plates under parabolic edge compression using Rayleigh-Ritz method. Very recently, Eftekhari and Jafari [17] employed the mixed finite element and differential quadrature method for free and forced vibration and buckling analysis of rectangular plates.

Beside the above considerations, in many practical applications, particularly in aerospace industry, modern missile technology and microelectronics, plate type structural elements have to work under high temperature environment which causes non-homogeneity in the material i.e. mechanical properties of the material vary with space variables. However, many structural components possess initial non-homogeneity due to the inclusion of foreign materials or imperfection or being composite materials. Plywood, timber and fibre-reinforced plastic etc. form an important class of non-homogeneous materials. These materials are of considerable interest to design engineers in various technological situations [18–21]. Thus, their design requires an accurate analysis for their vibration characteristics. Up till now, several studies have been devoted to the dynamic behaviour of non-homogeneous plates of various geometries and reported in references [22–30], to mention a few. In these references various model such as linear, quadratic, exponential etc. for the Young modulus and density of the plate material have been considered. Recently, in two papers, namely Lal and Dhanpati [31] and Kumar and Lal [32] studied the combined effect of constant biaxial/uniaxial in-plane forces and non-homogeneity of the plate material on the transverse vibrations of rectangular plates of unidirectional/bidirectional varying thickness using quintic spline technique and Rayleigh-Ritz method, respectively.

The present paper analyzes the effect of linearly varying in-plane force together with non-homogeneity of the plate material on the transverse vibration of rectangular plates on the basis of classical plate theory. The two opposite edges $y = 0$ and $y = b$ are taken simply supported and these are subjected to linearly varying in-plane force. Non-homogeneity of the plate material is assumed to arise due to exponential variation in Young's modulus and density of the plate along axial direction. The Poisson ratio ν is assumed to remain constant. The governing partial differential equation of motion for such plates has been reduced to an ordinary differential equation using the sine function for mode shapes between the simply supported edges. This resulting equation has been solved numerically employing differential quadrature method for three different combinations of clamped, simply supported and free boundary conditions at the other two edges. The effect of non-homogeneity parameter, density parameter, aspect ratio, in-plane force parameter and loading parameter on the natural frequencies has been illustrated for the first three modes of vibration. Three dimensional mode shapes have been presented for specified plates.

2. Mathematical formulation

Consider a non-homogeneous isotropic rectangular plate of length a, breadth b, thickness h and density ρ. The plate is referred to a system of rectangular Cartesian co-ordinates (x, y, z), the middle surface being $z = 0$ and origin is at the one of the corners of the plate. The x- and y-axes are taken along the edges of the plate, the axis of z is perpendicular to the xy-plane and a linearly varying compressive in-plane force N_y is applied along the two opposite simply supported edges $y = 0$ and $y = b$ as shown in Fig. 1. Following Leissa and Kang [6] with the incorporation of non-homogeneity and assuming $q = N_{xy} = N_x = 0$, the differential equation governing the transverse vibration of such plates is given by

$$\begin{aligned} &D\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4}\right) + 2\frac{\partial D}{\partial x}\frac{\partial^3 w}{\partial x^3} + \frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2\frac{\partial D}{\partial x}\frac{\partial^3 w}{\partial x \partial y^2} + 2D\frac{\partial^4 w}{\partial x^2 \partial y^2} \\ &\quad + \rho h \frac{\partial^2 w}{\partial t^2} - N_y \frac{\partial^2 w}{\partial y^2} = 0, \end{aligned} \tag{1}$$

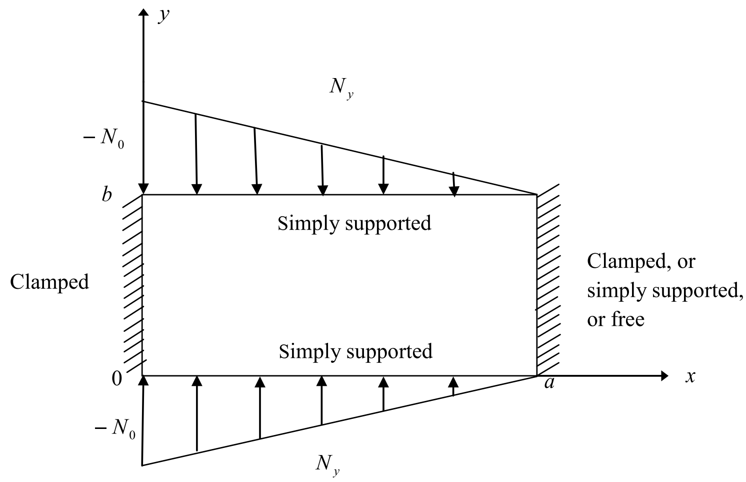

Fig. 1. Rectangular plate under compressive force $N_y = -N_0(1 - \gamma x/a), \gamma = 1$.

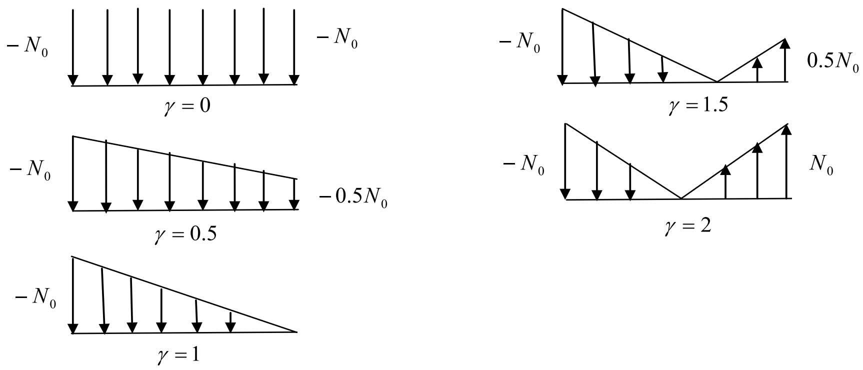

Fig. 2. Example of varying in-plane force N_y along the edge $y = b$.

where $D = Eh^3/12(1 - \nu^2)$, $N_y = -N_0(1 - \gamma x/a)$, $w(x, y, t)$ is the transverse deflection, t the time, D the flexural rigidity, E Young's modulus, ν Poisson ratio, N_0 the intensity of compressive force at the edge $x = 0$ and γ is the loading parameter.

For different values of loading parameter γ, one can obtain various particular cases e.g. $\gamma = 0$ gives the case of uniformly distributed compressive force while for $\gamma = 1$, the compressive force varies linearly from $-N_0$ at $x = 0$ to zero at $x = a$. Except these values of γ in the range $0 < \gamma \leqslant 2$, various combinations of bending and compression are obtained as shown in Fig. 2.

For a harmonic solution, the deflection w is assumed to be

$$w(x, y, t) = \overline{w}(x) \sin(p\pi y/b) e^{i\omega t}, \tag{2}$$

where p is a positive integer and ω is the frequency in rad/s.

Further, for elastically non-homogeneous material, it is assumed that the Young's modulus E and density ρ are the functions of space variable x only.

Introducing the non-dimensional variables $X = x/a, Y = y/b, \bar{h} = h/a, W = \bar{w}/a$, Eq. (1) reduces to

$$\begin{aligned} &E\bar{h}^3 W^{iv} + 2E'\bar{h}^3 W^{'''} + (E^{''}\bar{h}^3 - 2\lambda^2 E\bar{h}^3)W^{''} - 2E'\bar{h}^3\lambda^2 W' \\ &\quad + (E\bar{h}^3\lambda^4 - \nu E^{''}\lambda^2\bar{h}^3 - 12\bar{h}\rho\omega^2 a^2(1 - \nu^2) - N_0(1 - \gamma X)(1 - \nu^2)12\lambda^2)W = 0 \end{aligned} \tag{3}$$

where $\lambda^2 = \pi^2 p^2 a^2/b^2$ and prime denotes differentiation with respect to X. Following references [24,29] for exponential variation in Young's modulus E and density ρ of the plate material in X direction as follows:

$$E = E_0 e^{\mu X}, \quad \rho = \rho_0 e^{\beta X} \tag{4}$$

where E_0 and ρ_0 are the Young's modulus and density of the plate material at $X = 0$, respectively, μ is non-homogeneity parameter and β is the density parameter.

Equation (3) now, reduces to

$$A_0 W^{iv} + A_1 W''' + A_2 W'' + A_3 W' + A_4 W = 0 \tag{5}$$

where

$$A_0 = 1, A_1 = 2\mu, A_2 = \mu^2 - 2\lambda^2, A_3 = -2\mu\lambda^2$$
$$A_4 = \lambda^4 - \nu\mu^2\lambda^2 - \Omega^2 e^{(\beta-\mu)X} - \lambda^2 N_0^*(1 - \gamma X)e^{-\mu X}$$
$$\Omega^2 = 12\rho_0(1 - \nu^2)a^2\omega^2/E_0\bar{h}^3, N_0^* = 12N_0(1 - \nu^2)/aE\bar{h}^3$$

The solution of Eq. (5) together with the boundary conditions at the edges $X = 0$ and $X = 1$ constitutes a two-point boundary value problem. Due to the presence of variable coefficients in Eq. (5) its closed form solution is not possible. Hence, an approximate solution is obtained by employing the differential quadrature method.

3. Method of solution: Differential quadrature method

Let $X_1, X_2, \ldots, X_N$ be the N grid points in the applicability range [0, 1] of the plate. According to the DQM, the n^{th} order derivative of $W(X)$ with respect to X can be expressed discretely at the points X_i as

$$\frac{d^n W(X_i)}{dX^n} = \sum_{j=1}^{N} c_{ij}^{(n)} W(X_j), \quad i = 1, 2\ldots, N \tag{6}$$

where $c_{ij}^{(n)}$ are the weighting coefficients at discrete point X_i and given by

$$c_{ij}^{(1)} = \frac{M^{(1)}(X_i)}{(X_i - X_j)M^{(1)}(X_j)}, \quad i, j = 1, 2, \ldots, N; i \neq j \tag{7}$$

$$M^{(1)}(X_i) = \prod_{\substack{j=1\\ j\neq i}}^{N} (X_i - X_j), \tag{8}$$

$$c_{ij}^{(n)} = n\left(c_{ii}^{(n-1)}c_{ij}^{(1)} - \frac{c_{ij}^{(n-1)}}{(X_i - X_j)}\right) \quad \text{for } i, j = 1, 2\ldots, N, j \neq i \text{ and } n = 2, 3, 4 \tag{9}$$

$$c_{ii}^{(n)} = -\sum_{\substack{j=1\\ j\neq i}}^{N} c_{ij}^{(n)} \quad i = 1, 2, \ldots, N \text{ and } n = 1, 2, 3, 4 \tag{10}$$

Discretizing Eq. (5) at the grid points $X_i, i = 3, 4, \ldots, N - 2$, it reduces to,

$$A_0 W^{iv}(X_i) + A_1 W'''(X_i) + A_2 W''(X_i) + A_3 W'(X_i) + A_{4,i} W(X_i) = 0 \tag{11}$$

Substitution for $W(X)$ and its derivatives into Eq. (11) at the i^{th} grid point, gives

$$\sum_{j=1}^{N}\left(A_0 c_{ij}^{(4)} + A_1 c_{ij}^{(3)} + A_2 c_{ij}^{(2)} + A_3 c_{ij}^{(1)}\right)W(X_j) + A_{4,i}W(X_i) = 0 \tag{12}$$

For $i = 3, 4, \ldots, (N-2)$, one obtains a set of $(N-4)$ equations in terms of unknowns $W_j\,(\equiv W(X_j))\,, j = 1, 2, \ldots, N$, which can be written in the matrix form as

$$[A][W^*] = [0], \tag{13}$$

where A and W^* are matrices of order $(N-4) \times N$ and $(N \times 1)$ respectively.

Here, the $(N-2)$ internal grid points chosen for collocation are the zeroes of shifted Chebyshev polynomial of order $(N-2)$ with orthogonality range [0, 1] given by

$$X_{k+1} = \frac{1}{2}\left[1 + \cos\left(\frac{2k-1}{N-2}\frac{\pi}{2}\right)\right], \quad k = 1, 2, \ldots, N-2 \tag{14}$$

4. Boundary conditions and frequency equations

The three different combinations of boundary condition namely, C-C, C-S, and C-F have been considered in which first symbol represents the condition at the edge $X = 0$ and the second symbol at the edge $X = 1$ and C, S, F stand for clamped, simply supported and free edge, respectively. The relations that should be satisfied at clamped, simply supported and free edges are

$$W = \frac{dW}{dX} = 0; \quad W = \frac{d^2W}{dX^2} - \nu\lambda^2 W = 0; \text{ and} \tag{15}$$

$$W = \frac{d^2W}{dX^2} - \nu\lambda^2 W = \frac{d^3W}{dX^3} - (2-\nu)\lambda^2\frac{dW}{dX} = 0, \text{ respectively.} \tag{16}$$

Applying the boundary conditions for C-C plate, one can obtains a set of four homogeneous equations, which can be written as

$$[B^{CC}][W^*] = [0], \tag{17}$$

Equation (13) together with the Eq. (17) gives a complete set of N equations in N unknowns that can be denoted as

$$\begin{bmatrix} A \\ B^{CC} \end{bmatrix}[W^*] = [0]\,, \tag{18}$$

For a non-trivial solution of Eq. (18), the frequency determinant must vanish and hence,

$$\begin{vmatrix} A \\ B^{CC} \end{vmatrix} = 0 \tag{19}$$

Similarly for C-S and C-F plates, frequency determinants can be written as

$$\begin{vmatrix} A \\ B^{CS} \end{vmatrix} = 0 \tag{20}$$

and

$$\begin{vmatrix} A \\ B^{CF} \end{vmatrix} = 0, \tag{21}$$

respectively.

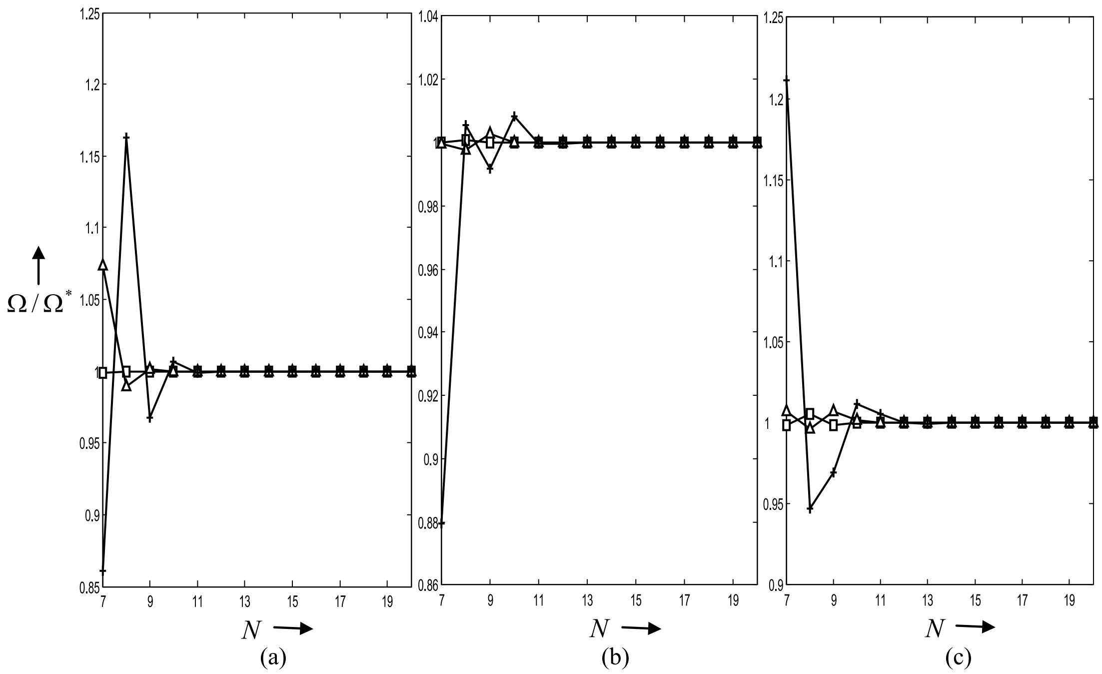

Fig. 3. Convergence of normalized frequency parameter Ω/Ω^* (a) C-C (b) C-S (c) C-F, with grid refinement for $a/b = 1, N_0^* = 20, \gamma = 1$. □, first mode; △, second mode; +, third mode. Ω^* – the DQ result using 20 grid points.

5. Numerical results and discussions

Frequency Eqs (19)–(21) have been solved to obtain the numerical values of frequency parameter Ω for various values of plate and other parameters taken here. Following references [6,28–32], numerical results have been computed to investigate the effect of in-plane force parameter $N_0^*(= -70, -50, -30, 0, 30, 50, 70)$, loading parameter $\gamma(= 0.0, 0.5, 1.0, 1.5, 2.0)$, aspect ratio $a/b(= 0.5, 1.0, 1.5, 2.0)$, non-homogeneity parameter $\mu(= -0.5, -0.3, -0.1, 0.0, 0.1, 0.3, 0.5)$ and density parameter $\beta(= -0.5, -0.3, -0.1, 0.0, 0.1, 0.3, 0.5)$, on natural frequencies for the first three modes of vibration for $p = 1$.

To choose an appropriate number of grid points N, convergence studies have been carried out for various set of plate parameters. The normalized frequency parameter Ω/Ω^* for the first three modes of vibration for a specified plate i.e. $a/b = 1, N_0^* = 20, \gamma = 1, \mu = \beta = -0.5$ is shown in Fig. 3. For this data the maximum deviations were observed. The frequency parameter Ω converges with the increasing number of grid points and the nature of convergence is oscillatory for all the three boundary conditions. The value of N has been fixed as 17, since there was no further improvement in the values of Ω even at the fourth place of decimal for all three plates.

The results have been reported to six significant digits in Tables 1 and 2 and Figs 4–12. It is observed that the frequency parameter Ω decreases in the order of boundary conditions C-C, C-S, and C-F for the same set of values of other parameters.

Figure 4(a) shows the plots for frequency parameter Ω versus in-plane force parameter N_0^* for aspect ratio $a/b = 1$, non-homogeneity parameter $\mu = 0.5$, density parameter $\beta = 0.5$ and loading parameter $\gamma = 0, 1, 2$ for the first mode of vibration. It is observed that the frequency parameter Ω decrease with the increasing values of in-plane force parameter N_0^* for $\gamma = 0$ and $\gamma = 1$ for all the three plates. The rate of decrease of Ω with increasing values of N_0^* for a C-F plate is higher than that for a C-S and C-C plates when $\gamma = 0$ and it is higher for C-S plate than that for C-F and C-C when $\gamma = 1$. But, for $\gamma = 2$ the frequency parameter Ω increases with the increasing values of N_0^* for C-S and C-F plates and remains almost same for C-C plate. The rate of increase of frequency parameter Ω with N_0^* is in the order of boundary conditions C-F > C-S. In case of second and third modes of vibration the rate of decrease of Ω with decreases with the increase in the number of modes in the order of boundary conditions C-F > C-S > C-C, Figs 4(b) and (c).

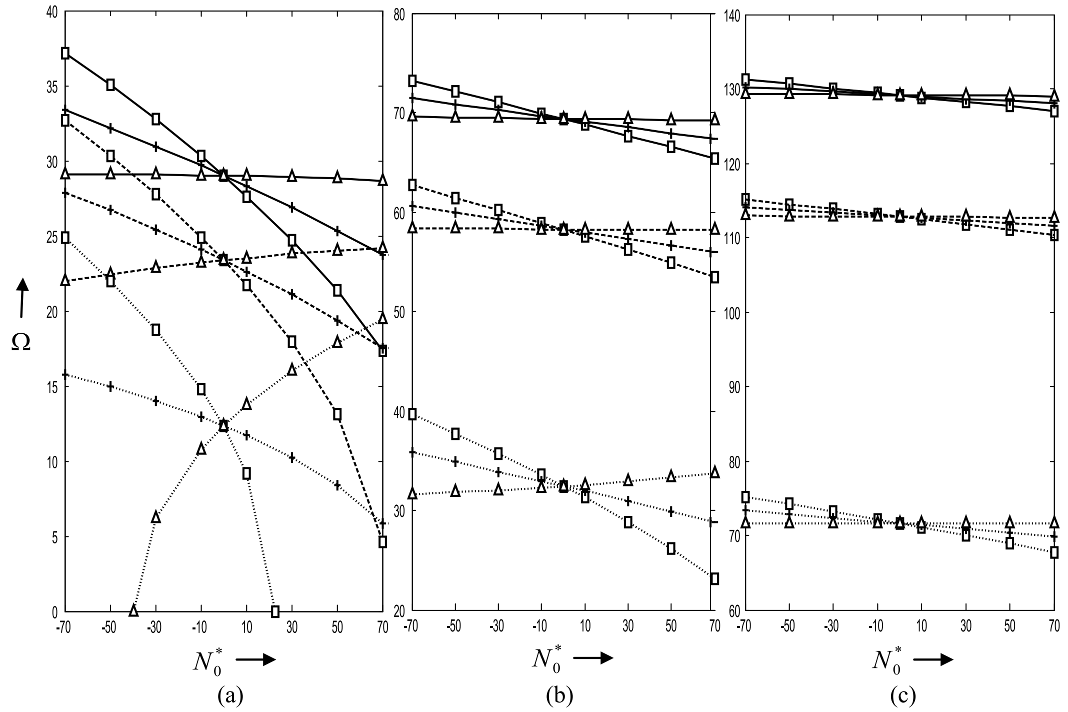

Fig. 4. Frequency parameter Ω for (a) first mode (b) second mode (c) third mode. _____, C-C; - - - - -, C-S; . . . . . ., C-F; $a/b = 1$, $\mu = \beta = 0.5$; □, $\gamma = 0$; +, $\gamma = 1$; △, $\gamma = 2$.

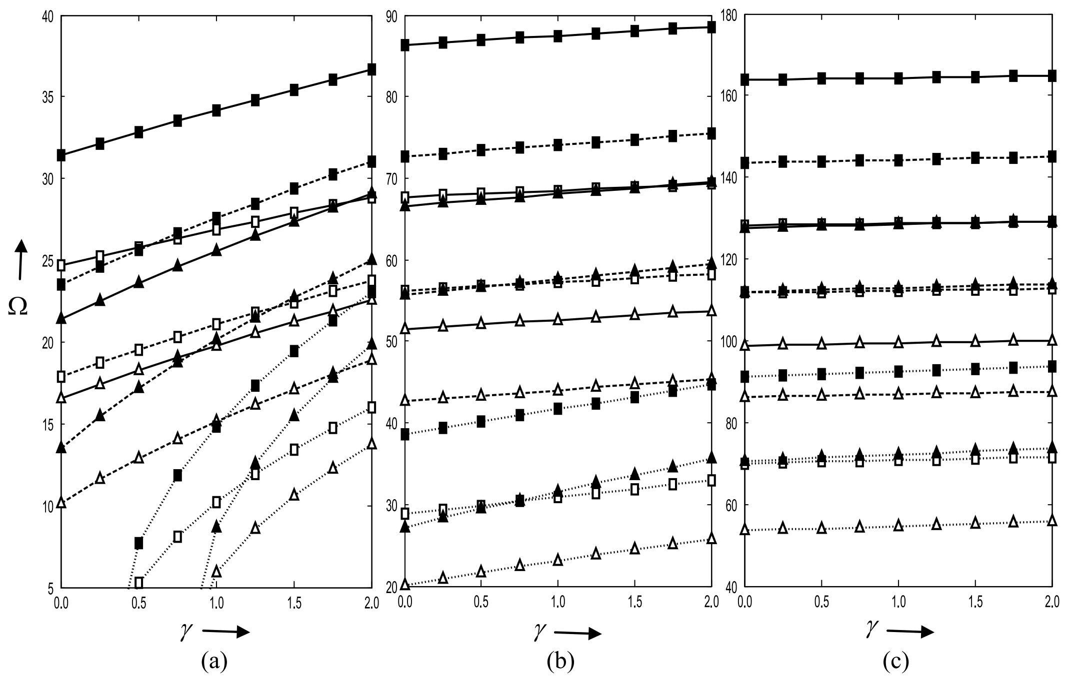

Fig. 5. Frequency parameter Ω for (a) first mode (b) second mode (c) third mode. _____, C-C; - - - - -, C-S; . . . . . ., C-F; $N_0^* = 30$, $a/b = 1$; □, $\mu = 0.5$, $\beta = 0.5$; ■, $\mu = 0.5$, $\beta = -0.5$; △, $\mu = -0.5$, $\beta = 0.5$; ▲, $\mu = -0.5$, $\beta = -0.5$.

Table 1
Values of lowest critical buckling loads N^*_{cr} for C-C, C-S, and C-F plates

Boundary conditions	γ	0.0	0.5	1.0	1.5	2.0
	μ			$a/b = 0.5$		
C-C	-0.5	178.890	240.479	363.108	690.683	1797.39
	-0.3	197.799	264.857	396.839	741.166	1879.81
	0.0	229.871	306.006	453.312	824.632	2012.26
	0.3	267.000	353.370	517.693	918.421	2156.51
	0.5	294.940	388.845	565.540	987.317	2260.02
C-S	-0.5	96.6013	135.681	225.025	551.639	1794.94
	-0.3	106.002	148.360	244.102	584.845	1875.29
	0.0	121.743	169.500	275.646	638.949	2002.67
	0.3	139.682	193.466	311.066	698.683	2138.65
	0.5	153.000	211.186	337.050	741.921	2234.30
C-F	-0.5	10.4609	17.4672	51.3814	547.997	1767.45
	-0.3	11.4614	19.0965	55.5814	579.687	1843.16
	0.0	13.1856	21.8973	62.7265	630.801	1962.95
	0.3	15.2300	25.2083	71.0635	686.533	2090.74
	0.5	16.8055	27.7530	77.3985	726.444	2180.71
				$a/b = 1.0$		
C-C	-0.5	66.0304	88.8009	133.780	248.635	581.822
	-0.3	73.0507	97.8211	146.139	266.621	609.811
	0.0	84.9225	112.992	166.738	296.213	654.752
	0.3	98.6082	130.368	190.086	329.261	703.601
	0.5	108.866	143.324	207.348	353.407	738.559
C-S	-0.5	43.9425	61.4897	100.463	223.056	581.521
	-0.3	48.6598	67.8000	109.728	237.970	609.222
	0.0	56.6536	78.4334	125.180	262.420	653.439
	0.3	65.8938	90.6433	142.707	289.622	701.089
	0.5	72.8365	99.7673	155.676	309.444	734.907
C-F	-0.5	11.7789	19.4580	51.6029	221.883	578.063
	-0.3	13.4111	22.0653	57.3747	236.329	605.274
	0.0	16.3096	26.6648	67.2763	259.863	648.764
	0.3	19.8554	32.2440	78.8859	285.850	695.741
	0.5	22.6487	36.6065	87.7078	304.669	729.170

Figure 5(a) shows the behaviour of frequency parameter Ω with the increasing values of loading parameter γ for a fixed value of in-plane force parameter $N_0^* = 30$, aspect ratio $a/b = 1.0$, two values of non-homogeneity parameter $\mu = \pm 0.5$ and density parameter $\beta = \pm 0.5$ for the first mode of vibration. It is found that the effect of non-homogeneity parameter μ is more pronounced for smaller values of loading parameter $\gamma(< 1)$ while for density parameter β it is more pronounced for larger values of $\gamma(> 1)$ keeping other parameters fixed, for all the three plates. The frequency parameter Ω increases with the increasing values of loading parameter γ. The rate of increase of frequency parameter Ω with loading parameter γ is in the order of boundary conditions C-F $>$ C-S $>$ C-C. A similar inference can be drawn from Figs 5(b) and (c) when the plate is vibrating in the second and third modes, respectively except that the effect of loading parameter γ and the rate of increase of Ω with γ decrease with the increase in number of modes.

Figure 6(a) depicts the behaviour of frequency parameter Ω with increasing values of aspect ratio a/b for in-plane force parameter $N_0^* = 30$, loading parameter $\gamma = 1.0$, two values of non-homogeneity parameter $\mu = \pm 0.5$ and density parameter $\beta = \pm 0.5$ for the first mode of vibration. It is observed that the frequency parameter Ω increases with the increasing values of aspect ratio a/b for the same set of values of plate parameters for all the three plates. The rate of increase of frequency parameter Ω with increasing values of aspect ratio a/b for C-F plate is higher than that for C-S and C-C plates when $\mu = 0.5$ and for C-S plate is higher than that for C-C and C-F plates when $\mu = -0.5$ for both the values of $\beta = \pm 0.5$. This rate is higher for $\mu = 0.5$ as compared to $\mu = -0.5$ for all the three plates. However, in case of second and third modes of vibration i.e. Figures 6(b) and (c), the rate of increase Ω with a/b for C-F plate is higher than that for C-S and C-C plates whatever be the values of other parameter.

The effect of non-homogeneity parameter μ on the frequency parameter Ω for in-plane force parameter $N_0^* = 30$, aspect ratio $a/b = 1.0$, two values of density parameter $\beta = \pm 0.5$ and loading parameter $\gamma = 0, 1$ has been shown

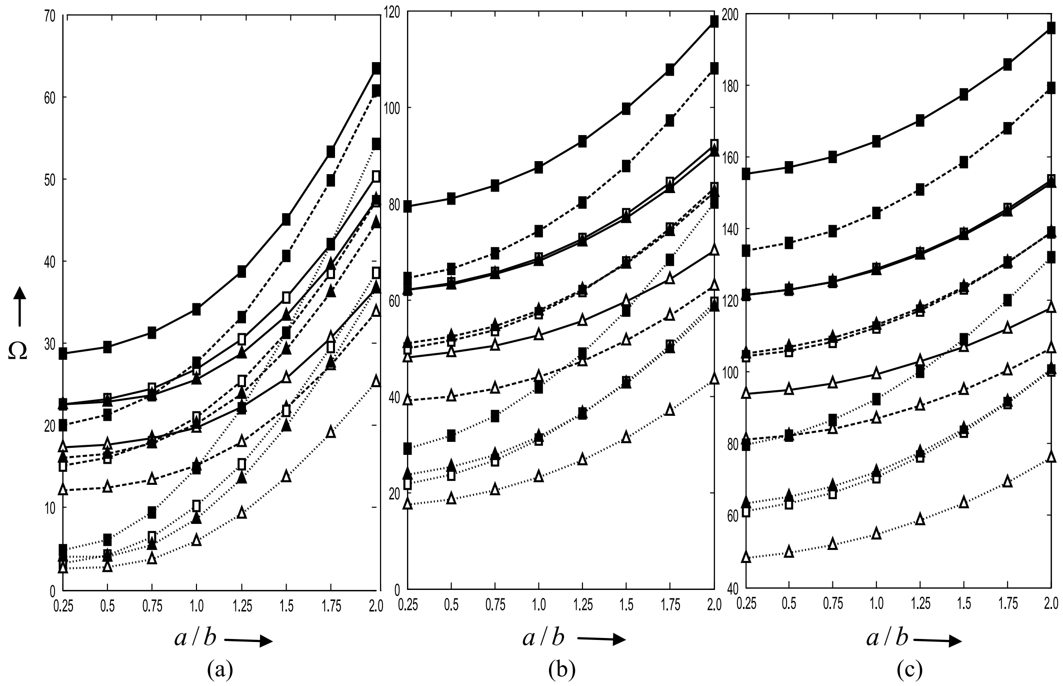

Fig. 6. Frequency parameter Ω for (a) first mode (b) second mode (c) third mode. _____, C-C; - - - - -, C-S; ……, C-F; $N_0^* = 30$, $\gamma = 1$; □, $\mu = 0.5$, $\beta = 0.5$; ■, $\mu = 0.5$, $\beta = -0.5$; △, $\mu = -0.5$, $\beta = 0.5$; ▲, $\mu = -0.5$, $\beta = -0.5$.

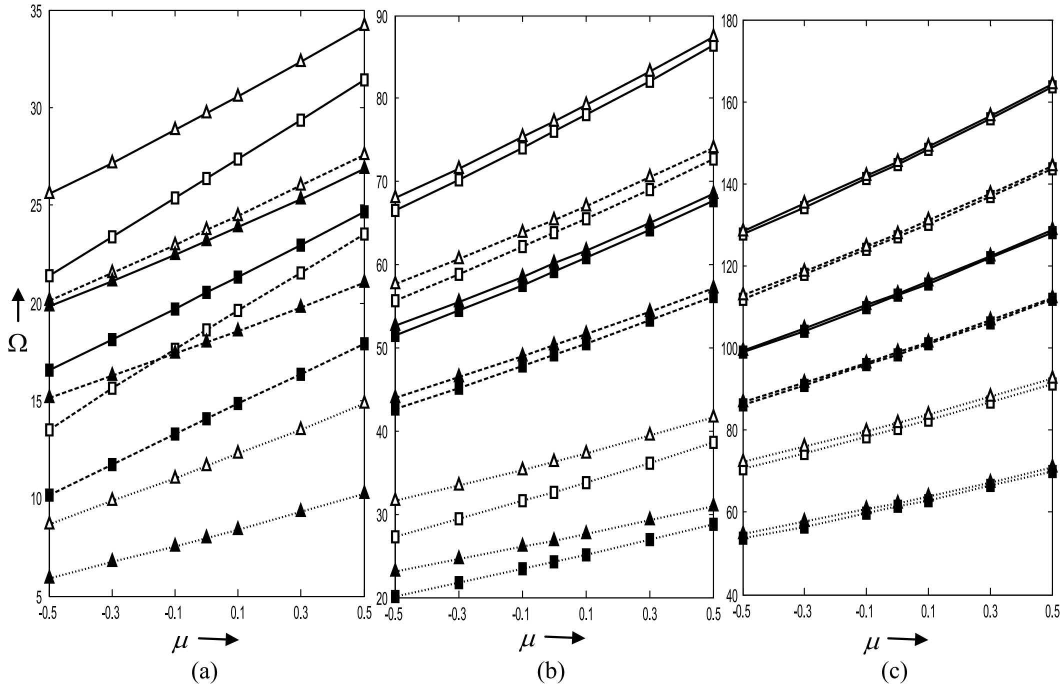

Fig. 7. Frequency parameter Ω for (a) first mode (b) second mode (c) third mode. _____, C-C; - - - - -, C-S; ……, C-F; $N_0^* = 30$, $a/b = 1$; □, $\beta = -0.5$, $\gamma = 0$; ■, $\beta = 0.5$, $\gamma = 0$; △, $\beta = -0.5$, $\gamma = 1$; ▲, $\beta = 0.5$, $\gamma = 1$.

Table 2
Values of lowest critical buckling loads N_{cr}^* for C-C, C-S, and C-F plates

Boundary conditions		C-C	C-S	C-F
μ	γ		$a/b = 0.5$	
−0.5	1.5	−12648.1	−3627.85	−47.9969
	2.0	−1370.77	−452.646	−17.0050
−0.3	1.5	−14989.5	−4240.25	−53.4283
	2.0	−1597.58	−515.897	−18.7818
0.0	1.5	−19344.3	−5358.23	−62.9721
	2.0	−2012.26	−628.192	−21.8737
0.3	1.5	−24972.4	−6771.17	−74.5623
	2.0	−2537.48	−765.610	−25.5844
0.5	1.5	−29612.4	−7914.61	−83.6780
	2.0	−2963.39	−873.968	−28.4737
			$a/b = 1.0$	
−0.5	1.5	−3563.18	−1216.68	−52.4509
	2.0	−447.959	−192.307	−19.2732
−0.3	1.5	−4219.87	−1426.22	−60.8963
	2.0	−521.240	−221.192	−22.2149
0.0	1.5	−5440.13	−1810.38	−76.3502
	2.0	−654.752	−273.071	−27.5458
0.3	1.5	−7015.42	−2298.44	−95.9753
	2.0	−823.159	−337.435	−34.2372
0.5	1.5	−8312.96	−2695.16	−111.945
	2.0	−959.261	−388.766	−39.6298

in Fig. 7. From the Fig. 7(a), when the plate is vibrating in the first mode of vibration it is found that the frequency parameter Ω increases with the increasing values of non-homogeneity parameter μ for both values of $\beta = \pm 0.5$ and for all the three plates. The rate of increase of frequency parameter Ω with increasing value of non-homogeneity parameter μ is in the order of boundary conditions C-C > C-S > C-F. It is higher for $\beta = -0.5$ as compared to $\beta = 0.5$ for all the three plates. For the density parameter $\beta = \pm 0.5$ and loading parameter $\gamma = 0$, the frequency parameter Ω increases with the increasing values of non-homogeneity parameter μ for both the C-C and C-S plates. Here, no frequencies for C-F plate are obtained as the values of critical buckling loads $N_{cr}^* (= N_0^*)$ are less than 30. In case of second and third modes of vibration, the behaviour of frequency parameter Ω is observed to increases continuously with the increasing values of non-homogeneity parameter μ for the both values of density parameter $\beta = \pm 0.5$ and loading parameter $\gamma = 0, 1$. The rate of increase of frequency parameter Ω with μ for $\beta = 0.5$ is smaller than that for $\beta = -0.5$ for all the three plates shown in Figs 7(b) and (c).

Figure 8(a) shows the behaviour of frequency parameter Ω with increasing values of density parameter β for in-plane force parameter $N_0^* = 30$, aspect ratio $a/b = 1.0$, two values of density parameter $\mu = \pm 0.5$ and loading parameter $\gamma = 0, 1$ for the first mode of vibration. It is found that the frequency parameter Ω decreases with the increasing values of density parameter β for the both values of $\mu = \pm 0.5$ and $\gamma = 1$ for all the three plates. The rate of decrease of frequency parameter Ω with increasing value of density parameter β for C-C plate is higher than that for C-S and C-F plates. For non-homogeneity parameter $\mu = \pm 0.5$ and loading parameter $\gamma = 0$, the frequency parameter Ω decreases with increasing values of density parameter β for both the C-C and C-S plates and no frequencies are found for C-F plate due to occurrence of critical buckling loads $N_{cr}^* (= N_0^*) < 30$. Figures 8(b) and (c) showing the behaviour of frequency parameter Ω with β for second and third modes of vibration, it is found that the frequency parameter Ω decreases with the increasing values of density parameter β for both values of non-homogeneity parameter $\mu = \pm 0.5$ and loading parameter $\gamma = 0, 1$. The rate of decrease of frequency parameter Ω with β is higher for $\mu = 0.5$ as compared to $\mu = -0.5$ for all the three plates.

By allowing the frequency approaches to zero, the values of lowest critical buckling loads N_{cr}^* for different values of aspect ratio $a/b = 0.5, 1.0$, non-homogeneity parameter $\mu = -0.5, 0.0, 0.5$ and loading parameter $\gamma = 0.0, 0.5, 1.0, 1.5, 2.0$ are reported in Tables 1 and 2. The analysis shows that the value of N_{cr}^* does not depend upon the density parameter β. From Table 1, it is clear that the values of critical buckling loads increase with the increasing value of loading parameter γ and non-homogeneity parameter μ for all the three plates. For C-C and C-S plates the values of critical buckling loads decrease with the increasing values of aspect ratio a/b. However, in case of C-F plate the values of critical buckling loads increase when $\gamma \leqslant 1$ and decrease when $\gamma > 1$ for the same

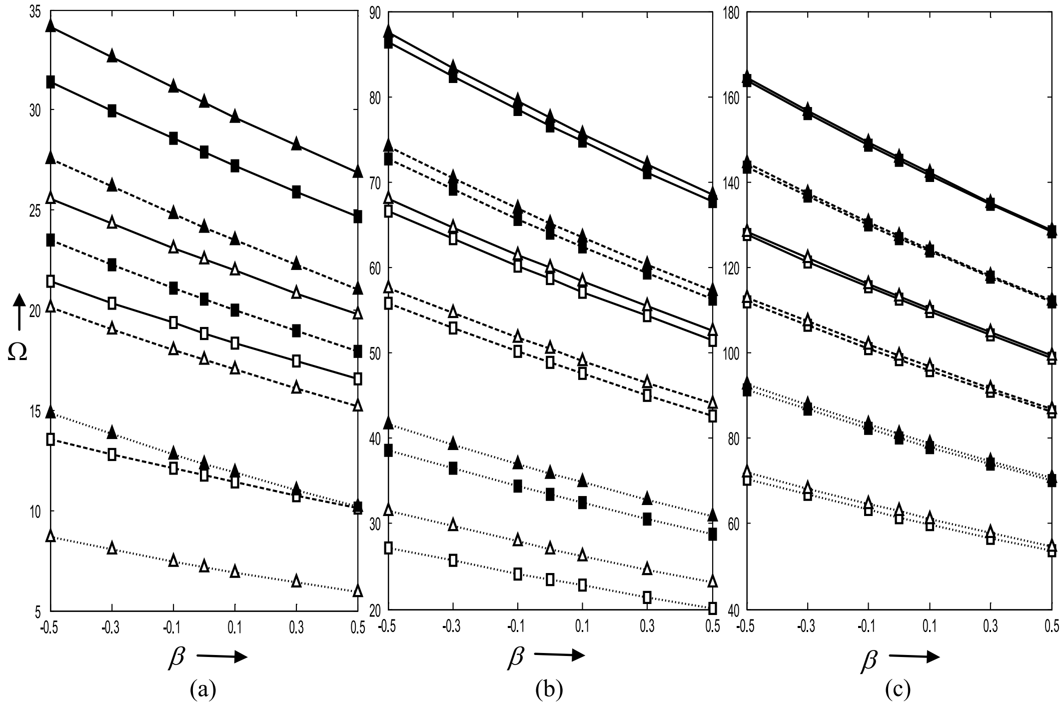

Fig. 8. Frequency parameter Ω for (a) first mode (b) second mode (c) third mode. _____, C-C; - - - - -, C-S; ……, C-F; $N_0^* = 30$, $a/b = 1$; □, $\mu = -0.5$, $\gamma = 0$; ■, $\mu = 0.5$, $\gamma = 0$; △, $\mu = -0.5$, $\gamma = 1$; ▲, $\mu = 0.5$, $\gamma = 1$.

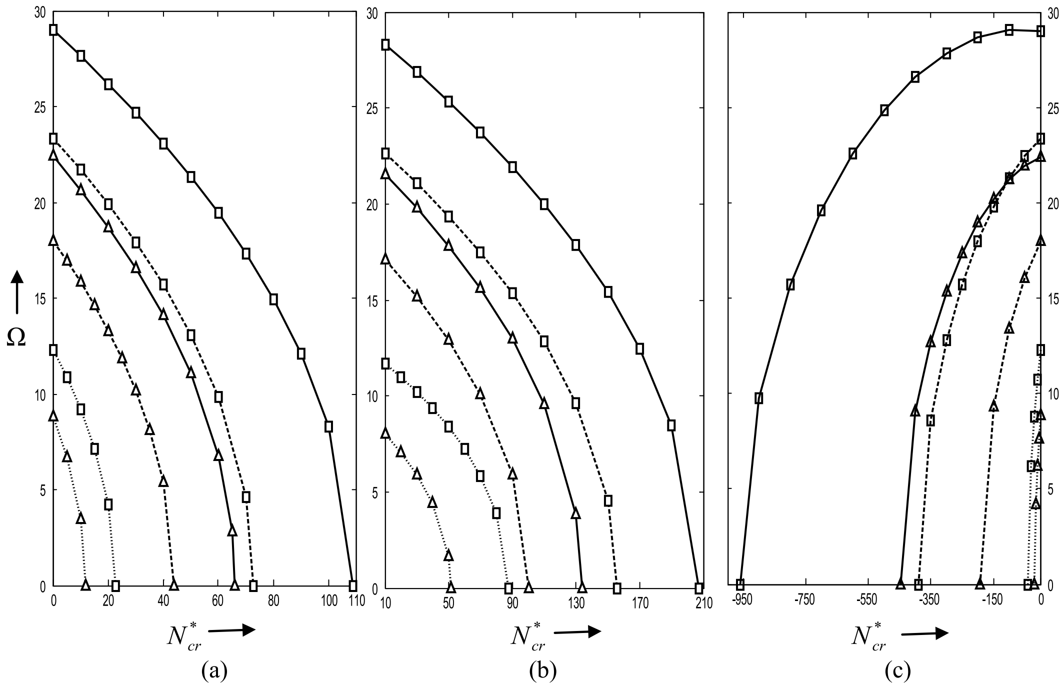

Fig. 9. Critical buckling loads N_{cr}^* for (a) $\gamma = 0$ (b) $\gamma = 1$ (c) $\gamma = 2$. _____, C-C; - - - - -, C-S; ……, C-F; $a/b = 1$, $\beta = 0.5$; □, $\mu = 0.5$; △, $\mu = -0.5$.

Table 3
Convergence of frequency parameter Ω for homogeneous ($\mu = \beta = 0$) isotropic C-C plate; for $b/a = 1, N_0^* = 0, p = 1$

Ref.	N	I mode	N	II mode	N	III mode
Present	10	28.9509	12	69.327	14	129.096
33	40	28.9499	40	69.3796	–	–
11	13	28.9509	14	69.327	17	129.09
6	22	28.95	29	69.33	36	129.1

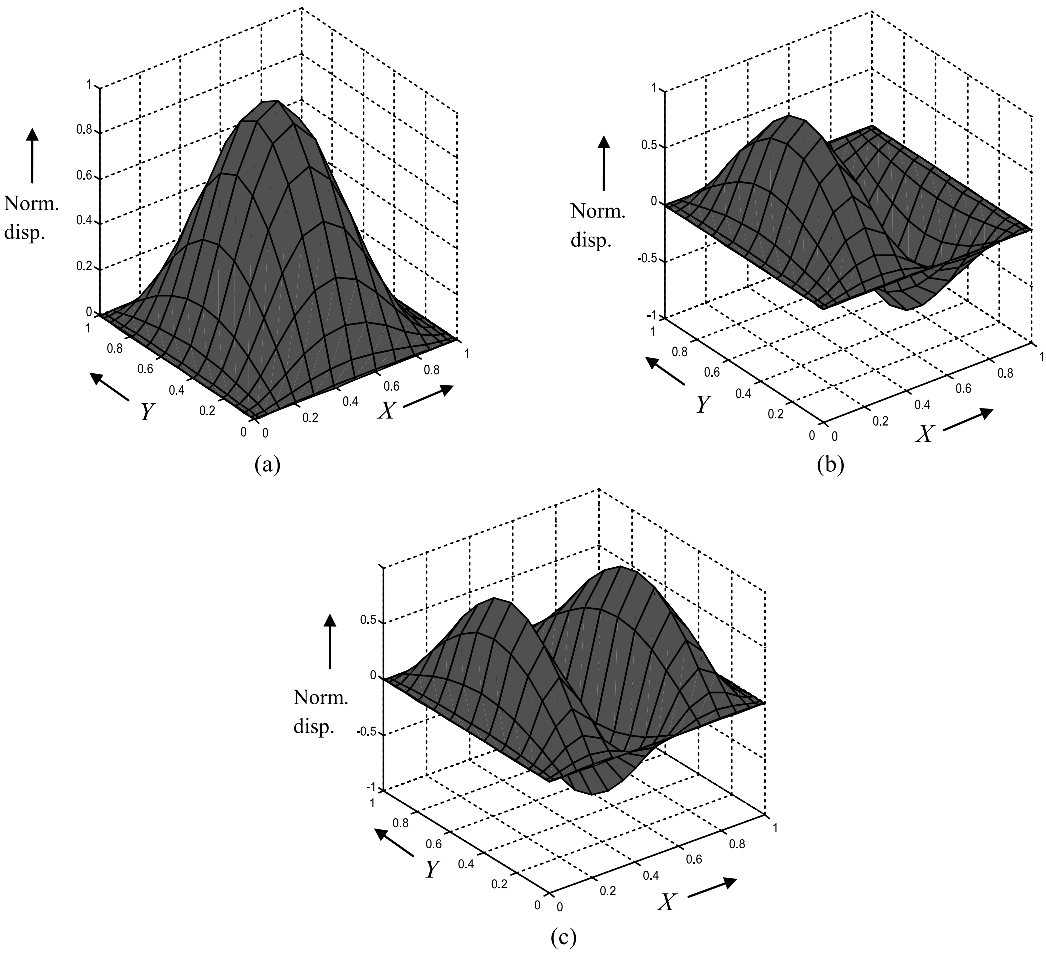

Fig. 10. Vibration modes of C-C plate (a) first mode (b) second mode (c) third mode; for $a/b = 1, \mu = \beta = 0.5, \gamma = 1, N_0^* = 30$.

non-homogeneity parameter μ. For $\gamma = 1.5, 2$, (Fig. 2), as the plate undergoes to compressive and tensile forces simultaneously, one obtains two values of N_{cr}^* (one positive and other negative). The negative values are reported in Table 2. It is observed that for $\gamma = 1.5$, the positive values of N_{cr}^* are smaller in magnitude as compared to the negative values of N_{cr}^* for C-C and C-S plates while for C-F plate the behaviour is just the reverse. In case of $\gamma = 2$, the positive values of N_{cr}^* are greater in magnitude as compared to negative values of N_{cr}^* for all three plates. The graphs for critical buckling loads N_{cr}^* for $\gamma = 0, 1, 2$ are shown in Fig. 9.

For the specified plate i.e. $\mu = \beta = 0.5, a/b = 1, \gamma = 1, N_0^* = 30$, three dimensional mode shapes for all the three plates are shown in Figs 10–12.

A comparison for the convergence study of the frequency parameter Ω with N for an unloaded ($N_0^* = 0$), homogeneous ($\mu = \beta = 0$), C-C square plate ($a/b = 1$) with quintic spline technique Lal et al. [33], differential quadrature method Wang et al. [11], power series method Leissa and Kang [6] for $p = 1$ has been shown in Table 3. The result shows that the present approach has faster rate of convergence. A comparison of critical buckling

Table 4
Comparison of critical buckling loads N_{cr}^* for homogeneous ($\mu = \beta = 0$) isotropic C-C plate, $p = 1$

	$\gamma = 0$			$\gamma = 1$			$\gamma = 2$		
$Ref.\ b/a$	0.4	0.5	0.6	0.4	0.5	0.6	0.4	0.5	0.6
Present	93.247	75.910	69.632	174.4	145.20	134.76	400.39	391.55	411.79
31	93.209	75.887	69.604	–	–	–	–	–	–
11	93.247	75.910	69.632	174.4	145.2	134.8	400.4	391.5	411.8
6	93.25	75.91	69.63	174.4	145.2	134.8	400.4	391.5	411.8

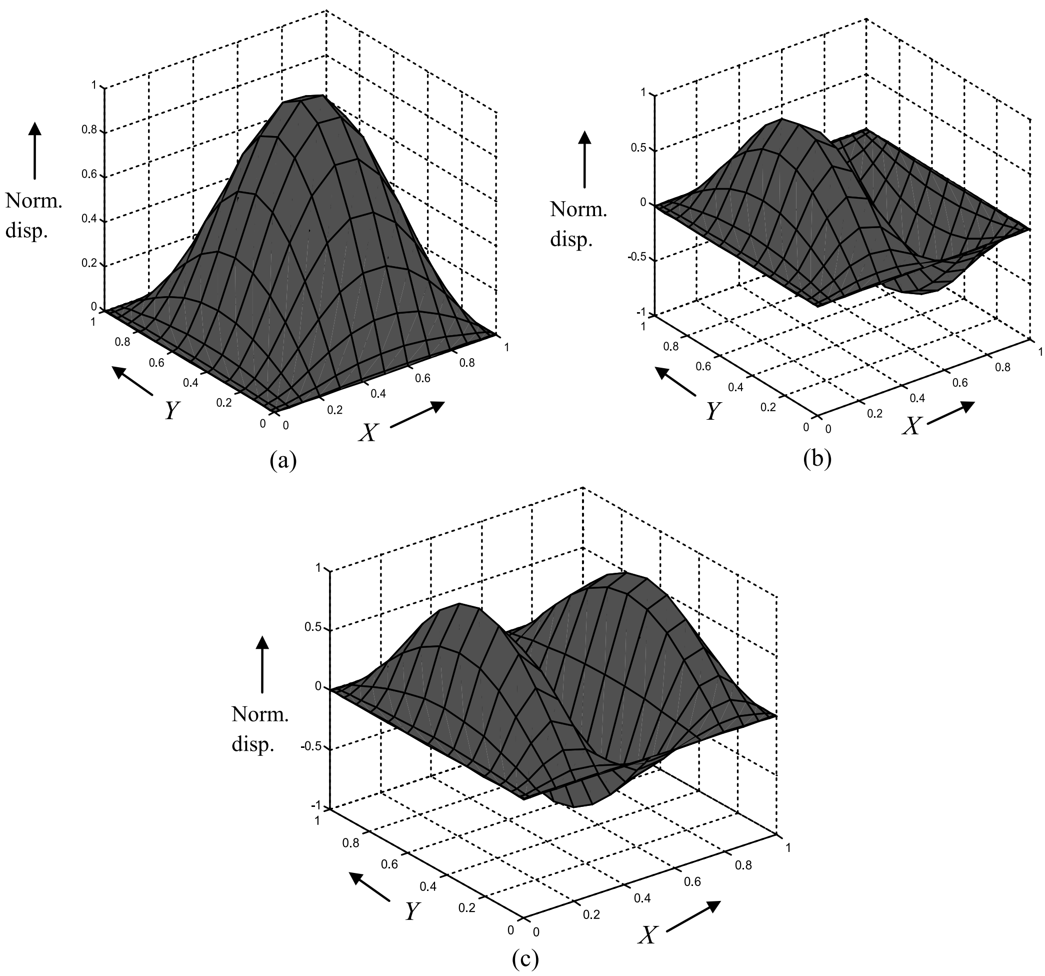

Fig. 11. Vibration modes of C-S plate (a) first mode (b) second mode (c) third mode; for $a/b = 1, \mu = \beta = 0.5, \gamma = 1, N_0^* = 30$.

loads N_{cr}^* for homogeneous ($\mu = 0, \beta = 0$) C-C plate for the values of loading parameter $\gamma = 0, 1, 2$, aspect ratio $b/a = 0.4, 0.5, 0.6$ and $p = 1$ with Lal and Dhanpati [31] obtained by quintic spline technique, differential quadrature method Wang et al. [11], and power series method Leissa and Kang [6] has been presented in Table 4. An excellent agreement of the results shows the versatility of the technique.

On the suggestion of one of the learned reviewers, the practical situation arising due to $\mu \neq 0$ and $\beta = 0$ has been analyzed. Physically, this consideration gives rise a type of non-homogeneity arising due to the change in Young's modulus of the material only. It is found that the values of frequency parameter Ω increases with the increasing values of non-homogeneity parameter μ for all the three boundary conditions for the same set of values of other parameters. It also increases with the increasing values of aspect ratio a/b. However, the values of frequency parameter Ω decreases with the increasing values of loading parameter γ for all the three boundary conditions

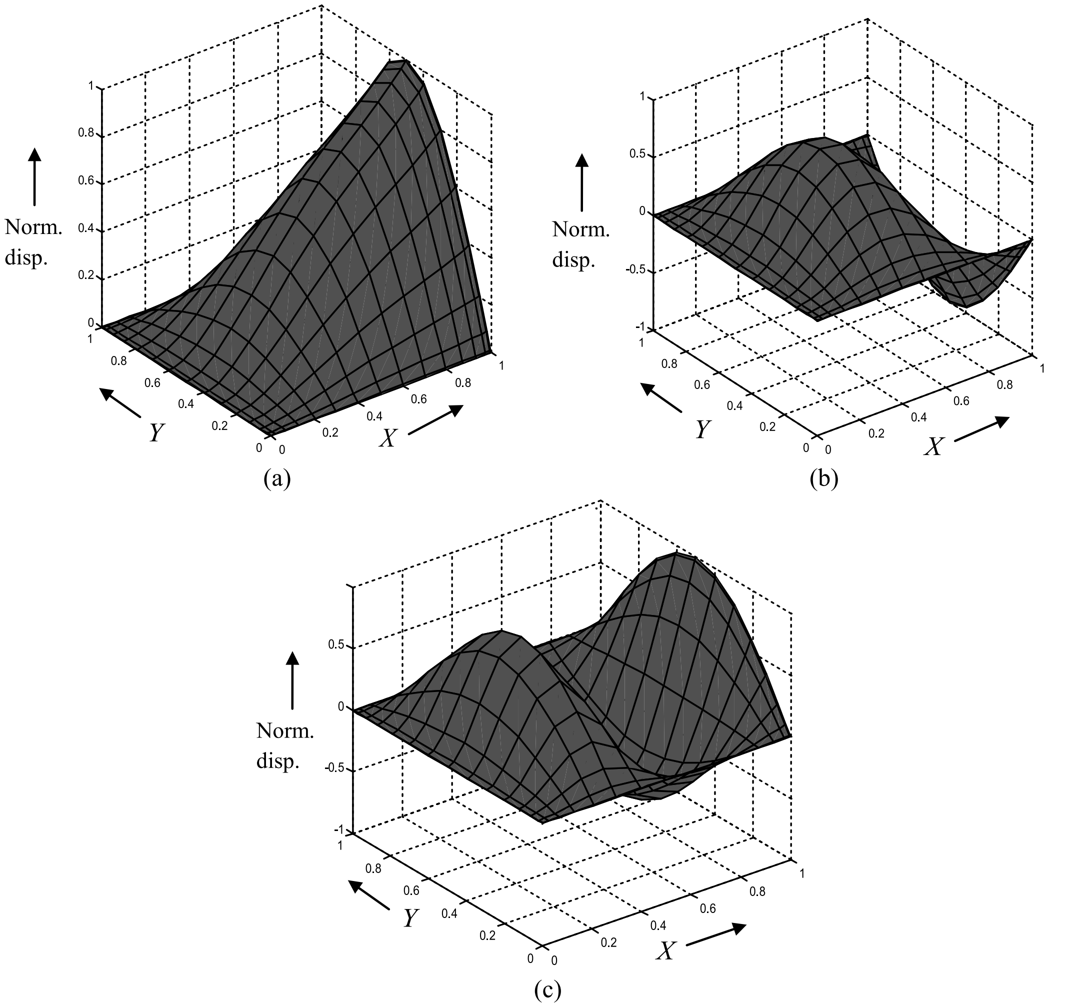

Fig. 12. Vibration modes of C-F plate (a) first mode (b) second mode (c) third mode; for $a/b = 1, \mu = \beta = 0.5, \gamma = 1, N_0^* = 30$.

whatever be the set of values of other parameter. Similarly, it is noticed that the frequency parameter Ω decreases with the increasing value of in-plane force parameter N_0^*, keeping other parameters fixed for all the three plates. The percentage change in the values of lowest frequency parameter Ω increases with the increasing values of loading parameter γ as well as in-plane force parameter N_0^*, whatever be the set of values of plate parameters in the order of boundary conditions C-C > C-S > C-F.

6. Conclusions

Differential quadrature method has been used to study the combine effect of linearly varying in-plane force and non-homogeneity of the plate material on the transverse vibration of thin rectangular plates of uniform thickness. The non-homogeneity is assumed to arise due to exponential variation in young modulus and density of the plate material along axial direction. It is observed that the values of frequency parameter Ω increases with the increasing values of in-plane force parameter N_0^* as the plate become more and more stiff towards the edge $x = 0$ to $x = a$, keeping other plate parameters fixed. However, the values of frequency parameter Ω decreases as the plate becomes more and more dense towards the edge $x = a$ for fixed values of other plate parameters and this further decreases with the increasing values of in-plane force parameter N_0^*. The frequency parameter Ω also increases with the increasing value of aspect ratio a/b and decreases with the increasing value of loading parameter γ for the same set of values of other parameters. A case of pure in-plane bending has been arisen for homogeneous $(\mu = \beta = 0)$ C-C plate when $\gamma = 2$. The value of critical buckling loads N_{cr}^* increases with the increasing value of loading parameter γ. It is found that for $\gamma > 1.5$ there exist two values of critical buckling loads N_{cr}^* (one positive and another negative), as the plate undergoes to compressive and tensile forces simultaneously for all the three boundary conditions. The percentage variation in the value of lowest frequency parameter Ω are -11.8 to 13.3, -10.9 to 12.1, -10.9 to 13.0 for C-C, C-S and C-F boundary conditions, respectively, when the non-homogeneity arises due to the change in only μ (i.e. $\beta = 0$) from -0.5 to 0.5 with respect to $\mu = 0$ for $\gamma = 0, N_0^* = 0, a/b = 0.5$. This effect increases with the increasing values of aspect ratio a/b in the order of boundary conditions C-C < C-S < C-F. These variations are 13.1 to -11.9, 15.0 to -13.4, 21.9 to -18.1 for C-C, C-S and C-F boundary conditions, respectively, when the non-homogeneity arises due to the change in only density parameter β (i.e. $\mu = 0$) from -0.5 to 0.5 with respect to $\beta = 0$ for the increasing values of aspect ratio a/b. This effect remains almost same for all the three boundary condition. In case of critical buckling loads N_{cr}^*, the percentage variation are -22.2 to 28.2, -22.4 to 28.6 and -27.8 to 38.9 for C-C, C-S and C-F plates, respectively, when μ changes from -0.5 to 0.5 with respect to $\mu = 0$ for $\gamma = 0$ and $a/b = 1$. These variations remain unchanged due to variation in density parameter β. The corresponding changes become -19.8 to 24.4, -19.9 to 24.6 and -23.3 to 30.4 for $\gamma = 1$. Almost similar percentage variations were obtained for second and third modes of vibration. The present analysis will be of great use to the design engineers in obtaining the desired frequency by varying one or more plate parameters considered here.

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References

[1] S. Way, Stability of rectangular plates under shear and bending forces, *ASME Journal of Applied Mechanics* **3** (1936), A131–A135.

[2] N. Grossman, Elastic stability of simply supported flat rectangular plates under critical combinations of transverse compression and longitudinal bending, *Journal of the Aeronautical Sciences* **16** (1949), 272–276.

[3] S. Timoshenko and J. Gere, *Theory of Elastic Stability*, 2nd edition, McGraw-Hill Book Company, Inc. New York, 1963.

[4] T.M. Wang and J.M. Sussman, Elastic stability of a simply supported plate under linearly variable compressive stresses, *AIAA Journal* **5** (1967), 1362–1364.

[5] R.H. Gutierrez and P.A.A. Laura, Use of differential quadrature method when dealing with transverse vibrations of a rectangular plate subjected to a non-uniform stress distribution field, *Journal of Sound and Vibration* **220**(4) (1999), 765–769.

[6] A.W. Leissa and J.H. Kang, Exact solution for vibration and buckling of an SS-C-SS-C rectangular plate loaded by linearly varying in plane stresses, *International Journal of Mechanical Sciences* **44** (2002), 1925–1945.

[7] C.W. Bert and K.K. Devarakond, Buckling of rectangular plates subjected to nonlinearly distributed in-plane loading, *International Journal of Solids and Structures* **40** (2003), 4097–4106.

[8] H. Hu, A. Badir and A. Abatan, Buckling behaviour of a graphite/epoxy composite plate under parabolic variation of axial loads, *International Journal of Mechanical Sciences* **45** (2003), 1135–1147.

[9] S. Naguleswaran, Transverse vibration of uniform Euler-Bernoulli beam under linearly varying axial force, *Journal of Sound and Vibration* **275** (2004), 47–57.

[10] J.H. Kang and A.W. Leissa, Exact solutions for the buckling of rectangular plates having linearly varying in-plane loading on two opposite simply supported edges, *International Journal of Solids and Structures* **42** (2005), 4220–4238.

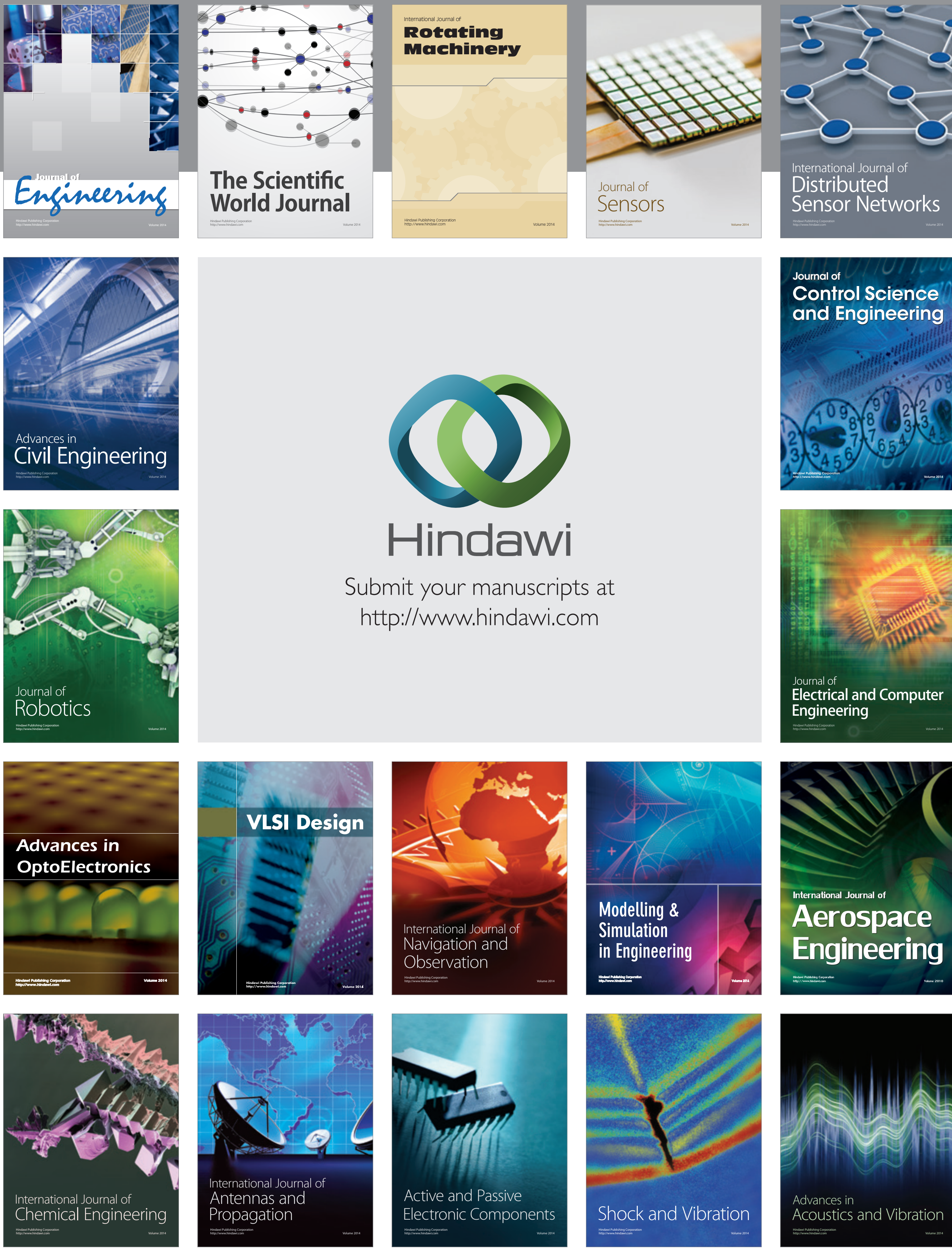
[11] X. Wang, L. Gan and Y. Wang, A differential quadrature analysis of vibration and buckling of an SS-C-SS-C rectangular plate loaded by linearly varying in-plane stress, *Journal of Sound and Vibration* **298** (2006), 420–431.

[12] B. Jana and K. Bhaskar, Analytical solutions for buckling of rectangular plates under non-uniform biaxial compression or uniaxial compression with in-plane lateral restraint, *International Journal of Mechanical Sciences* **49** (2007), 1104–1112.

[13] X. Wang, L. Gan and Y. Zhang, Differential quadrature analysis of the buckling of thin rectangular plates with cosine-distributed compressive loads on two opposite edges, *Advances in Engineering Software* **30** (2008), 497–504.

[14] K.K.V. Devarakonda and C.W. Bert, Buckling of rectangular plate with nonlinearly distributed compressive loading on two opposite sides, *Mechanics of Advanced Material and Structures* **11** (2004), 433–444.

[15] O. Civalek, A. Korkmaz and C. Demir, Discrete singular convolution approach for buckling analysis of rectangular Kirchhoff plates subjected to compressive loads on two-opposite edges, *Advances in Engineering Software* **41** (2010), 557–560.
[16] Y. Tang and X. Wang, Buckling of symmetrically laminated rectangular plates under parabolic edge compressions, *International Journal of Mechanical Sciences* **53** (2011), 91–97.
[17] S.A. Eftekhari and A.A. Jafari, Mixed finite element and differential quadrature method for free and forced vibration and buckling analysis of rectangular plates, *Appl Math Mech-Engl* Ed. **33** (2012), 81–98.
[18] V.A. Lomakin, *The Elasticity Theory of Non-Homogeneous Materials (in Russian)*, Nauka, Moscow, 1976.
[19] L.P. Khoroshun and S.Y. Kozlov, *The Generalized Theory of Plates and Shells Non-Homogeneous in Thickness Direction (in Russian)*, Naukova Dumka, Kiev, 1988.
[20] D.V. Babich and L.P. Khoroshun, Stability and natural vibrations of shells with variable geometric and mechanical parameters, *International Applied Mechanics* **37** (2001), 837–869.
[21] A.H. Sofiyev, M.H. Omurtag and E. Schnack, The vibration and stability of orthotropic conical shells with non-homogeneous material properties under a hydrostatic pressure, *Journal of Sound and Vibration* **319** (2009), 963–983.
[22] R.K. Bose, Note on forced vibration of a thin non-homogeneous circular plate with central hole, *Indian Journal of Physics* **41** (1967), 886–890.
[23] S.K. Biswas, Note on the torsional vibration of a finite circular cylinder of non-homogeneous material by a particular type of twist on one of the plane surface, *Indian Journal of Physics* **43** (1969), 320–341.
[24] G.V. Rao, B.P. Rao and I.S. Raju, Vibrations of inhomogeneous thin plates using a high-precision triangular element, *Journal of Sound and Vibration* **34**(3) (1974), 444–445.
[25] S. Chakraverty, R. Jindal and V.K. Agarwal, Vibration of non-homogeneous orthotropic elliptic and circular plates with variable thickness, *Journal of Sound and Vibration* **129** (2007), 256–259.
[26] R. Lal and S. Sharma, Axisymmetric vibration of non-homogeneous polar orthotropic annular plate of variable thickness, *Journal of Sound and Vibration* **272**(1–2) (2004), 245–265.
[27] U.S. Gupta, R. Lal and S. Sharma, Vibration analysis of non-homogeneous circular plate of nonlinear thickness variation by differential quadrature method, *Journal of Sound and Vibration* **298** (2006), 892–906.
[28] J.S. Tomar, D.C. Gupta and N.C. Jain, Vibration of non-homogeneous plates of variable thickness, *Journal of the Acoustical Society of America* **72** (1982), 851–855.
[29] R. Lal and Dhanpati, Transverse vibrations of non-homogeneous orthotropic rectangular plates of variable thickness: A spline technique, *Journal of Sound and Vibration* **306** (2007), 203–214.
[30] R. Lal and Y. Kumar, Boundary characteristic orthogonal polynomials in the study of transverse vibrations of non-homogeneous rectangular plates with bilinear thickness variation, *Shock and Vibration* **19** (2012), 349–364.
[31] R. Lal and Dhanpati, Quintic splines in the study of buckling and vibration of non-homogeneous orthotropic rectangular plates with variable thickness, *Int J Appl Math Mech* **3**(3) (2007), 18–35.
[32] Y. Kumar and R. Lal, Buckling and vibration of orthotropic non-homogeneous rectangular plates with bilinear thickness variation, *Journal of Applied Mechanics* **78** (2011), 061012-11.
[33] R. Lal, Dhanpati and Y. Kumar, Buckling and vibration of non-homogeneous orthotropic rectangular plates of varying thickness under biaxial compression, *Int J Appl Math Mech* **4**(4) (2008), 93–107.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