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Research Article

Chaotic Motions of the Duffing-Van der Pol Oscillator with External and Parametric Excitations

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The chaotic motions of the Duffing-Van der Pol oscillator with external and parametric excitations are investigated both analytically and numerically in this paper. The critical curves separating the chaotic and nonchaotic regions are obtained. The chaotic feature on the system parameters is discussed in detail. Some new dynamical phenomena including the controllable frequency are presented for this system. Numerical results are given, which verify the analytical ones.

1. Introduction

Dufffing-Van der Pol oscillator has a wide usage in many fields. The dynamics of Duffing-Van der Pol oscillator has been investigated widely in these years. Using the Melnikov method, Ravisankar et al. [1] studied horseshoe chaos in Duffing-Van der Pol oscillator driven by different periodic forces. Melnikov threshold curve was drawn in a parameter space. With the second-order averaging method and Melnikov method, Jing et al. [2] investigated chaotic motions in Duffing-Van der Pol equation with fifth nonlinear-restoring force and two external forcing terms. Numerical simulations were given to show the consistence with the theoretical analysis and exhibit the more new complex dynamical behaviors. With the singularity analysis, bifurcation properties of Duffing-Van der Pol system with two parameters under multifrequency excitations were studied by Qin and Chen [3]. Using the residue harmonic method, Leung et al. [4] investigated periodic bifurcation of Duffing-Van der Pol oscillators having fractional derivatives and time delay. It was shown that jumps and hysteresis phenomena can be delayed or removed. By using a simple transformation, the first integrals and the solutions of the Duffing-Van der Pol type equation under certain conditions were obtained by Udwadia and Cho [5]. Using the residue harmonic homotopy, a generalized Duffing-Van der Pol oscillator with nonlinear fractional order damping was investigated by Leung et al. [6]. Nonlinear dynamic behaviors of the harmonically forced oscillator were further explored by the harmonic balance method along with the polynomial homotopy continuation technique. By using the GYC partial region stability theory, Ge and Li [7] studied the synchronization of new Mathieu-Van der Pol systems with new Duffing-Van der Pol systems. With the second-order averaging method, the conditions for the existence and the bifurcations of harmonics for the damped and driven Duffing-Van der Pol system were obtained by Zhang et al. [8]. By the averaging method together with truncation of Taylor expansions, Li et al. [9] investigated the dynamics of Duffing-Van der Pol oscillators under linear-plus-nonlinear position feedback control with two time delays. By applying phase diagrams, potential diagram, Poincaré maps, bifurcation diagrams, and maximal Lyapunov exponent diagrams, the nonlinear behavior and the complex state of the Duffing-Van der Pol equation with fifth nonlinear-restoring force and two external periodic excitations were investigated by Shi et al. [10]. By using Melnikov analysis and numerical simulations, the dynamical behaviors including chaos, period-doubling cascades, and strange attractors of the extended Duffing-Van der Pol system were investigated by Yu et al. [11, 12]. With numerical methods, Yu et al. [13] also investigated the dynamical behavior of the extended Duffing-Van der Pol oscillator with ϕ^6 potential. Different routes to chaos and rich dynamical phenomena were observed. Patel and Sharma [14] revisited the stochastic Duffing-Van der Pol "filtering" in the

Fokker-Planck setting in lieu of the filtering in the Kushner setting. With the help of the modified quasiconservative averaging, Li et al. [15] investigated the stochastic responses of Duffing-Van der Pol vibroimpact system under additive colored noise excitation.

In this paper, the chaotic motions of the Duffing-Van der Pol oscillator with external and parametric excitations are studied analytically with the Melnikov method. The critical curves separating the chaotic and nonchaotic regions are obtained. The chaotic feature on the system parameters is discussed in detail and some new dynamical phenomena are presented. Numerical simulations verify the analytical results.

2. Formulation of the Problem

Consider the Duffing-Van der pol oscillator with external and parametric excitations

$$\ddot{x} + p\dot{x}\left(1 - x^2\right) - \alpha x + \beta x^3 = F(t), \tag{1}$$

where p is a damping parameter, $\alpha > 0$, $\beta > 0$ are real parameters, and $F(t) = f(1 + \delta x)\cos\omega t$ is the external and parametric forces.

Assume the damping and excitation terms p, f are small, setting $p = \varepsilon\overline{p}$, $f = \varepsilon\overline{f}$, where ε is a small parameter; then (1) can be written as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \alpha x - \beta x^3 + \varepsilon\left[-\overline{p}y\left(1 - x^2\right) + \overline{f}(1 + \delta x)\cos\omega t\right]. \end{aligned} \tag{2}$$

Using the transformations

$$x = u\sqrt{\frac{\alpha}{\beta}}, \qquad t = \sqrt{\frac{1}{\alpha}}\tau, \tag{3}$$

then (2) can be written as

$$\begin{aligned} u' &= v, \\ v' &= u - u^3 - \varepsilon\tilde{p}v\left(1 - \frac{\alpha}{\beta}u^2\right) + \varepsilon\tilde{f}\left(1 + \delta u\sqrt{\frac{\alpha}{\beta}}\right)\cos(\tilde{\omega}\tau), \end{aligned} \tag{4}$$

where $\tilde{p} = \overline{p}/\sqrt{\alpha}$, $\tilde{f} = \overline{f}/(\alpha\sqrt{\alpha}/\sqrt{\beta})$, $\tilde{\omega} = \omega/\sqrt{\alpha}$, and $'$ represents $d/d\tau$.

When $\varepsilon = 0$, the unperturbed system of (4) is

$$\begin{aligned} u' &= v, \\ v' &= u - u^3 \end{aligned} \tag{5}$$

which is a planar Hamiltonian system with the Hamiltonian

$$H(u, v) = \frac{v^2}{2} - \frac{u^2}{2} + \frac{u^4}{4}. \tag{6}$$

System (6) has three equilibrium points, where $(0, 0)$ is a saddle point, and $(\pm 1, 0)$ are all centers.

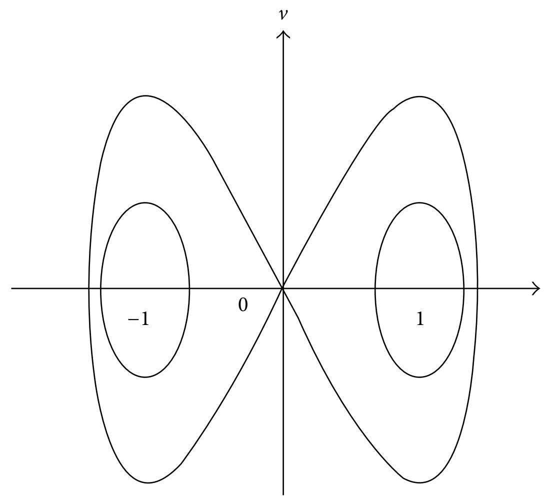


Figure 1: The phase portrait of system (5).

There exist homoclinic orbits connecting $(0, 0)$ to itself with the expressions [16]

$$\begin{aligned} u_{\text{hom}}(\tau) &= \pm\sqrt{2}\,\text{sech}(\tau), \\ v_{\text{hom}}(\tau) &= \mp\sqrt{2}\,\text{sech}(\tau)\tanh(\tau) \end{aligned} \tag{7}$$

and closed periodic orbits around $(\pm 1, 0)$ with the expressions [16]

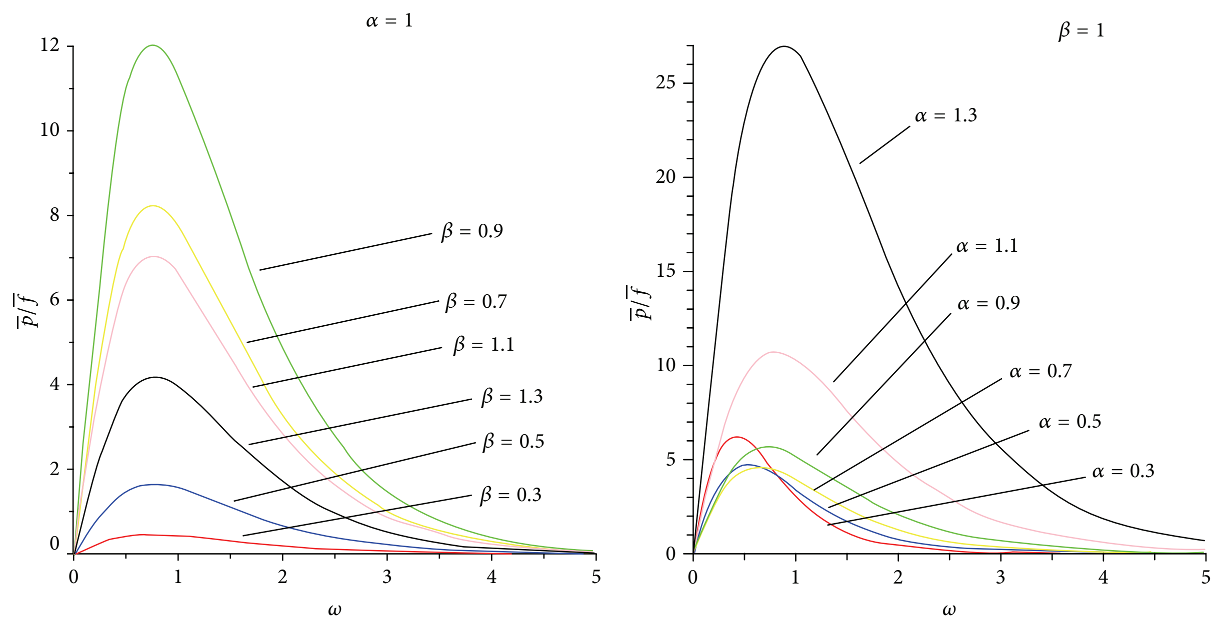
$$\begin{aligned} u_k(\tau) &= \frac{k}{\sqrt{2k^2 - 1}}cn\left(\sqrt{\frac{1}{2k^2 - 1}}\tau, k\right), \\ v_k(\tau) &= \frac{-k}{\sqrt{2k^2 - 1}}sn\left(\sqrt{\frac{1}{2k^2 - 1}}\tau, k\right)dn\left(\sqrt{\frac{1}{2k^2 - 1}}\tau, k\right); \end{aligned} \tag{8}$$

see Figure 1, where sn, cn, dn are Jacobi elliptic functions, and $1/\sqrt{2} < k < 1$ is the modulus of the Jacobi elliptic functions. The period of the closed orbit is $T_k = 4\sqrt{2k^2 - 1}K(k)$, where $K(k)$ is the complete elliptic integral of the first kind.

3. Chaotic Motions of the System

Melnikov method [17] is an analytical tool to study chaotic systems. Recently, chaotic motions of many systems, for example, ϕ^6-Rayleigh oscillator [18], Duffing oscillator [19], Gylden's problem [20], and nonsmooth systems [21], have been investigated by the Melnikov method. In this section, we use the Melnikov method to investigate the chaotic motions of system (4). We compute the Melnikov functions of system (4) along the homoclinic orbit (7) as follows:

$$\begin{aligned} M(\tau_0) &= \int_{-\infty}^{+\infty} -\tilde{p}v_{\text{hom}}^2(\tau)\left(1 - \frac{\alpha}{\beta}u_{\text{hom}}^2\right)(\tau)\,d\tau \\ &\quad + \int_{-\infty}^{+\infty} \tilde{f}\cos\tilde{\omega}(\tau + \tau_0)\,v_{\text{hom}}(\tau)\,d\tau \end{aligned}$$



(a) (b)

FIGURE 2: The critical curves for chaotic motions of system (2) in the case of $\delta = 0$.

$$
\begin{aligned}
&+\delta \int_{-\infty}^{+\infty} \tilde{f} \sqrt{\frac{\alpha}{\beta}} u_{\mathrm{hom}}(\tau)\, v_{\mathrm{hom}}(\tau) \cos \tilde{\omega}(\tau+\tau_0)\, d\tau \\
&\equiv -\tilde{\mu} I_0 + \tilde{g}(I_1 + \delta I_2) \sin \tilde{\omega} \tau_0,
\end{aligned}
\tag{9}
$$

where

$$
\begin{aligned}
I_0 &= \int_{-\infty}^{+\infty} v_{\mathrm{hom}}^2(\tau)\left(1 - \frac{\alpha}{\beta} u_{\mathrm{hom}}^2(\tau)\right) d\tau = \frac{4}{3} - \frac{16\alpha}{15\beta}, \\
I_1 &= \int_{-\infty}^{+\infty} v_{\mathrm{hom}}(\tau) \sin \tilde{\omega}\tau\, d\tau = \sqrt{2}\pi\tilde{\omega} \operatorname{sech}\left(\frac{\pi\tilde{\omega}}{2}\right) \\
&= \sqrt{2}\pi \frac{\omega}{\sqrt{\alpha}} \operatorname{sech}\left(\frac{\pi\omega}{2\sqrt{\alpha}}\right), \\
I_2 &= \int_{-\infty}^{+\infty} \sqrt{\frac{\alpha}{\beta}} u_{\mathrm{hom}}(\tau)\, v_{\mathrm{hom}}(\tau) \sin \tilde{\omega}\tau\, d\tau \\
&= -2\sqrt{\frac{\alpha}{\beta}} \int_{-\infty}^{+\infty} \operatorname{sech}^2(\tau) \tanh^2(\tau) \sin \tilde{\omega}\tau\, d\tau \\
&= \frac{2\sqrt{(\alpha/\beta)}\, e^{-\pi\omega/2\sqrt{\alpha}} \left(\omega^2 - \alpha\right)\pi}{\left(e^{-\pi\omega/\sqrt{\alpha}} - 1\right)\alpha}.
\end{aligned}
\tag{10}
$$

By Melnikov analysis, we estimate the condition for transverse intersection and chaotic separatrix motion as follows:

$$
\frac{\tilde{p}}{\tilde{f}} < \frac{|I_1 + \delta I_2|}{|I_0|};
\tag{11}
$$

that is,

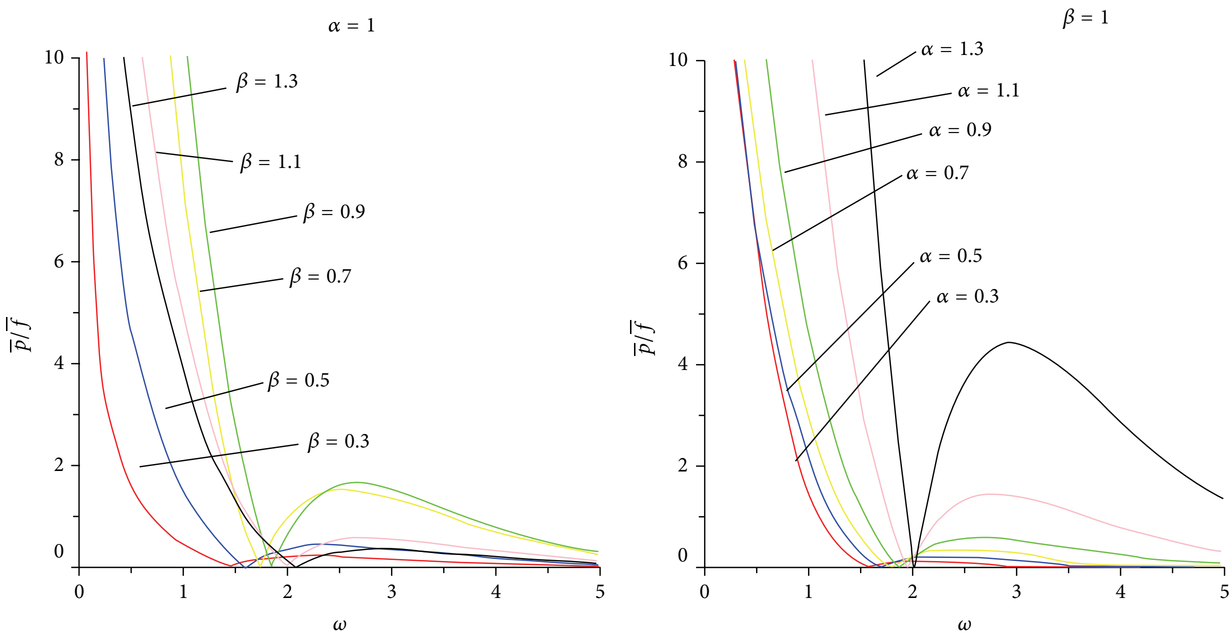
$$
\frac{\bar{p}}{\bar{f}} < \frac{|I_1 + \delta I_2|}{|I_0|} \cdot \frac{\sqrt{\beta}}{\alpha}.
\tag{12}
$$

First, taking $\delta = 0$, which is the case of periodic external excitation, letting $\alpha = 1$, for different values of β, we get the critical curves separating the chaotic regions (below) and nonchaotic regions (above) as in Figure 2(a). Next, Letting $\beta = 1$, the critical curves for different values of α are shown in Figure 2(b).

Secondly, taking $\delta = 1$, which is the case of both periodic external and parametric excitations, letting $\alpha = 1$, for different values of β, we get the critical curves as in Figure 3(a). Next, Letting $\beta = 1$, the critical curves for different values of α are shown in Figure 3(b).

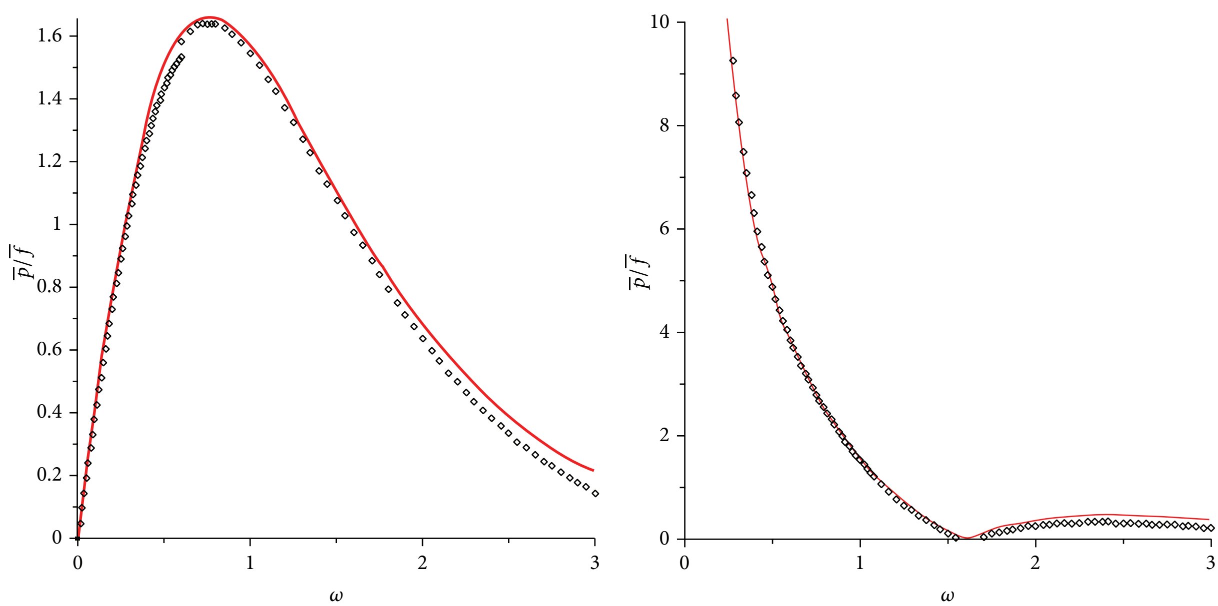
From Figures 2-3 we can obtain the following conclusions.

(1) For the case of periodic external excitation, the critical curves have the classical bell shape; this means that, with the excitations possessing sufficiently small or very large periods, the systems are not chaotically excited. When α is fixed, for each $\beta \in (0, 0.8)$, the larger the values of β, the larger the critical values for chaotic motions, while for $\beta > 0.8$, the larger the values of β, the smaller the critical values for chaotic motions. On the other hand, when β is fixed, if $\alpha < 0.9$, for the case of small values of ω, that is, the period of the excitation is large, the critical value for chaotic motions decreases as α increases; when ω crosses a critical value, the case is opposite, so for the case of large values of ω, that is, the period of the excitation is small, the critical value for chaotic motions increases



(a) (b)

Figure 3: The critical curves for chaotic motions of system (2) in the case of $\delta = 1$.



(a) (b)

Figure 4: The theoretical and numerical critical values for chaos of system (2) in the case of (a) $\delta = 0$ and (b) $\delta = 1$.

as α increases; if $\alpha \geq 0.9$, for each excitation frequency ω, the critical value increases as α increases.

(2) For the case of parametric excitations, the critical curve first decreases quickly to zero and then increases; at last it decreases to zero as ω increases from zero. There exists a controllable frequency ω excited at which chaotic motions do not take place no matter how large the excitation amplitude is. When α (β) is fixed, the controllable frequency increases as β (α) increases.

4. Numerical Simulations

First, choosing the system parameters $\varepsilon = 0.01$, $\alpha = 1$, $\delta = 0$, $\beta = 0.5$, and $\omega \in (0, 3)$, which is the case of external excitations, the critical values $\overline{p}/\overline{f}$ for chaotic motions are shown in

Figure 4(a), where the red curve is the theoretical predictions, and the black points are the numerical predictions.

Next, choosing the system parameters $\varepsilon = 0.01$, $\alpha = 1$, $\delta = 1$, $\beta = 0.5$, and $\omega \in (0, 3)$, which is the case of both external and parametric excitations, the critical values $\overline{p}/\overline{f}$ for chaotic motions are shown in Figure 4(b), where the red curve is the theoretical predictions and the black points are the numerical predictions. From Figure 4, we can see that the difference of the critical values for chaotic motions between the theoretical and numerical predictions is very small, so numerical simulations agree with the analytical results.

5. Conclusions

Using the Melnikov and numerical methods, the chaotic motions for the Duffing-Van der Pol oscillator with external and parametrical excitations are investigated in this paper. The critical curves separating the chaotic and nonchaotic regions are obtained. It is shown that there exists a controllable frequency ω for the system with parametric excitations. When the system parameter α (β) is fixed, the controllable frequency increases as β (α) increases. Some new dynamical behaviors are presented.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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