Research Article

Dynamic Analysis of Planar 3-RRR Flexible Parallel Robots with Dynamic Stiffening

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In consideration of the second-order coupling quantity of the axial displacement caused by the transverse displacement of flexible beam, the first-order approximation coupling model of planar 3-RRR flexible parallel robots is presented, in which the rigid body motion constraints, elastic deformation motion constraints, and dynamic constraints of the moving platform are considered. Based on the different speed of the moving platform, numerical simulation results using the conventional zero-order approximation coupling model and the proposed firstorder approximation coupling model show that the effect of “dynamic stiffening” term on dynamic characteristics of the system is insignificant and can be neglected, and the zero-order approximation coupling model is enough precisely for catching essentially dynamic characteristics of the system. Then, the commercial software ANSYS 13.0 is used to confirm the validity of the zero-order approximation coupling model.

1. Introduction

In recent several decades, many researchers have paid more attention to the light flexible robots with high-speed, high-acceleration, and high-precision which are widely used in the assembly industry, the aerospace industry and the precision machining, and the measurement field. Essentially, the flexible parallel robots mechanism belongs to flexible multibody system. Dynamic modeling of flexible multibody system is a challenging task, in which not only rigid-flexible coupling effect must be studied but also elastic deformation coupling must be analyzed carefully.

At present, dynamic modeling and control of the flexible multibody system have received considerable attention as seen in survey papers [1–4]. Unfortunately, most of published works in this area addressed the manipulators with one flexible link. Comparing with single-link flexible manipulator, two-link flexible manipulator, or four-bar linkage flexible mechanism [5, 6], the research works on the flexible parallel robots are rather few.

Recently, few works have been done on dynamic modeling and control of complex mechanisms. Lee and Geng [7] developed a dynamic model of a flexible Stewart platform using Lagrange equations. Wang et al. [8–10] studied dynamic modeling and control of planar 3-Prismatic-joint-Revolute-joint-and-Revolute-joint (3-PRR) parallel robots. Zhang et al. [11, 12] studied dynamic modeling method and dynamic characteristics of planar 3-RRR flexible parallel robots. Q. H. Zhang and X. M. Zhang [13] also studied dynamic performance of planar 3-RRR flexible parallel robots under uniform temperature change.

For a simple flexible hub-beam system, when the system operates at a high speed, the conventional hybrid coordinate model, namely, zero-order approximation coupling model (ZOAC model) that assumes axial and transverse displacement of a point on the neutral axis of a beam are uncoupled is an invalid modeling method [14]. Kane et al. [15] found that when the nominal motion of rigid-flexible rotating beam system is very large, the deformation of the flexible system deformation exists significantly warp by

the conventional dynamic model and experimental method, and he thought the premature linearization would lead to lack of “dynamic stiffening” term and firstly proposed “dynamic stiffening” phenomenon. Since then, many scholars start to study the so-called “dynamic stiffening” terms of rigid-flexible coupling system and different methods are used to capture it [14, 16–18]. Mayo et al. [16] reviewed different formulations to account for the geometric stiffening or dynamic stiffening effect arising from deflections large enough to cause significant changes in the configuration of the system. Ligris et al. [17] used two different methods to calculate dynamic stiffening of the flexible system: one method is from the deformation energy and the other is from foreshortening. Zhang et al. [18] indicated that when the rotational frequency of slender beams meets or exceeds the fundamental frequency, the elastic deformation of flexible beam that is obtained by the conventional ZOAC model tends to diverge. Yu and Hong [19] summarized several methodologies for analyzing dynamic stiffening of flexible system. In consideration of structural geometric nonlinear effects, Wu and Haug [20] established a substructure synthesis formulation of flexible multibody system. Yang [21] studies dynamic model of rigid-flexible system from both theory and experiment. In consideration of dynamic stiffening effect, Liu and Lu [22] established the rigid-flexible coupling dynamic equations of three-dimensional hub-beams system using velocity variational principle and finite element method.

Since dynamic stiffening phenomenon was proposed by Kane et al., many relevant literatures were published [14, 16–22], but they mainly focus on the flexible rotation beam system. To my knowledge, dynamic stiffening phenomenon of flexible parallel robots has never been studied. Because dynamic stiffening term was not neglected in the high-speed flexible rotation beam system, we do not know that how dynamic stiffening of flexible parallel robots influences its own dynamic characteristics. So, it is important for us to study dynamic stiffening of flexible parallel robots.

The remainder of the paper is organized as follows. Section 2 introduces dynamic modeling of planar 3-RRR flexible parallel robots with dynamic stiffening. In consideration of the second-order coupling quantity of the axial displacement caused by the transverse displacement of flexible beam, based on the finite element method (FEM) and the Lagrange equation, the first-order approximation coupling model (FOAC model) of planar 3-RRR flexible parallel robots is presented. Section 3 discusses the constraint equations of planar 3-RRR flexible parallel robots system which include rigid body motion constraints, elastic deformation motion constraints, and dynamic constraints of the moving platform. In Section 4, numerical results are presented based on the ZOAC model and FOAC model. One finds that the influence of dynamic stiffening of planar 3-RRR flexible parallel robots on dynamic characteristics can be neglected, and the ZOAC model can effectively reflect dynamic characteristics of the system (Figure 7). The commercial ANSYS 13.0 software is used to confirm the validity of the ZOAC model. Finally, the conclusions are given in Section 5.

FIGURE 1: Beam element deformation.

2. Dynamic Modeling of Planar 3-RRR Flexible Parallel Robots with Dynamic Stiffening

The flexible link can be modeled by connecting a series of beam elements. Figure 1 shows a beam element before and after deformation. The $O-XY$ is the global fixed frame and the $A-xy$ is the local moving frame with Ax axis coincident with the neutral line of the beam element. Its original point A is located at one node of the beam element before deformation. B is another node of the beam element. The $O-x'y'$ system is an intermediate coordinate frame whose origin is rigidly attached to the origin of the $O-XY$ and whose axes are parallel to the axes of the local moving frame $A-xy$. φ is the angle between the global fixed frame $O-XY$ and the intermediate coordinate frame $O-x'y'$. Considering the general point C in the beam element, let point C_0 be the corresponding point on the neutral line. Points C' and C'_0 are their positions after deformation, respectively. The elastic deformation of the point C_0 in the $A-xy$ is given by

$$\mathbf{v}_0(x, t) = \begin{bmatrix} v_{10} \\ v_{20} \end{bmatrix} = \bar{\mathbf{N}}(x) \mathbf{e}_f, \quad (1)$$

where $\mathbf{e}_f = [e_1 \ e_2 \ e_3 \ e_4 \ e_5 \ e_6 \ e_7 \ e_8]^T$ is the nodal displacement vector, in which e_1 and e_5 are the axial displacements of two nodes A and B , respectively; e_2 and e_6 are the transverse displacements; e_3 and e_7 are elastic rotational angles; e_4 and e_8 are section curvatures; and the superscript T indicates matrix transpose. $\bar{\mathbf{N}}(x)$ is the shape function matrix. Assuming that the axial displacement v_{10} of point C_0 is a linear function, the transverse displacement v_{20} is a fifth-order hermit function, yielding

$$\bar{\mathbf{N}} = \begin{bmatrix} \bar{\mathbf{N}}_1(x) \\ \bar{\mathbf{N}}_2(x) \end{bmatrix} = \begin{bmatrix} f_1 & 0 & 0 & 0 & f_5 & 0 & 0 & 0 \\ 0 & f_2 & f_3 & f_4 & 0 & f_6 & f_7 & f_8 \end{bmatrix}, \quad (2)$$

where

$$\begin{aligned}
 f_1 &= 1 - \xi, \\
 f_2 &= 1 - 10\xi^3 + 15\xi^4 - 6\xi^5, \\
 f_3 &= l(\xi - 6\xi^3 + 8\xi^4 - 3\xi^5), \\
 f_5 &= \xi, \\
 f_4 &= \frac{l^2(\xi^2 - 3\xi^3 + 3\xi^4 - \xi^5)}{2}, \\
 f_6 &= 10\xi^3 - 15\xi^4 + 6\xi^5, \\
 \xi &= \frac{x}{l}, \\
 f_7 &= l(-4\xi^3 + 7\xi^4 - 3\xi^5), \\
 f_8 &= \frac{l^2(\xi^3 - 2\xi^4 + \xi^5)}{2}.
 \end{aligned} \tag{3}$$

l is length of the beam element.

The deformation displacements of C' in the A - xy can be written as [14]

$$\mathbf{v} = \begin{bmatrix} v_1(x, t) \\ v_2(x, t) \end{bmatrix} = \begin{bmatrix} v_{10} - y \frac{\partial v_{20}}{\partial x} - \int_0^x \frac{1}{2} \left(\frac{\partial v_{20}}{\partial \xi} \right)^2 d\xi \\ v_{20} \end{bmatrix}; \tag{4}$$

then, the displacement of C' can be expressed in the global fixed frame by

$$\mathbf{r}_{e,C'} = \mathbf{r}_{e,A} + \mathbf{R}(\mathbf{e}_0 + \mathbf{v}), \tag{5}$$

where $\mathbf{R}$ is the planar transformation matrix defined as

$$\mathbf{R} = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}; \tag{6}$$

$\mathbf{e}_0$ is the location coordinates vector of the point C in the A - xy system and is given by $[x \ y]^T$.

Taking the first derivative on (5) and considering (1), the velocity vector of the point C in the O - XY system can be written as

$$\begin{aligned}
 \dot{\mathbf{r}}_{e,C'} &= \dot{\mathbf{r}}_{e,A} + \mathbf{R}_\varphi \left(\mathbf{e}_0 + \mathbf{N} \mathbf{e}_f + \begin{bmatrix} w_c \\ 0 \end{bmatrix} \right) \dot{\varphi} \\
 &\quad + \mathbf{R} \left(\mathbf{N} - \begin{bmatrix} \mathbf{e}_f^T \mathbf{H}_e \\ \mathbf{0} \end{bmatrix} \right) \dot{\mathbf{e}}_f \\
 &= \left[\mathbf{I} \ \mathbf{R}_\varphi \left(\mathbf{e}_0 + \mathbf{N} \mathbf{e}_f + \begin{bmatrix} w_c \\ 0 \end{bmatrix} \right) \ \mathbf{R} \left(\mathbf{N} - \begin{bmatrix} \mathbf{e}_f^T \mathbf{H}_e \\ \mathbf{0} \end{bmatrix} \right) \right] \\
 &\quad \times \begin{bmatrix} \dot{\mathbf{r}}_{e,A} \\ \dot{\varphi} \\ \dot{\mathbf{e}}_f \end{bmatrix} \\
 &= \mathbf{S} \mathbf{u},
 \end{aligned} \tag{7}$$

where

$$\mathbf{S} = \left[\mathbf{I} \ \mathbf{R}_\varphi \left(\mathbf{e}_0 + \mathbf{N} \mathbf{e}_f + \begin{bmatrix} w_c \\ 0 \end{bmatrix} \right) \ \mathbf{R} \left(\mathbf{N} - \begin{bmatrix} \mathbf{e}_f^T \mathbf{H}_e \\ \mathbf{0} \end{bmatrix} \right) \right], \tag{8}$$

$$\mathbf{u} = \left[\mathbf{r}_{e,A}^T \ \varphi \ \mathbf{e}_f^T \right]^T,$$

$\dot{\mathbf{r}}_{e,A}$ is the velocity vector of the original point A in the O - XY system,

$$\begin{aligned}
 \mathbf{H}_e &= \int_0^x \frac{\partial \bar{\mathbf{N}}_2}{\partial \xi} \frac{\partial \bar{\mathbf{N}}_2^T}{\partial \xi} d\xi, \\
 \mathbf{R}_\varphi &= \frac{\partial \mathbf{R}}{\partial \varphi}, \quad \mathbf{N} = \begin{bmatrix} \bar{\mathbf{N}}_1^T - y \frac{\partial \bar{\mathbf{N}}_2^T}{\partial x} \\ \bar{\mathbf{N}}_2^T \end{bmatrix},
 \end{aligned} \tag{9}$$

$\mathbf{I}$ is 2×2 unit matrix, and parameter $w_c = -(1/2) \mathbf{e}_f^T \mathbf{H}_e \mathbf{e}_f$ is the second-order coupling term that is the axial shrinking quantity caused by the transverse displacement v_2 . In the ZOAC model, the small deformation assumption is adopted, so $v_1(x, t) = v_{10} - y(\partial v_{20}/\partial x)$ is taken, and w_c is neglected.

2.1. Kinetic Energy of the Beam Element. Using (7), kinetic energy of the beam element which includes translational kinetic energy of the beam element, rotational kinetic energy of the cross-section, and kinetic energy of lumped mass at the end points of beam can be written as

$$\begin{aligned}
 T_e &= \frac{1}{2} \int_V \rho \dot{\mathbf{r}}_{e,C'}^T \dot{\mathbf{r}}_{e,C'} dV \\
 &\quad + \frac{1}{2} \int_0^l \dot{\theta}^2(x, t) dJ_c + \frac{1}{2} (m_L \dot{\mathbf{r}}_{A'}^T \dot{\mathbf{r}}_{A'} + m_R \dot{\mathbf{r}}_{B'}^T \dot{\mathbf{r}}_{B'}) \\
 &\quad + \frac{1}{2} (J_L \dot{\theta}^2(0, t) + J_R \dot{\theta}^2(l, t)) \\
 &= \frac{1}{2} \dot{\mathbf{u}}^T \mathbf{m}_e \dot{\mathbf{u}},
 \end{aligned} \tag{10}$$

where

$$\begin{aligned}
 \mathbf{m}_e &= \begin{bmatrix} \mathbf{m}_{err} & \mathbf{m}_{er\varphi} & \mathbf{m}_{erf} \\ & m_{e\varphi\varphi} & \mathbf{m}_{e\varphi f} \\ \text{Symmetry} & & m_{eff} \end{bmatrix}, \\
 \mathbf{m}_{err} &= \left(\int_V \rho dV + m_L + m_R \right) \mathbf{I},
 \end{aligned}$$

$$\begin{aligned}
\mathbf{m}_{er\varphi} &= \mathbf{R}_\varphi \mathbf{N}_1 + \mathbf{R}_\varphi \mathbf{N}_2 \mathbf{e}_f + \mathbf{R}_\varphi \int_V \rho \begin{bmatrix} w_c \\ 0 \end{bmatrix} dV + \mathbf{R}_\varphi \bar{\mathbf{N}}_c \mathbf{e}, \\
\mathbf{m}_{erf} &= \mathbf{R} \mathbf{N}_2 - \mathbf{R} \int_V \rho \begin{bmatrix} \mathbf{e}_f^T \mathbf{H}_e \\ 0 \end{bmatrix} dV + \mathbf{R} \bar{\mathbf{N}}_c, \\
m_{e\varphi\varphi} &= \mathbf{N}_3 + 2\mathbf{N}_4 \mathbf{e}_f + \mathbf{e}_f^T \mathbf{N}_8 \mathbf{e}_f \\
&\quad + \mathbf{e}_f^T \mathbf{N}_5 \mathbf{e}_f + \rho I l + \mathbf{e}^T \bar{\mathbf{N}}_c \mathbf{e} + J_L + J_R, \\
\mathbf{m}_{e\varphi f} &= \mathbf{N}_6 + \mathbf{e}_f^T \mathbf{N}_7 + \mathbf{e}_f^T \mathbf{N}_9 + \mathbf{e}^T \bar{\mathbf{N}}_c + \mathbf{C}_1, \\
\mathbf{m}_{eff} &= \mathbf{N}_5 + \mathbf{C}_2, \\
\mathbf{N}_1 &= \int_V \rho \mathbf{e}_0 dV, \\
\mathbf{N}_2 &= \int_V \rho \mathbf{N} dV, \quad \mathbf{N}_3 = \int_V \rho \mathbf{e}_0^T \mathbf{e}_0 dV, \\
\mathbf{N}_4 &= \int_V \rho \mathbf{e}_0^T \mathbf{N} dV, \\
\mathbf{N}_5 &= \int_V \rho \mathbf{N}^T \mathbf{N} dV, \quad \mathbf{N}_6 = \int_V \rho \mathbf{e}_0^T \bar{\mathbf{I}} \mathbf{N} dV, \\
\mathbf{N}_7 &= \int_V \rho \mathbf{N}^T \bar{\mathbf{I}} \mathbf{N} dV, \\
\mathbf{N}_8 &= - \int_V \rho x \mathbf{H}_e dV, \quad \mathbf{N}_9 = \int_V \rho y \mathbf{H}_e dV, \\
\bar{\mathbf{N}}_c &= m_L \mathbf{N}(0) + m_R \mathbf{N}(l), \\
\bar{\mathbf{N}}_c &= m_L \mathbf{N}(0)^T \mathbf{N}(0) + m_R \mathbf{N}(l)^T \mathbf{N}(l), \\
\bar{\mathbf{N}}_c &= m_L \mathbf{N}(0)^T \bar{\mathbf{I}} \mathbf{N}(0) + m_R \mathbf{N}(l)^T \bar{\mathbf{I}} \mathbf{N}(l), \\
\mathbf{C}_1 &= \int_0^l \rho I \mathbf{N}_2'^T dx + J_L \mathbf{N}_2'^T(0) + J_R \mathbf{N}_2'^T(l), \\
\mathbf{C}_2 &= \int_0^l \rho I \mathbf{N}_2' \mathbf{N}_2'^T dx + \bar{\mathbf{N}}_c \\
&\quad + J_L \mathbf{N}_2'(0) \mathbf{N}_2'^T(0) + J_R \mathbf{N}_2'(l) \mathbf{N}_2'^T(l).
\end{aligned} \tag{11}$$

$\bar{\mathbf{I}}$ is skew symmetric matrix, $\theta(x, t) = \varphi + \partial v_2(x, t)/\partial x$ is absolute rotation angle of microsegment dx which distances x from the original point A . m_L , J_L , m_R , and J_R are lumped mass and lumped moment of inertia at left and right end points of the beam element, respectively. $\mathbf{N}_2' = \partial \mathbf{N}_2 / \partial x$. It should be mentioned that the high-order terms related with w_c , such as $v_{10} w_c$, $\dot{w}_c^2$, $v_2 \dot{w}_c$, and $\dot{v}_2 w_c$, are omitted since w_c , $\dot{w}_c$, v_{10} , v_2 , and $\dot{v}_2$ are very small.

2.2. Strain Energy of the Beam Element. Nonlinear terms in the strain-displacement relationship are neglected [16]. Then, strain energy of the beam element can be written as

$$\begin{aligned}
U_e &= \frac{1}{2} \int_0^l EI(x) (v_{20}''(x, t))^2 dx \\
&\quad + \frac{1}{2} \int_0^l ES(x) (v_{10}'(x, t))^2 dx \\
&= \frac{1}{2} \mathbf{u}^T \mathbf{k}_e \mathbf{u},
\end{aligned} \tag{12}$$

where $v_{10}'(x, t) = \partial v_{10}(x, t)/\partial x$, $v_{20}''(x, t) = \partial^2 v_{20}(x, t)/\partial x^2$, and E is elastic modulus of materials. $I(x)$, $S(x)$ are cross-sectional moment of inertia and cross-sectional area of the beam element, respectively. Consider

$$\begin{aligned}
\mathbf{k}_e &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{k}_{eff} \end{bmatrix}, \\
\mathbf{k}_{eff} &= \int_0^l EI(x) \mathbf{N}_2''^T(x) \mathbf{N}_2''(x) dx \\
&\quad + \int_0^l ES(x) \mathbf{N}_1'^T(x) \mathbf{N}_1'(x) dx.
\end{aligned} \tag{13}$$

2.3. Dynamic Equation of the Beam Element. According to Lagrange's equation, dynamic equation of the beam element can be written as

$$\mathbf{m}_e \ddot{\mathbf{u}} + \mathbf{k}_e \mathbf{u} = \mathbf{p}_{e,e} + \mathbf{p}_{e,v}, \tag{14}$$

where $\mathbf{p}_{e,e}$ and $\mathbf{p}_{e,v}$ are the generalized external forces and the quadratic velocity vector that contains the gyroscopic and the Coriolis force components, respectively.

The elemental dynamic equation has been established in the A - xy system. Before forming the dynamic equation of the system, (14) must be expressed in the O - XY system. Define the coordinate transformation matrix $\mathbf{B}$ as follows:

$$\mathbf{B} = \begin{bmatrix} \cos \varphi & \sin \varphi & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sin \varphi & \cos \varphi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos \varphi & \sin \varphi & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{15}$$

Let

$$\mathbf{e}_f = \mathbf{B} \mathbf{E}_e, \tag{16}$$

where $\mathbf{E}_e$ is the elemental nodal coordinate vector in the O - XY system. Taking the first and the second derivative on (16) with respect to time yields

$$\begin{aligned}
\dot{\mathbf{e}}_f &= \mathbf{B}_\varphi \mathbf{E}_e \dot{\varphi} + \mathbf{B} \dot{\mathbf{E}}_e, \\
\ddot{\mathbf{e}}_f &= \mathbf{B}_{\varphi\varphi} \mathbf{E}_e \dot{\varphi}^2 + 2\mathbf{B}_\varphi \dot{\mathbf{E}}_e \dot{\varphi} + \mathbf{B}_\varphi \mathbf{E}_e \ddot{\varphi} + \mathbf{B} \ddot{\mathbf{E}}_e,
\end{aligned} \tag{17}$$

where $\mathbf{B}_\varphi = \partial \mathbf{B} / \partial \varphi$, $\mathbf{B}_{\varphi\varphi} = \partial^2 \mathbf{B} / \partial \varphi^2$. Substitute (16) and (17) into (14) and premultiply by matrix

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{E}_e^T \mathbf{B}_\varphi \\ \mathbf{0} & \mathbf{0} & \mathbf{B}^T \end{bmatrix}. \tag{18}$$

Then, dynamic equation of the beam element can be expressed in the O - XY system by

$$\mathbf{M}_e \ddot{\mathbf{U}}_e + \mathbf{K}_e \mathbf{U}_e = \mathbf{Q}_{e,e} + \mathbf{Q}_{e,v}, \tag{19}$$

FIGURE 2: The sketch of planar 3-RRR parallel robot.

where

$$\mathbf{U}_e = [\mathbf{r}_{e,A}^T \quad \varphi \quad \mathbf{E}_e^T]^T, \quad (20)$$

where $\mathbf{M}_e$ and $\mathbf{K}_e$ are 8×8 the element generalized mass and stiffness matrixes in the O - XY system, respectively. $\mathbf{Q}_{e,v}$ is 8×1 the quadratic velocity vector in the O - XY system. $\mathbf{Q}_{e,e}$ is 8×1 the generalized external forces vector in the O - XY system.

3. Governing Dynamic Equation of Planar 3-RRR Flexible Parallel Robots

In this section, according to the above dynamic equation of the beam element, governing dynamic equation of the planar 3-RRR parallel robots can be established, in which constraint equations are considered.

3.1. The Sketch of Planar 3-RRR Parallel Robots. The sketch of planar flexible 3-RRR parallel robots is constructed by the regular triangle moving platform $C_1C_2C_3$, the static platform, and three symmetrical kinematic chains $A_1B_1C_1$, $A_2B_2C_2$, and $A_3B_3C_3$, as shown in Figure 2. Each kinematic chain has one active revolute (R) joint followed by two consecutive passive revolute (R) joints. The active revolute joints are installed at A_i , $i = 1, 2, 3$. A_1 , A_2 and A_3 are the regular triangle's three vertices. $A_1B_1 = A_2B_2 = A_3B_3$, and $B_1C_1 = B_2C_2 = B_3C_3$. The vertexes O and P are centers of the regular triangles $A_1A_2A_3$ and $C_1C_2C_3$, respectively. The O - XY is the global fixed frame. Parameters α_i , β_i ($i = 1, 2, 3$) are the angles between the X -axis of the fixed frame and linkages A_iB_i , B_iC_i ($i = 1, 2, 3$), respectively. θ is the angle between the X -axis of the fixed frame and side C_1C_2 of the regular triangle $C_1C_2C_3$. L_1 , L_2 , L_3 , and L_4 are the length of the segments A_2B_2 , B_2C_2 , C_2P , and OA_2 .

3.2. Constraint Equations. Different from the simple structure, constraint relationships of planar 3-RRR of flexible parallel robots which include the rigid body motion constraints, the elastic deformation motion constraints, and

dynamic constraints of the moving platform are very complex and can be used to eliminate the correlation of generalized coordinates.

3.2.1. Constraint Equations of the Rigid-Body Motion. Generalized coordinates are formed by rigid-body motion coordinates and elastic coordinates. As shown in Figure 2, rigid-body motion coordinates that include drive joint rotation angles $\alpha = [\alpha_1 \quad \alpha_2 \quad \alpha_3]^T$, passive joint rotation angles $\beta = [\beta_1 \quad \beta_2 \quad \beta_3]^T$, translation displacement and rotation angle of the moving platform $\mathbf{X}_p = [X_p \quad Y_p \quad \theta]^T$, and rigid body motion coordinate vector α , β , $\mathbf{X}_p$ are not independent. Because three kinematic chains of 3-RRR parallel robots are full symmetrical, one chain is studied. As shown in Figure 3, a closed-loop vector equation can be established

$$\overrightarrow{OA_i} + \overrightarrow{A_iB_i} + \overrightarrow{B_iC_i} + \overrightarrow{C_iP} - \overrightarrow{OP} = \vec{0}. \quad (21)$$

Projecting (21) into X and Y component yields

$$\begin{aligned} X_{A_i} + L_1 \cos \alpha_i + L_2 \cos \beta_i \\ + L_3 \cos \left(\frac{\pi}{6} + \frac{2(i-1)\pi}{3} + \theta \right) - X_p = 0, \\ Y_{A_i} + L_1 \sin \alpha_i + L_2 \sin \beta_i \\ + L_3 \sin \left(\frac{\pi}{6} + \frac{2(i-1)\pi}{3} + \theta \right) - Y_p = 0, \end{aligned} \quad (22)$$

$i = 1, 2, 3,$

where (X_{A_i}, Y_{A_i}) and (X_p, Y_p) are the coordinates of the points A_i and P in the O - XY system, respectively.

3.2.2. Constraint Equations of the Elastic Deformation Motion. As shown in Figure 3, center point P and vertex C_i of the moving platform are moved to the points P' , C'_i because of elastic deformation motion of the flexible links A_iB_i , B_iC_i ($i = 1, 2, 3$). Setting P - xy is the local coordinate system with original point P , and P' - $x'y'$ is the elastic coordinate system with original point P' . $\mathbf{T}$, $\bar{\mathbf{T}}$ and $\bar{\mathbf{T}}$ are the coordinate transformation matrixes for p - $xy \rightarrow O$ - XY , p' - $x'y' \rightarrow p$ - xy and p' - $x'y' \rightarrow O$ - XY , respectively. Then, $\bar{\mathbf{T}} = \mathbf{T}\bar{\mathbf{T}}$. U_{i22} , U_{i23} , U_{i24} are elastic displacements and elastic rotational angle at the end point C_i of the flexile link B_iC_i , and the section curvature is equal to zero at the point C_i . ΔX_p , and ΔY_p , ε are translational and rotational declinations of the moving platform because of elastic vibration of the flexible links. Assuming that the moving platform is rigid, so $PC_i = P'C'_i$. Noting that

$$\mathbf{T} = \begin{bmatrix} \cos \theta & -\sin \theta & X_p \\ \sin \theta & \cos \theta & Y_p \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{\mathbf{T}} = \begin{bmatrix} 1 & -\varepsilon & \Delta X_p \\ \varepsilon & 1 & \Delta Y_p \\ 0 & 0 & 1 \end{bmatrix}. \quad (23)$$

FIGURE 3: The flexible kinematic chain.

Assuming that $(X_{C_i} \ Y_{C_i})$ and $(X'_{C'_i} \ Y'_{C'_i})$ are the coordinates of the points C_i and C'_i in the O - XY system yields

$$\begin{bmatrix} X_{C'_i} \\ Y_{C'_i} \\ 1 \end{bmatrix} = \bar{\mathbf{T}} \begin{bmatrix} x'_{C'_i} \\ y'_{C'_i} \\ 1 \end{bmatrix}_{P'} \quad (24)$$

$$\begin{bmatrix} X_{C_i} \\ Y_{C_i} \\ 1 \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_{C_i} \\ y_{C_i} \\ 1 \end{bmatrix}_P$$

where $(x_{C_i}, y_{C_i})_P = (x'_{C'_i}, y'_{C'_i})_{P'} = (L_3 \cos(\pi/2 + 2i\pi/3), L_3 \sin(\pi/2 + 2i\pi/3))$. $(\)_P$ means the description in the p - xy .

Setting

$$\mathbf{T}_1 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad (25)$$

yields

$$\begin{bmatrix} U_{i22} \\ U_{i23} \end{bmatrix} = \begin{bmatrix} X_{C'_i} \\ Y_{C'_i} \end{bmatrix} - \begin{bmatrix} X_{C_i} \\ Y_{C_i} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{T}_1 \end{bmatrix} \begin{bmatrix} -y_{C_i} \\ x_{C_i} \end{bmatrix}_P \begin{bmatrix} \Delta X_P \\ \Delta Y_P \\ \varepsilon \end{bmatrix} \quad (26)$$

$$\varepsilon = \sum_{i=1}^3 U_{i24},$$

where $\mathbf{I}$ is 2×2 unit matrix.

3.2.3. Dynamic Constraints of the Moving Platform. Assuming that $\mathbf{F}_i$ is the generalized joint constraint antforce that the passive joint C_i are applied to the moving platform. M_v, J_v are the mass and moment of inertia of the moving platform. Then, dynamic constraint of the moving platform can be expressed by

$$\begin{bmatrix} M_v & & \\ & M_v & \\ & & J_v \end{bmatrix} \begin{bmatrix} \Delta \ddot{X}_P \\ \Delta \ddot{Y}_P \\ \ddot{\varepsilon} \end{bmatrix} = \sum_{i=1}^3 \mathbf{F}_i - \begin{bmatrix} M_v \ddot{X}_P \\ M_v \ddot{Y}_P \\ J_v \ddot{\varepsilon} \end{bmatrix}. \quad (27)$$

4. Governing Dynamic Equation of the Flexible Robots System

Considering constraint equations ((22)–(27)) and assembling all the element dynamic equations with respect to the compatibility at the nodes, then the governing dynamic equation that describes the dynamic characteristic of system can be formed as

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{Q}_e + \mathbf{Q}_v, \quad (28)$$

where $\mathbf{U}$ is the generalized coordinate vector of planer 3-RRR flexible parallel robots in the O - XY system which includes rigid-body motion coordinates, node elastic deformation coordinates, and the declinations of the moving platform. Equation (28) is the nonlinear and rigid-flexible coupling

FIGURE 4: (a) and (b) are elastic displacements of the moving platform along with X, Y direction, respectively; (c) is elastic rotational angle of the moving platform in XY planar, when $w = 10\pi$ rad/s.

differential equation and can be decomposed into the rigid subsection and the flexible subsection as

$$\mathbf{M}_r(\mathbf{U}) \ddot{\mathbf{a}} = \mathbf{Q}_v^r(\mathbf{U}, \dot{\mathbf{U}}) + \boldsymbol{\tau}, \quad (29)$$

$$\mathbf{M}_f(\boldsymbol{\alpha}) \ddot{\mathbf{U}} + \mathbf{C}_f(\boldsymbol{\alpha}, \dot{\mathbf{a}}) \dot{\mathbf{U}} + \mathbf{K}_f(\boldsymbol{\alpha}, \dot{\mathbf{a}}, \ddot{\mathbf{a}}) \mathbf{U} = \mathbf{Q}_f(\boldsymbol{\alpha}, \dot{\mathbf{a}}, \ddot{\mathbf{a}}). \quad (30)$$

Now, one will emphatically study flexible subsection (30), in which generalized stiffness matrix $\mathbf{K}_f(\boldsymbol{\alpha}, \dot{\mathbf{a}}, \ddot{\mathbf{a}})$ can be expressed by

$$\begin{aligned} \mathbf{K}(\boldsymbol{\alpha}, \dot{\mathbf{a}}, \ddot{\mathbf{a}}) &= \sum_e \mathbf{D}_e^T \mathbf{B}^T \left\{ \left[(\mathbf{N}_5 + \mathbf{C}_2) \mathbf{B}_{\varphi\varphi} - 2 \left(\mathbf{N}_7 + \widetilde{\mathbf{N}}_c \right) \mathbf{B}_{\varphi} \right. \right. \\ &\quad \left. \left. - (\mathbf{N}_5 + \widetilde{\mathbf{N}}_c) \mathbf{B} \right] \dot{\boldsymbol{\varphi}}^2 \right. \\ &\quad \left. + \left[(\mathbf{N}_5 + \mathbf{C}_2) \mathbf{B}_{\varphi} + \left(\mathbf{N}_7^T + \widetilde{\mathbf{N}}_c^T \right) \mathbf{B} \right] \ddot{\boldsymbol{\varphi}} \right. \\ &\quad \left. + \mathbf{k}_{ff} \mathbf{B} \right\} \mathbf{D}_e \end{aligned}$$

FIGURE 5: (a) and (b) are elastic displacements of along with X, Y direction which are located at the 1/3 length of the active link 1, respectively, when $w = 10\pi$ rad/s.

FIGURE 6: (a) is maximum stress response of all flexible links through numerical calculation and (b) is ANSYS results, when $w = 10\pi$ rad/s.

$$\begin{aligned}
 & + \sum_e \mathbf{D}_e^T \mathbf{B}^T (g_{1e} \tilde{\mathbf{H}}_e + \mathbf{N}_9^T \dot{\varphi} - \mathbf{N}_8 \dot{\varphi}^2) \mathbf{B} \mathbf{D}_e \\
 & = \mathbf{K}_1 + \mathbf{K}_d,
 \end{aligned}$$

(31)

where $\tilde{\mathbf{H}}_e = \int_V -\rho \mathbf{H}_e^T dV$, $\mathbf{R}^T \tilde{\mathbf{r}}_{e,A} = \begin{bmatrix} g_{1e} \\ g_{2e} \end{bmatrix}$, $\mathbf{D}_e$ is the coordinate compatibility matrix, $\sum_e$ means summing to all elements, and φ is a function of rigid body motion variable α . The underline term is the dynamic stiffening term [14]. In the conventional

ZOAC model, the term is equal to zero. In general, when the system operates at a high-speed and high-acceleration such as the flexible hub-beam [14–22], the term cannot be neglected.

5. Numerical Simulation

In this section, dynamic responses of planar 3-RRR flexible parallel robots are studied using the ZOAC model and FOAC model, comparing influences of the dynamic stiffening term

FIGURE 7: (a), (b), (c), and (d) are the first four-order modal shapes of the system, respectively.

TABLE 1: Parameters of the flexible links of 3-RRR.

Parameter	Link A_iB_i	link B_iC_i
Length (m)	0.245	0.242
Width (m)	0.025	0.025
Thickness (m)	0.003	0.003
Density ρ (kg/m ³)	2712	
Young's Modulus (Pa) E	7.102×10^{10}	
Poisson's Ratio μ	0.3	

on dynamic response under the different speeds of the moving platform.

Assuming that all links that include three active links and three passive links are flexible and that other parts are rigid, every flexible link is divided into three equal length beam elements. Assume that the material of 3-RRR system

is the aluminum alloy, the thickness of the moving platform $c = 0.0034$ m, $L_1 = 0.245$ m, $L_2 = 0.242$ m, $L_3 = 0.108$ m, $L_4 = 0.4$ m, the lumped mass of the joint $M_v = 0.02$ kg, and the lumped moment of inertia $J_v = 0.00005$ kgm². The other parameters of flexible links are defined as in Table 1.

It is assumed that the trajectory of the moving platform is described as

$$\begin{aligned}
 X_P &= 0.04 \cos(w\pi t) - 0.02, \\
 Y_P &= 0.04 \sin(w\pi t), \\
 \theta &= \frac{\pi}{4},
 \end{aligned} \tag{32}$$

where w is angular frequency that can reflect moving speed of the motion platform. Due to the requirement of following numerical simulation, the parameter w is chosen to be

FIGURE 8: (a) and (b) are elastic displacements of the moving platform along with X, Y direction, respectively, (c) is elastic rotational angle of the moving platform in XY planar, and (d) is maximum stress response of all flexible links through numerical calculation, when $w = 350\pi$ rad/s.

10π rad/s and 350π rad/s, respectively. Let the moving platform continuously operate four motion cycles along with the given circle trajectory Equation (32). Firstly, considering the parameter $w = 10\pi$ rad/s in (32), in this case, the acceleration of the moving platform is 4 g (g is weight acceleration), and the maximum absolute angle speed of the drive joints w_i , $i = 1, 2, 3$ reaches to 5.46, 5.19, and 5.2 rad/s, respectively. Figure 4 shows elastic displacements and rotation angle of the moving platform with respect to four motion cycles based on the ZOAC model and the FOAC model when $w = 10\pi$ rad/s, respectively. Figure 5 shows elastic displacement responses which are located at the 1/3 length of active link 1.

In all figures, the slide line with $\circ$ is a result using the ZOAC model, the slide line with $+$ is a result using the FOAC model, and the slide line with $*$ is a result using ANSYS simulation. Figure 6(a) shows numerical result of the maximum stress response of flexible links; meanwhile, the simplified 3D model is developed by SolidWorks software and then imported into Workbench environment which is a simulation module of ANSYS 13.0 software for transient dynamic analysis as shown in Figure 6(b). All curves show periodical change in Figures 4–6.

Form Figures 4, 5, and 6(a), we can find that the numerical results are very close using the ZOAC model and the FOAC

FIGURE 9: (a) and (b) are elastic displacements of along with X,Y direction which are located at the 1/3 length of the active link 1, respectively, when $\omega = 350\pi$ rad/s.

model. The influence of dynamic stiffening on dynamic characteristic of the system can be neglected when $\omega = 10\pi$ rad/s. Reviewing Figures 6(a) and 6(b), one can find that numerical results are similar with ANSYS simulation results, and the change law of the curves are consistent and show periodic variation. Then, the validity of ZOAC model and FOAC model is verified.

To illustrate the validity of the theory model, modal analysis module of Workbench is used to calculate natural frequencies of the system. Theory results using the theory model and ANSYS result using ANSYS-Workbench software are listed in Table 2. From Table 2, we can find that the difference of the first natural frequency is 0.99% and very small, and the difference of the second natural frequency is 8.91%, the difference of the third natural frequency is big and reaches to 37.14%, the fourth-order is 11.74%, meanwhile, all the first three-order natural frequencies are smaller than the fourth-order natural frequency using two methods. The existence of difference can be explained as follows: (1) all components are flexible in the ANSYS simulation, but, just only the links are flexible in theory model, (2) Meshing is different, and (3) solver may be different. Form the difference toleration and the consistent of the natural frequency change, ANSYS result can be used to verify the validity of the theory model.

To further verify, influence of stiffening dynamic of 3-RRR flexible parallel robots is slight and can be neglected in the mathematics model. Let $\omega = 350\pi$ rad/s in (32). Then, the acceleration of the moving platform is 4836 g (Unbelievable), and the maximum absolute angle speed of the drive joints ω_i , $i = 1, 2, 3$ reaches to 191.02 (30.04 Hz),

TABLE 2: The first four-order natural frequencies of the system (Hz).

	ω_1	ω_2	ω_3	ω_4
Theory result	27.177	38.146	65.794	307.4
ANSYS result	27.446	34.74	41.359	348.29
Difference	0.99%	8.93%	37.14%	11.74%

181.77 (28.93 Hz), and 182.58 rad/s (29.06 Hz) and exceeds the first natural frequency 27.177 Hz (or 27.446 Hz), as shown in Figures 8 and 9. Figures 8(a), 8(b), and 8(c) are elastic displacements and rotation angle of the moving platform with respect to four motion cycles using the ZOAC model and the FOAC model when $\omega = 350\pi$ rad/s, respectively. Figure 8(d) is the maximum stress response of flexible links with respect to four motion cycles. Figures 9(a) and 9(b) are elastic displacement responses along with X- axis and Y-axis which are located at the 1/3 length of active link 1, respectively.

From Figures 8 and 9, we find that although the acceleration of the moving platform has reached to 4836 g m/s² and the angle speed of the drive joints has exceeded the first natural frequency of the system, all curves using the ZOAC model are still very close to the relevant curves using the FOAC model. The influence of dynamic stiffening is still very slight. At the moment, elastic displacements and rotational angle of the moving platform and flexible link 1 increase two orders of magnitude more than $\omega = 10\pi$ rad/s. The maximum stress of flexible links changes from 67.99 MPa to 125600 MPa and increases four orders of magnitude. When

increasing w in (32) again, there are just only increasing vibration amplitude of flexible links and the moving platform and maximum stress amplitude of flexible links; numerical results of the ZOAC model are still close to those of the FOAC model, and the difference between the ZOAC model and FOAC model is very slight.

From the above analysis, we can conclude that the influence of dynamic stiffening of planar 3-RRR flexible parallel robots is very slight and can fully be neglected for elastic dynamic modeling and the conventional ZOAC model can completely reveal dynamic characteristic of the system. The reasons can be explained as follows. (1) Reviewing $\mathbf{K}_d = \sum_e \mathbf{D}_e^T \mathbf{B}^T (g_{1e} \tilde{\mathbf{H}}_e + \mathbf{N}_9^T \dot{\boldsymbol{\varphi}} - \mathbf{N}_8 \dot{\boldsymbol{\varphi}}^2) \mathbf{B} \mathbf{D}_e$, where $\dot{\boldsymbol{\varphi}}, g_{1e}$ may take positive or negative value, the values of the first two terms may be positive or negative in bracket, and the third term is positive. Then, the value of the expression in brackets will be small and its sign is indeterminate. After summing with respect to all elements, $\mathbf{K}_d$ is much smaller than $\mathbf{K}_1$, so that $\mathbf{K}_d$ can be neglected in (31). (2) Because of the coupling among flexible links, the influence of dynamic stiffening which is related to rigid-body motion will be weakened.

6. Conclusions

The FOAC model of planar 3-RRR flexible parallel robots with six flexible links was established through the finite element method and the Lagrange equation, in which dynamic stiffening term was considered. The explicit expression of dynamic stiffening was presented. The constraint equations of the system that include the rigid body motion constraints, elastic deformation motion constraints, and dynamics constraints of the moving platform were studied in detail. For comparing the FOAC model with the conventional ZOAC model, numerical simulation was given under two different speeds of the moving platform. The results indicated that the conventional ZOAC model can fully reveal dynamic characteristic of the system and that the influence of dynamic stiffening is very slight and can be neglected for dynamic modeling of planar 3-RRR flexible parallel robots. So, the modeling process can be simplified. Meanwhile, the commercial software ANSYS 13.0 is used to confirm the validity of the zero-order approximation coupling model.

Appendix

$$\begin{aligned} \mathbf{p}_{e,v} &= -\dot{\mathbf{m}}\dot{\mathbf{u}} + \frac{\partial}{\partial \mathbf{u}} \left(\frac{1}{2} \dot{\mathbf{u}}^T \mathbf{m} \dot{\mathbf{u}} \right) \\ &= \begin{bmatrix} -\dot{\boldsymbol{\varphi}}^2 \mathbf{R}_{\varphi\varphi} \mathbf{N}_1 - \dot{\boldsymbol{\varphi}}^2 \mathbf{R}_{\varphi\varphi} \mathbf{N}_2 \mathbf{e}_f - 2\dot{\boldsymbol{\varphi}} \mathbf{R}_{\varphi\varphi} \mathbf{N}_2 \dot{\mathbf{e}}_f - \dot{\boldsymbol{\varphi}}^2 \mathbf{R}_{\varphi\varphi} \int_V \rho \begin{bmatrix} w_c \\ 0 \end{bmatrix} dV \\ -\mathbf{R}_{\varphi} \int_V \rho \begin{bmatrix} -\mathbf{e}_f^T \mathbf{H}_e \dot{\mathbf{e}}_f \\ 0 \end{bmatrix} dV - \mathbf{R} \int_V \rho \begin{bmatrix} -\dot{\mathbf{e}}_f^T \mathbf{H}_e \dot{\mathbf{e}}_f \\ 0 \end{bmatrix} dV - \dot{\boldsymbol{\varphi}}^2 \mathbf{R}_{\varphi\varphi} \bar{\mathbf{N}}_c \mathbf{e} - 2\dot{\boldsymbol{\varphi}} \mathbf{R}_{\varphi\varphi} \bar{\mathbf{N}}_c \dot{\mathbf{e}}_f \\ -2(\mathbf{N}_4 \dot{\mathbf{e}}_f + \mathbf{e}_f^T \mathbf{N}_8 \dot{\mathbf{e}}_f + \mathbf{e}_f^T \mathbf{N}_5 \dot{\mathbf{e}}_f + \mathbf{e}^T \bar{\mathbf{N}}_c \dot{\mathbf{e}}_f) \dot{\boldsymbol{\varphi}} - (\dot{\mathbf{e}}_f^T \mathbf{N}_7 + \dot{\mathbf{e}}_f^T \mathbf{N}_9 + \dot{\mathbf{e}}_f^T \bar{\mathbf{N}}_c) \dot{\mathbf{e}}_f \\ 2\mathbf{N}_7 \dot{\mathbf{e}}_f \dot{\boldsymbol{\varphi}} + (\mathbf{N}_4^T + \mathbf{N}_8 \mathbf{e}_f + \mathbf{N}_5 \mathbf{e}_f) \dot{\boldsymbol{\varphi}}^2 + 2\bar{\mathbf{N}}_c \dot{\mathbf{e}}_f \dot{\boldsymbol{\varphi}} + \bar{\mathbf{N}}_c \mathbf{e} \dot{\boldsymbol{\varphi}}^2 \end{bmatrix}, \\ \mathbf{M}_f(\boldsymbol{\alpha}) &= \sum_e \mathbf{D}_e^T \mathbf{B}^T (\mathbf{N}_5 + \mathbf{C}_2) \mathbf{B} \mathbf{D}_e, \\ \mathbf{C}_f(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}) &= \sum_e \mathbf{D}_e^T \mathbf{B}^T 2 \left[(\mathbf{N}_5 + \mathbf{C}_2) \mathbf{B}_{\varphi} - (\mathbf{N}_7 + \bar{\mathbf{N}}_c) \mathbf{B} \right] \dot{\boldsymbol{\varphi}} \mathbf{B} \mathbf{D}_e, \\ \mathbf{Q}_f(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}, \ddot{\boldsymbol{\alpha}}) &= \sum_e \mathbf{D}_e^T \mathbf{B}^T \left[\mathbf{N}_4^T \dot{\boldsymbol{\varphi}}^2 + \mathbf{N}_c \mathbf{P}_0 \dot{\boldsymbol{\varphi}}^2 - (\mathbf{N}_2^T + \mathbf{N}_c^T) \mathbf{R}^T \ddot{\mathbf{r}}_{e,A} - \mathbf{N}_6^T \ddot{\boldsymbol{\varphi}} - \bar{\mathbf{N}}_c^T \mathbf{P}_0 \ddot{\boldsymbol{\varphi}} - \mathbf{C}_1^T \ddot{\boldsymbol{\varphi}} \right], \\ \mathbf{P}_0 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T. \end{aligned} \quad (\text{A.1})$$

Conflict of Interests

The authors declared that they have no conflicts of interest to this work.

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