Research Article

Free Vibration Analysis of Moderately Thick Rectangular Plates with Variable Thickness and Arbitrary Boundary Conditions

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A generalized Fourier series solution based on the first-order shear deformation theory is presented for the free vibrations of moderately thick rectangular plates with variable thickness and arbitrary boundary conditions, a class of problem which is of practical interest and fundamental importance but rarely attempted in the literatures. Unlike in most existing studies where solutions are often developed for a particular type of boundary conditions, the current method can be generally applied to a wide range of boundary conditions with no need of modifying solution algorithms and procedures. Under the current framework, the one displacement and two rotation functions are generally sought, regardless of boundary conditions, as an improved trigonometric series in which several supplementary functions are introduced to remove the potential discontinuities with the displacement components and its derivatives at the edges and to accelerate the convergence of series representations. All the series expansion coefficients are treated as the generalized coordinates and solved using the Rayleigh-Ritz technique. The effectiveness and reliability of the presented solution are demonstrated by comparing the present results with those results published in the literatures and finite element method (FEM) data, and numerous new results for moderately thick rectangular plates with nonuniform thickness and elastic restraints are presented, which may serve as benchmark solution for future researches.

1. Introduction

In comparison with vibration analysis of plates with uniform thickness, far less attention has been paid to the vibration problems of plates with variable thickness, which are commonly widely used in engineering applications as a stand-alone structure or a constituent structural component. By carefully designing the thickness distribution, a substantial increase in stiffness, buckling, and vibration capacities of the plate may be obtained over its uniform thickness counterpart. In addition, the moderately thick plates with variable thickness in these practical applications often work in complex environments and suffer from arbitrary edge restraints. Therefore a thorough dynamic study of moderately thick rectangular plates with variable thickness and arbitrary edge restraints is essential to assess and use the full potentials of plates.

Over past decades, extensive investigations have been carried out to determine the vibration characteristics

(natural frequencies, mode shapes, and so on) of moderately thick rectangular plates. A comprehensive review on the relevant studies done before 1995 was presented by Liew et al. [1]. The Rayleigh-Ritz method based on polynomials with properties corresponding to those of Timoshenko beam functions was used by Chung et al. [2] to study the free vibrations of orthotropic rectangular Mindlin plates with edges elastically restrained against rotation. Later, Cheung and Zhou [3] extended this solution for the free vibration problems of moderately rectangular plates. Wang [4] presented an exact formula for the vibration frequencies of simply supported Mindlin plates with the corresponding simply supported thin plate frequencies. Saha et al. [5] utilized the variational method to investigate the vibration characteristics of isotropic Mindlin plates with edges elastically restrained against rotation and translation. Gorman [6] presented accurate eigenvalues for shear-deformable plates resting on uniform elastic foundations with modified superposition-Galerkin

method. Subsequently, Gorman [7, 8] utilized the superposition method to investigate the vibration problems of elastic restrained and point supported Mindlin plates. In his superposition method, the solution satisfied the differential equations exactly but approximated the boundary conditions. Xiang [9] and Xiang and Wei [10] employed the Levy solution approach in conjunction with the domain decomposition technique to analytically solve the free vibrations of rectangular Mindlin plates with two opposite edges simply supported. In [10], the influence of the step thickness ratios on the vibration behavior of rectangular Mindlin plates is studied. Yeh et al. [11] proposed a hybrid method which combines the finite difference method and the differential transformation method to analyse the free vibration of clamped and simply supported rectangular thin plates. Xiang et al. [12] extended the DSC-Ritz element method to solve the free vibration analysis of moderately thick rectangular plates with mixed supporting edges. Thai et al. [13–15] and Nguyen-Thoi et al. [16–19] have recently proposed the isogeometric analysis (IGA) and smoothed finite element method (S-FEM) to analyse the cracked Mindlin plate, laminated composite, and sandwich plates. In their research, the structural boundary was taken as uniform classic boundary conditions. In addition, Nguyen-Thanh et al. [20] utilized the alpha finite element method (α -FEM) in static, free vibration, and buckling analyses of Mindlin-Reissner plates using triangular elements only.

A Rayleigh-Ritz formulation based on pb -2 functions was utilized by Liew et al. [21] to study the vibration behaviors of Mindlin plates with elastic restrained edges. Zhou et al. [22, 23] investigated the similar vibration problems in terms of a set of static Timoshenko beam functions. In their study, the free vibrations of rectangular Mindlin plates with internal line supports were also solved. Shen et al. [24] presented a new set of admissible functions, which satisfied both geometrical and natural boundary conditions, for the free and forced vibration problems of moderately thick plates with four free edges. Xing and Liu [25] presented the closed-form solutions for vibration problems of Mindlin plates with any combinations of simply supported and clamped edge conditions. Apart from aforementioned solutions, some other methods such as finite strip element method [26], spline strip method [27], finite element method [28], meshless method [29], Ritz method [30], differential quadrature method (DQM) [31–33], and Green function method [34, 35] had been developed to determine the vibration behavior of moderately thick plates.

It is noteworthy that most of the previous researches on moderately thick rectangular plates are confined to the uniform thickness and classical boundary conditions, such as free, simply supported, clamped edges and their combinations. Only few studies have been found in the published literature about the vibrations of moderately thick rectangular plates with variable thickness and elastic restraints. A linear finite strip element method based on Mindlin's plate theory was proposed by Gagnon et al. [26] to solve the vibration problems of rectangular thick plates, in which the thickness can vary in any direction. Huang et al. [36] developed a discrete method for solving the vibration problems of orthotropic rectangular plates with variable thickness and

general classical boundary supports. Extended Kantorovich method was utilized to investigate the free vibrations of rectangular thick plates with variable thickness and different classical boundary conditions by Shufrin and Eisenberger [37]. Eftekhari and Jafari [38] proposed an efficient and accurate variational formulation for the vibration problems of variable thin and thick plates with elastic edge restraints.

Although a large number of studies have been carried out based on Mindlin's plate theory and methods, it appears that the information available about the vibration characteristics of nonuniform thickness moderately thick rectangular plates is very limited. Most of the contributions to moderately thick rectangular plates with classic boundary supports and elastic edges are confined to uniform thickness or bilinearly varying thickness. However, the engineering practices contain a variety of possible boundary conditions such as elastic restraints and nonlinearly varying thickness. The existing results are simply too scarce for engineering applications and comparative studies. Moreover, most of the available solution procedures in the open literature are often only customized for a specific set of restraint conditions, which may not be appropriate for practical application because there are hundreds of different combinations of boundary conditions for a plate. It is desirable to develop a unified and efficient method which is capable of dealing with more complicated problems involving arbitrary elastic edge restraints and nonuniform thickness.

In view of those technical limitations and practical needs, this investigation sets out to present a modified Fourier solution technique for the free vibration analysis of moderately thick rectangular plates with variable thickness and arbitrary boundary conditions and to provide a unified and reasonable accurate alternative to other analytical and numerical techniques. This paper can be considered as an extension of the modified Fourier series method previously developed for modeling plates [39–41] and shells [42, 43]. The one displacement and two rotation functions are invariably expressed as the superposition of a 2D Fourier cosine series and four supplementary functions in the form of the product of a polynomial function and a single cosine series expansion, with all these unknown expansion coefficients treated as the generalized coordinates and determined using the Rayleigh-Ritz procedure. The change of the boundary conditions can be easily achieved by only varying the stiffness of the three sets of boundary springs along all edges of the rectangular plates without involving any change to the solution procedure. The current results are checked against with FEM results or existing results published in the literature for both uniform and nonuniform (linear and nonlinear variation) thickness plate cases, with good agreement achieved.

2. Theoretical Formulations

To perform the free vibration analysis of moderately thick rectangular plates with thickness varying in two directions subjected to the general elastic boundary conditions, the combination of the artificial spring technique together with Rayleigh-Ritz method is feasible. Consider a moderately thick rectangular plate, with the dimension of $a \times b \times h(x, y)$,

FIGURE 1: The general elastic boundary conditions of moderately thick rectangular plates with varying thickness in two directions.

and the coordinate of the moderately thick rectangular plate with elastically restrained edges is depicted in Figure 1. Three groups of boundary restraining springs (translation, rotational, and torsional springs) are arranged at all sides of the plate to separately simulate the boundary force. By assigning the stiffness of the boundary springs various values, it is equivalent to impose different boundary force on the midsurface of the plate. For example, the clamped boundary condition can be readily obtained by setting the spring coefficients into infinity (a very large number in practical calculation) for the translation, rotations, and torsional restraining springs along each edge.

Based on the Mindlin plate theory, the displacements vectors with three directions are

$$\begin{aligned} u(x, y, z, t) &= z\psi_x(x, y, t), \\ v(x, y, z, t) &= z\psi_y(x, y, t), \\ w(x, y, z, t) &= w(x, y, t), \end{aligned} \quad (1)$$

where u , v , and w represent the x , y , and z direction displacement functions and the ψ_x and ψ_y are the slop due to bending along in the respective planes. The relationship w with the slops ψ_x and ψ_y is $\psi_x = -dw/dx$ and $\psi_y = -dw/dy$.

For the moderately thick rectangular plates, making use of the strain-stress relationship defined in elasticity theory, the normal, shear strains and transverse shear strains can be expressed as follows:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \frac{z\partial\psi_x}{\partial x} \\ \frac{z\partial\psi_y}{\partial y} \\ z\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\right) \\ \psi_x + \frac{\partial w}{\partial x} \\ \psi_y + \frac{\partial w}{\partial y} \end{Bmatrix},$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \frac{E}{2(1-\mu^2)} \begin{bmatrix} 2 & 2\mu & 0 & 0 & 0 \\ 2\mu & 2 & 0 & 0 & 0 \\ 0 & 0 & 1-\mu & 0 & 0 \\ 0 & 0 & 0 & \kappa(1-\mu) & 0 \\ 0 & 0 & 0 & 0 & \kappa(1-\mu) \end{bmatrix} \times \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}, \quad (2)$$

where ε_{xx} , ε_{yy} , and γ_{xy} are the normal and shear strains in the x , y , and z coordinate system. The transverse shear strains γ_{xz} and γ_{yz} are constant through the thickness. The σ_{xx} and σ_{yy} are the normal stresses in the x , y directions; τ_{xz} , τ_{yz} , and τ_{xy} are shear stresses in the x , y , and z coordinate system. The E is Young's modulus, μ is Poisson's ration, and κ is the shear correction factor to account for the fact.

In terms of transverse displacements and slope, the bending and twisting moments and the transverse shearing forces in plates can be expressed as

$$\begin{aligned} M_{xx} &= \int_{-h/2}^{h/2} \sigma_{xx} z dz = D \left(\frac{\partial\psi_x}{\partial x} + \mu \frac{\partial\psi_y}{\partial y} \right), \\ M_{yy} &= \int_{-h/2}^{h/2} \sigma_{yy} z dz = D \left(\frac{\partial\psi_y}{\partial y} + \mu \frac{\partial\psi_x}{\partial x} \right), \\ M_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} z dz = \frac{D(1-\mu)}{2} \left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x} \right), \\ Q_x &= \kappa \int_{-h/2}^{h/2} \tau_{xz} dz = \kappa G h(x, y) \left(\psi_x + \frac{\partial w}{\partial x} \right), \\ Q_y &= \kappa \int_{-h/2}^{h/2} \tau_{yz} dz = \kappa G h(x, y) \left(\psi_y + \frac{\partial w}{\partial y} \right), \end{aligned} \quad (3)$$

where $h(x, y)$ is the thickness function, $D = Eh(x, y)^3 / (12(1-\mu^2))$ is the flexural rigidity, and $G = E/(2(1+\mu))$ is the shear modulus.

The boundary conditions for a general supported moderately thick rectangular plate can be expressed as the following forms based on the force equilibrium relationship on the four sides:

on $x = 0$,

$$k_{x0}w = -Q_x, \quad K_{x0}\psi_x = -M_{xx}, \quad K_{yx0}\psi_y = -M_{xy},$$

on $x = a$,

$$k_{xa}w = Q_x, \quad K_{xa}\psi_x = M_{xx}, \quad K_{yxa}\psi_y = M_{xy},$$

on $y = 0$,

$$k_{y0}w = -Q_y, \quad K_{y0}\psi_y = -M_{yy}, \quad K_{xy0}\psi_x = -M_{xy},$$

on $y = b$,

$$k_{yb}w = Q_y, \quad K_{yb}\psi_y = M_{yy}, \quad K_{xyb}\psi_x = M_{xy}, \quad (4)$$

where k_{x0} and k_{xa} (k_{y0} and k_{yb}) are linear spring constants, K_{x0} and K_{xa} (K_{y0} and K_{yb}) are the rotational spring constants, and K_{yx0} and K_{yxa} (K_{xy0} and K_{xyb}) are the torsional spring constants at $x = 0$ and a ($y = 0$ and b), respectively. Therefore, arbitrary boundary conditions of the plate can be generated by assigning the linear springs, rotational springs, and torsional springs at proper stiffness. For instance, a clamped boundary (C) is achieved by simply setting the stiffness of the entire springs equal to infinity (which is represented by a very large number, 10^{14}). Inversely, a free boundary (F) is gained by setting the stiffness of the entire springs equal to zero.

Thus the total potential energy of the spring restrained plate which is composed of two parts, namely, the strain energy of the moderately thick rectangular plates and the potential energy stored in the boundary springs, can be expressed as

$$\begin{aligned} U = & \frac{1}{2} \int_0^a \int_0^b \left\{ D \left[\left(\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right)^2 - 2(1-\mu) \right. \right. \\ & \times \left. \left(\frac{\partial \psi_x}{\partial x} \frac{\partial \psi_y}{\partial y} - \frac{1}{4} \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right)^2 \right) \right] \\ & + \kappa Gh(x, y) \\ & \times \left[\left(\psi_x + \frac{\partial w}{\partial x} \right)^2 + \left(\psi_y + \frac{\partial w}{\partial y} \right)^2 \right] \Big\} dx dy \\ & + \frac{1}{2} \int_0^a \left[(k_{y0}w^2 + K_{y0}\psi_y^2 + K_{xy0}\psi_x^2) \Big|_{y=0} \right. \\ & \left. + (k_{yb}w^2 + K_{yb}\psi_y^2 + K_{xyb}\psi_x^2) \Big|_{y=b} \right] dx \\ & + \frac{1}{2} \int_0^b \left[(k_{x0}w^2 + K_{x0}\psi_x^2 + K_{yx0}\psi_y^2) \Big|_{x=0} \right. \\ & \left. + (k_{xa}w^2 + K_{xa}\psi_x^2 + K_{yxa}\psi_y^2) \Big|_{x=a} \right] dy. \end{aligned} \quad (5)$$

As the springs are considered with no mass while retaining certain stiffness, the total kinetic energy of the moderately thick rectangular plates is

$$T = \frac{\rho \omega^2}{2} \int_0^a \int_0^b \left[\frac{h(x, y) w^2 + h^3(x, y) (\psi_x^2 + \psi_y^2)}{12} \right] dx dy, \quad (6)$$

where ρ is the mass density and ω denotes the natural frequency of the plate.

In view of satisfying arbitrarily supported boundary conditions of the moderately thick rectangular plate, the admissible functions expressed in the form of the improved Fourier series are introduced to remove the potential discontinuities with the functions and their derivatives. Thus, the moderately thick rectangular plate displacements and rotation are expressed as

$$\begin{aligned} \psi_x(x, y) = & \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \cos \lambda_{am} x \cos \lambda_{bn} y \\ & + \sum_{l=1}^2 \zeta_b^l(y) \sum_{m=0}^{\infty} a_m^l \cos \lambda_{am} x \end{aligned} \quad (7)$$

$$\begin{aligned} \psi_y(x, y) = & \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} B_{mn} \cos \lambda_{am} x \cos \lambda_{bn} y \\ & + \sum_{l=1}^2 \zeta_b^l(y) \sum_{m=0}^{\infty} c_m^l \cos \lambda_{am} x \\ & + \sum_{l=1}^2 \zeta_a^l(x) \sum_{n=0}^{\infty} b_n^l \cos \lambda_{bn} y, \end{aligned} \quad (8)$$

$$\begin{aligned} w(x, y) = & \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn} \cos \lambda_{am} x \cos \lambda_{bn} y \\ & + \sum_{l=1}^2 \zeta_b^l(y) \sum_{m=0}^{\infty} e_m^l \cos \lambda_{am} x \\ & + \sum_{l=1}^2 \zeta_a^l(x) \sum_{n=0}^{\infty} f_n^l \cos \lambda_{bn} y, \end{aligned} \quad (9)$$

where $\lambda_{am} = m\pi/a$, $\lambda_{bn} = n\pi/b$, and A_{mn} , B_{mn} , and C_{mn} are the Fourier coefficients of two-dimensional Fourier series expansions for the displacements and rotation functions, respectively. a_m^l , b_m^l , c_m^l , d_m^l , e_m^l , and f_m^l are the supplemented coefficients of the auxiliary functions, where $l = 1, 2$. The specific expressions of the auxiliary functions ζ_a^l and ζ_b^l are defined as

$$\begin{aligned} \zeta_a^1(x) &= \frac{a}{2\pi} \sin\left(\frac{\pi x}{2a}\right) + \frac{a}{2\pi} \sin\left(\frac{3\pi x}{2a}\right), \\ \zeta_a^2(x) &= -\frac{a}{2\pi} \cos\left(\frac{\pi x}{2a}\right) + \frac{a}{2\pi} \cos\left(\frac{3\pi x}{2a}\right), \\ \zeta_b^1(y) &= \frac{b}{2\pi} \sin\left(\frac{\pi y}{2b}\right) + \frac{b}{2\pi} \sin\left(\frac{3\pi y}{2b}\right), \\ \zeta_b^2(y) &= -\frac{b}{2\pi} \cos\left(\frac{\pi y}{2b}\right) + \frac{b}{2\pi} \cos\left(\frac{3\pi y}{2b}\right). \end{aligned} \quad (10)$$

As shown in (7)–(9), the supplementary functions $\zeta_a^1(x)$, $\zeta_a^2(x)$, $\zeta_b^1(y)$, and $\zeta_b^2(y)$ are used for the displacement and rotation expressions. The theoretical meaning of introducing these terms into the Fourier series is to remove the

potential discontinuities and their derivatives throughout the whole plate structure including the boundaries and then to effectively enhance the convergence of the results. To ensure this continuity of selection expressions and corresponding derivatives at any point on the plate, the first-order derivatives of the x and y directions should exist, as indicated by (4). This requirement is guaranteed by the selected supplementary functions because it is easy to verify that

$$\begin{aligned}\zeta_a^1(0) = \zeta_a^1(a) = \zeta_a^{1'}(a) = 0, \quad \zeta_a^{1'}(0) = 1, \\ \zeta_a^2(0) = \zeta_a^2(a) = \zeta_a^{2'}(0) = 0, \quad \zeta_a^{2'}(a) = 1.\end{aligned}\quad (11)$$

Similar conditions exist for the y -related polynomials, $\zeta_b^1(y)$ and $\zeta_b^2(y)$. It has to be mentioned that although the solution is theoretically exact for the superposition of infinite numbers of Fourier terms, in actual calculation we truncate the infinite series to M and N to obtain the results with acceptable accuracy.

Since the energy expressions and admissible function of the plate have been established, the remaining task is to determine the Fourier expanded coefficients and supplemented

coefficients in (7)–(9). The Lagrangian energy functional (L) of the plate is written as

$$L = T - U. \quad (12)$$

Then, the Lagrangian expression is minimized by taking its derivatives with respect to these coefficients:

$$\frac{\partial L}{\partial \Theta} = 0, \quad \Theta = \left\{ \begin{matrix} A_{mn} & a_m^1 & b_n^1 \\ B_{mn} & c_m^1 & d_n^1 \\ C_{mn} & e_m^1 & f_n^1 \end{matrix} \right\}. \quad (13)$$

Since the displacements and rotation components of the plate are chosen as M and N to obtain the results with acceptable accuracy, a total of $3 * (M + 1) * (N + 1) + 6 * (M + N + 2)$ equations are obtained.

They can be summed up in a matrix form

$$(\mathbf{K} - \omega^2 \mathbf{M}) \mathbf{E} = \mathbf{0}. \quad (14)$$

The unknown coefficients in the displacement expressions can be expressed in the vector form as $\mathbf{E}$, where

$$\mathbf{E} = \left\{ \begin{matrix} A_{00}, A_{01}, \dots, A_{m'0}, A_{m'1}, \dots, A_{m'n'}, \dots, A_{MN}, a_0^1, \dots, a_M^1, a_0^2, \dots, a_M^2, b_0^1, \dots, b_N^1, b_0^2, \dots, b_N^2 \\ B_{00}, B_{01}, \dots, B_{m'0}, B_{m'1}, \dots, B_{m'n'}, \dots, B_{MN}, c_0^1, \dots, c_M^1, c_0^2, \dots, c_M^2, d_0^1, \dots, d_N^1, d_0^2, \dots, d_N^2 \\ C_{00}, C_{01}, \dots, C_{m'0}, C_{m'1}, \dots, C_{m'n'}, \dots, C_{MN}, e_0^1, \dots, e_M^1, e_0^2, \dots, e_M^2, f_0^1, \dots, f_N^1, f_0^2, \dots, f_N^2 \end{matrix} \right\}^T. \quad (15)$$

In (14), $\mathbf{K}$ is the stiffness matrix for the plate, and $\mathbf{M}$ is the mass matrix. They can be expressed separately as

$$\begin{aligned}\mathbf{K} &= \begin{bmatrix} \mathbf{K}_{1-1} & \mathbf{K}_{1-2} & \mathbf{K}_{1-3} & \cdots & \mathbf{K}_{1-9} \\ \mathbf{K}_{1-2}^T & \mathbf{K}_{2-1} & \mathbf{K}_{2-3} & \cdots & \mathbf{K}_{2-9} \\ \mathbf{K}_{1-3}^T & \mathbf{K}_{2-3}^T & \mathbf{K}_{3-3} & \cdots & \mathbf{K}_{3-9} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{K}_{1-9}^T & \mathbf{K}_{2-9}^T & \mathbf{K}_{3-9}^T & \cdots & \mathbf{K}_{9-9} \end{bmatrix}, \\ \mathbf{M} &= \begin{bmatrix} \mathbf{M}_{1-1} & \mathbf{M}_{1-2} & \mathbf{M}_{1-3} & \cdots & \mathbf{M}_{1-9} \\ \mathbf{M}_{1-2}^T & \mathbf{M}_{2-1} & \mathbf{M}_{2-3} & \cdots & \mathbf{M}_{2-9} \\ \mathbf{M}_{1-3}^T & \mathbf{M}_{2-3}^T & \mathbf{M}_{3-3} & \cdots & \mathbf{M}_{3-9} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{1-9}^T & \mathbf{M}_{2-9}^T & \mathbf{M}_{3-9}^T & \cdots & \mathbf{M}_{9-9} \end{bmatrix}.\end{aligned}\quad (16)$$

The specific expressions for the elements in (16) are given in Appendix A. Moreover, all the necessary expressions which will be used in the calculations of the eigenvalues and eigenvectors are given in Appendix B.

Obviously, the natural frequencies and eigenvectors can now be readily obtained by solving a standard matrix eigenproblem. Since the components of each eigenvector are actually the expansion coefficients of the modified Fourier series, the corresponding mode shape can be directly determined from (14). In other words, once the coefficient eigenvector $\mathbf{E}$ is

determined for a given frequency, the displacement functions of the plate can be determined by substituting the coefficients into (9). When the forced vibration is involved, by adding the work done by external force in the Lagrangian energy function and summing the loading vector $\mathbf{F}$ on the right side of (14), the characteristic equation for the forced vibration of the moderately thick rectangular plates is readily obtained.

3. Numerical Examples and Discussion

In this section, a systematic comparison between the current solutions and theoretical results published by other researchers or finite element method (FEM) results is carried out to validate the excellent accuracy, reliability, and feasibility of the modified Fourier method. A comprehensive study on the effects of elastic restraint parameters and varying thickness in two directions is also reported. New results are obtained for plates subjected to general elastic boundary restraints with nonlinear variable thickness in both directions. The discussion is arranged as follows. Firstly, the convergence of the modified Fourier solution is checked. In addition, the influence of the stiffness of boundary spring components is studied. Secondly, the uniform thickness plates with various combinations of classical boundary conditions, elastic boundary conditions, and different structure parameters are examined. Thirdly, the nonuniform thickness plate with linear variation in both directions, various combinations of classical boundary conditions, conditions, and

TABLE 1: Frequency parameters $\Omega = \omega b^2(\rho h/D)^{1/2}/\pi^2$ for square moderately thick plates with uniform thickness and different boundary conditions.

Boundary conditions	M, N	Model sequence							
		1	2	3	4	5	6	7	8
C-C-C-C	$M = N = 2$	3.375	7.191	7.191	9.856	16.05	16.28	17.47	17.47
	$M = N = 4$	3.299	6.308	6.308	8.853	10.46	10.56	12.63	12.63
	$M = N = 6$	3.294	6.282	6.282	8.801	10.37	10.47	12.54	12.54
	$M = N = 8$	3.292	6.278	6.278	8.797	10.36	10.46	12.53	12.53
	$M = N = 10$	3.292	6.276	6.276	8.794	10.36	10.46	12.53	12.53
	$M = N = 12$	3.292	6.276	6.276	8.793	10.36	10.46	12.52	12.52
	$M = N = 14$	3.292	6.276	6.276	8.793	10.36	10.46	12.52	12.52
	$M = N = 16$	3.292	6.275	6.275	8.792	10.36	10.45	12.52	12.52
	Reference [4]	3.292	6.276	6.276	8.792	10.36	10.46	12.52	12.52
E-E-E-E	$M = N = 2$	0.6153	1.0861	1.0861	1.7898	2.7709	2.8417	3.5010	3.5010
	$M = N = 4$	0.6150	1.0799	1.0799	1.7785	2.7695	2.8405	3.4914	3.4914
	$M = N = 6$	0.6149	1.0796	1.0796	1.7773	2.7694	2.8403	3.4898	3.4898
	$M = N = 8$	0.6149	1.0796	1.0796	1.7771	2.7693	2.8403	3.4893	3.4893
	$M = N = 10$	0.6149	1.0796	1.0796	1.7770	2.7693	2.8403	3.4892	3.4892
	$M = N = 12$	0.6149	1.0796	1.0796	1.7770	2.7693	2.8403	3.4892	3.4892
	$M = N = 14$	0.6149	1.0796	1.0796	1.7770	2.7693	2.8403	3.4892	3.4892
	$M = N = 16$	0.6149	1.0796	1.0796	1.7770	2.7693	2.8403	3.4891	3.4891
	Reference [33]	0.6153	1.0861	1.0861	1.7898	2.7709	2.8417	3.5010	3.5010
	FEM	0.6150	1.7097	1.7097	1.7774	2.7694	2.8406	3.4898	3.4898

different structure parameters are examined. Then, numerical solutions for moderately thick rectangular plates with elastic boundary conditions are presented. The effects of the elastic restraint parameters are also investigated. Finally, the nonuniform thickness plate with nonlinear variation in both directions and arbitrary boundary conditions is also studied.

3.1. Convergence and Stiffness Value Study. Since the Fourier series is numerically truncated and only finite terms are counted in actual calculations, the proposed solution should be understood as a solution with arbitrary precision. In this subsection, a uniform thickness square moderately thick rectangular plate with completely clamped boundary condition (C-C-C-C) and four edges equally elastically restrained against linear spring constants and rotation spring constants supports (E-E-E-E, $K_T = k_i/(b^3/D)$, $K_R = K_i/(b/D)$, $K_T = 10$, and $K_R = 5$) has been selected to demonstrate the convergence and accuracy of the modified Fourier method. In Table 1, the first eight frequency parameters $\Omega = \omega b^2(\rho h/D)^{1/2}/\pi^2$ for the considered uniform thickness square moderately thick rectangular plate with C-C-C-C and E-E-E-E boundary conditions are examined. The table shows that the proposed method has fast convergence behavior. The maximum discrepancy in the worst case between the 6×6 truncated configuration and the 8×8 one is less than 0.064%. In order to fully illustrate the convergence of the present method, Figures 3, 4, and 5 present the 1st and 8th frequency parameters $\Omega = \omega b^2(\rho h/D)^{1/2}/\pi^2$ with various truncated numbers $M = N$ subjected to different boundary condition and aspect ratios. A highly desired convergence characteristic is observed such that (a) sufficiently accurate results can be obtained with only a small number of terms in the series expansions and (b) the solution is consistently refined as more

terms are included in the expansions. However, this should not constitute a problem in practice because one can always verify the accuracy of the solution by increasing the truncation number until a desired numerical precision is achieved. As a matter of fact, this “quality control” scheme can be easily implemented automatically. In modal analysis, the natural frequencies for higher-order modes tend to converge slower (see Table 1). Thus, an adequate truncation number should be dictated by the desired accuracy of the largest natural frequencies of interest. In view of the excellent numerical behavior of the current solution, the truncation numbers will be simply set as $M = N = 12$ in the following calculations.

As far as the accuracy of the present method is concerned, the converged solutions of the present method are in excellent agreement with both the results reported by reference data and the finite element results. For C-C-C-C boundary conditions, the max discrepancy between the present results and the reference data does not exceed 0.011% for the worst case and in most cases is 0%. Comparing the results with exact solutions [4], it is observed that eight terms are sufficient to obtain accurate results. Moreover, with regard to the E-E-E-E boundary condition, the max discrepancy between the present results and the reference data does not exceed 0.31% for the worst case and in most cases is 0.1%. Regarding the results with DQM solutions [33], it can be seen that the six terms are sufficient to obtain enough accurate results. In addition, it is clear that the results of the present approach with just 663 DOFs ($M \times N = 12 \times 12$) can predict the vibration characteristics accurately. Most of them are identical to those obtained from finite element method (FEM) with 102,01 DOFs (S4R: 0.01 m $\times$ 0.01 m). That is to say, it needs only 6.62% DOFs compared with FEM to obtain the same precision solutions for the considered case.

FIGURE 2: The effect of boundary spring stiffness on the natural frequencies Ω : (a) translation spring, (b) rotation spring, and (c) torsional spring.

On the same hardware (Intel i7-3.9 GHz), the computing time of the present formulation for the solution ($M \times N = 12 \times 12$) implemented in optimized MATLAB scripts is about 2.125 s, whereas the finite element solution consumes 34.578 s, that is, at least 16 times more CPU time than the present method for the same problem.

As mentioned earlier, in the current modeling framework, all the classical boundary conditions and their combinations can be conveniently viewed as special cases when the stiffness for the normal and tangential boundary springs becomes zero and/or infinitely large. Thus, the effects of the stiffness of the translation (k_i), rotation (K_i), and torsional springs ($K_{i,j}$) on the modal characteristics should be investigated. As shown in the Figures 2(a)–2(c), the first and the third frequency parameters are separately obtained by varying the stiffness of one group of the boundary springs from

extremely large (10^{14}) to extremely small (10^0) while assigning the other group of the springs infinite stiffness (10^{14}). It can be found in Figure 2(a) that the frequency parameter almost keeps at a level when the stiffness of the translation springs is larger than 10^{12} or smaller than 10^7 . In Figure 2(b), the influences of the rotation springs on frequency parameters are given. It is shown that the frequency curves change greatly within the stiffness range (10^6 to 10^{10}) while out of this range the frequency curves separately keep level. In Figure 2(c), the influences of the torsional springs on frequency parameters are given. It is shown that the frequency curves almost change when the stiffness changes in the whole range.

Based on the analysis, it can be found that the torsional springs almost have no effect on the structure. Also, the relationship between the rotation springs and twisting moment can be seen from the boundary condition expression. Then,

FIGURE 3: The effect of numerically truncated numbers $M = N$ on the natural frequencies Ω for C-C-C-C boundary condition.

FIGURE 4: The effect of numerically truncated numbers $M = N$ on the natural frequencies Ω for F-F-F-F boundary condition.

the twisting moments have small effect on the vibration characteristics of the structure. However, in this paper, in order to get a more accurate prediction of the vibration characteristics of the structure, the twisting moment on boundary edges is taken into account. In the latter study, in addition to the fact that free boundary is not unexpected considering the torsion spring, the other boundary conditions are introduced into a torsion spring, and the spring stiffness takes infinity. From Figure 2 analysis, it also concluded that the translation spring has a wider influence range than the rotation spring on the frequency parameters; that is, for the translation springs, the stable frequency parameters appear when the stiffness is more than 10^{12} or less 10^7 while for the rotation springs, when the stiffness value is assigned more than 10^{10} the frequency parameters become smooth. Thus, it is suitable to use 10^{14} to simulate the infinite stiffness value in the model validation parts and in the following examples.

3.2. Uniform Thickness Moderately Thick Plates with Classical and Elastic Boundary Conditions. In this subsection, the modified Fourier solution is applied to deal with vibration problems of uniform thickness moderately thick rectangular plates subject to the classical boundary conditions and arbitrary elastic boundary conditions. In present work, three groups of continuously distributed boundary springs are introduced to simulate the given or typical boundary conditions. As mentioned earlier, the stiffness of these boundary springs can take any value from zero to infinity to better model many real-world restraint conditions. Taking edge $x = 0$, for example, the corresponding spring stiffness for the three types of classical boundaries and elastic boundaries is

completely free: $Q_x = 0$, $M_{xx} = 0$, and $M_{xy} = 0$

$$k_{x0} = 0, \quad K_{x0} = 0, \quad K_{yx0} = 0, \quad (17a)$$

FIGURE 5: The effect of numerically truncated numbers $M = N$ on the natural frequencies Ω with four edges elastically restrained against translation and rotation ($K_T = k_i/(b^3/D)$, $K_R = K_i/(b/D)$, $K_T = 100$, and $K_R = 100$).

simply supported: $w = 0$, $M_{xx} = 0$, and $\psi_y = 0$

$$k_{x0} = 10^{14}, \quad K_{x0} = 0, \quad K_{yx0} = 10^{14}, \quad (17b)$$

completely clamped: $w = 0$, $\psi_x = 0$, and $\psi_y = 0$

$$k_{x0} = 10^{14}, \quad K_{x0} = 10^{14}, \quad K_{yx0} = 10^{14}, \quad (17c)$$

arbitrarily elastic boundary: $w \neq 0$, $\psi_x \neq 0$, $\psi_y \neq 0$, $Q_x \neq 0$, $M_{xx} \neq 0$, and $M_{xy} \neq 0$

$$k_{x0} = \Gamma_{wx0}, \quad K_{x0} = \Gamma_{xx0}, \quad K_{yx0} = 10^{14}. \quad (17d)$$

The appropriateness of the three classical boundaries defined in (17a)–(17c) will be proved by several examples given in following, the arbitrary elastic boundaries are also defined in (17d), and the Γ_{wi} ($\Gamma_{wi} = k_i/k_0$, $k_0 = 1 \times 10^9 \text{ N/m}^2$, $i = x0, xa, y0, yb$) and Γ_{xi} ($\Gamma_{xi} = K_i/K_0$, $K_0 = 1 \times 10^9 \text{ Nm/rad}$, $i = x0, xa, y0, yb$) elastic restraint parameters represent corresponding spring stiffness. For the sake of simplicity, a four-letter string is employed to represent the restraint condition of a plate, such that F-C-S-E identifies the plate with edges $x = 0$, $y = 0$, $x = a$, and $y = b$ having free, clamped, shear-supported restrained, and elastic boundary conditions, respectively.

As for the first case, a uniform thickness moderately thick plate with different classical boundaries and structure parameters is investigated here. In Table 2, the comparison of the first eight frequency parameters $\Omega = \omega b^2(\rho h/D)^{1/2}/\pi^2$ of the considered plate is presented. The S-S-S-S, C-F-F-F, S-S-F-F, F-F-F-F, and S-C-S-C boundary conditions are performed in the comparison. Excellent agreements are observed between the solutions obtained by the modified Fourier method, the referential data, and finite element method (FEM) results for the uniform thickness moderately thick rectangular plates. It is also verified that the definition of the three types of

classical boundaries in (17a)–(17c) is appropriate. In addition, the elastic boundary conditions (17d) are also verified. In the next two examples, we will account for the vibration of moderately thick plate with elastic edge supports. The first model considered is an S-S-S-S square moderately thick plate with all edges elastically rotationally restrained. That is, $K_{x0} = K_{xa} = K_{y0} = K_{yb} = \Gamma_x$, and all the other restraining springs are set to have an infinite stiffness (namely, represented by 10^{14} in numerical calculation). The six frequency parameters $\Omega = \omega b^2(\rho h/D)^{1/2}/\pi^2$ are given in Table 3 for several different restraining coefficient values; the finite element method (FEM) results calculated using ABAQUS are also listed in this table as a reference. The second model concerns a complete square moderately thick plate with all edges elastically restrained. That is, $k_{x0} = k_{xa} = k_{y0} = k_{yb} = \Gamma_w$ and $K_{x0} = K_{xa} = K_{y0} = K_{yb} = \Gamma_x$. The six frequency parameters $\Omega = \omega b^2(\rho h/D)^{1/2}/\pi^2$ are given in Table 4 for several different restraining coefficient values; the finite element method (FEM) results are also listed in Table 4 as a reference. It can be clearly seen that the comparison is extremely good, which implies that the current method is able to make correct predictions for the modal characteristics of moderately thick plate with not only classical boundary conditions but also elastic edge restraints.

The excellent agreements between the solution obtained by the modified Fourier method and the referential data for the moderately thick plate subjected to the combinations of classical boundary conditions and elastic boundary conditions given in Tables 2–4 indicate that the proposed method is sufficiently accurate to deal with uniform thickness moderately thick plate with arbitrary boundary conditions.

3.3. Linearly Variation Thickness Moderately Thick Plates with Classical and Elastic Boundary Conditions. In the theoretical formulations, this paper concerns the varying thickness moderately thick plates with classical and elastic boundary

TABLE 2: Frequency parameters $\Omega = \omega b^2(\rho h/D)^{1/2}/\pi^2$ for uniform thickness moderately thick plates with different classical boundaries and structure parameters.

Boundary condition	a/b	Model sequence							
		1	2	3	4	5	6	7	8
S-S-S-S	1	1.931	4.605	4.605	7.064	8.605	8.605	10.792	10.792
		(1.933 ^a)	(4.611)	(4.611)	(7.067)	(8.607)	(8.607)	(10.799)	(10.780)
		(1.931 ^b)	(4.605)	(4.605)	(7.064)	(8.605)	(8.605)	(10.792)	(10.792)
	3/2	1.408	2.648	4.127	4.605	5.261	7.064	7.153	8.184
		(1.410 ^a)	(2.651)	(4.128)	(4.608)	(5.267)	(7.074)	(7.155)	(8.188)
		(1.408 ^b)	(2.648)	(4.127)	(4.605)	(5.261)	(7.064)	(7.151)	(8.184)
C-F-F-F	1	0.348	0.817	2.035	2.583	2.862	4.816	5.478	5.774
		(0.348 ^a)	(0.817)	(2.038)	(2.586)	(2.865)	(4.821)	(5.504)	(5.793)
		(0.348 ^b)	(0.816)	(2.034)	(2.582)	(2.860)	(4.811)	(5.477)	(5.772)
	3/2	0.155	0.502	0.941	1.665	2.292	2.613	3.303	3.498
		(0.154 ^a)	(0.502)	(0.941)	(1.663)	(2.295)	(2.618)	(3.305)	(3.500)
		(0.154 ^b)	(0.501)	(0.940)	(1.662)	(2.292)	(2.612)	(3.298)	(3.494)
S-S-F-F	1	0.333	1.678	1.875	3.559	4.720	4.946	6.474	6.632
		(0.335 ^a)	(1.680)	(1.878)	(3.600)	(4.721)	(4.949)	(6.478)	(6.642)
		(0.333 ^b)	(1.677)	(1.874)	(3.557)	(4.718)	(4.945)	(6.472)	(6.631)
	3/2	0.222	0.938	1.631	2.363	2.573	4.082	4.482	4.753
		(0.223 ^a)	(0.940)	(1.632)	(2.364)	(2.575)	(4.092)	(4.483)	(4.759)
		(0.222 ^b)	(0.938)	(1.631)	(2.361)	(2.571)	(4.077)	(4.479)	(4.752)
F-F-F-F	1	1.290	1.919	2.363	3.235	3.235	5.605	5.605	5.646
		(1.291 ^a)	(1.920)	(2.365)	(3.239)	(3.239)	(5.607)	(5.607)	(6.648)
	3/2	0.865	0.948	1.961	2.166	2.461	2.843	3.551	4.026
		(0.865 ^a)	(0.949)	(1.963)	(2.168)	(2.465)	(2.844)	(3.556)	(4.034)
S-C-S-C	1	2.700	4.971	5.990	7.973	8.787	10.250	11.334	12.024
		(2.701 ^a)	(4.971)	(5.993)	(7.975)	(8.788)	(10.253)	(11.335)	(12.028)
	3/2	2.348	3.253	4.972	5.646	6.489	7.381	7.974	9.930
		(2.348 ^a)	(3.255)	(4.974)	(5.647)	(6.590)	(7.384)	(7.801)	(9.932)

^aResults in parentheses are taken from FEM.

^bResults in parentheses are taken from [11].

conditions. The varying thickness function $h(x, y)$ can be expressed as $h_0(1 + \alpha x^s)(1 + \beta y^t)$, in which the h_0 , α , and β represent the initial thickness, gradient in x direction, and gradient in y direction. When the indexes s and t take the value $s = t = 1$, the analytical model imitates the linearly variation thickness moderately thick plates structure. In order to unify the description and facilitate the analytical calculations of the involved integrals, all the thickness variation functions can be expanded into either 1D or 2D Fourier cosine series, resulting in

$$h(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} h_0 q_{mn} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}, \quad (18)$$

where

$$q_{mn} = \frac{4}{ab} \int_0^b \int_0^a (1 + \alpha x^s)(1 + \beta y^t) \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} dx dy. \quad (19)$$

In order to prove the validity of the present method for the vibration of linearly variation thickness moderately

thick plates with arbitrary boundary conditions, the typical classical boundary conditions as the first case will be considered. In Table 5, the comparison of the first six frequency parameters $\Omega = \omega a^2(\rho h_0/D_0)^{1/2}$ of the moderately thick plates with linearly varying thickness is presented. The S-S-S-S, C-F-F-F, S-S-F-F, C-C-C-C, and S-C-S-C boundary conditions are performed in the comparison. Excellent agreements are observed between the solutions obtained by the modified Fourier method and finite element method (FEM) results for the moderately thick plates with linear variation thickness. To investigate the influence of the aspect ratio on the uniform thickness and nonuniform thickness moderately thick plates, the effect on the frequency parameters for plates with S-S-S-S boundary conditions is presented in Figure 6. The thickness functions are h_0 and $h_0(1 + 0.5 \times x)(1 + 0.5 \times y)$, respectively. It is seen from Figure 6 that the influence of aspect ratios on the frequency parameters for nonuniform thickness moderately thick plates is more complicated.

In the next two examples, we also account for the vibrations of moderately thick plate with linear variation thickness and elastic edge supports. The first model considered is an S-S-S-S square moderately thick plate with all edges elastically

TABLE 3: Frequency parameters $\Omega = \omega b^2(\rho h/D)^{1/2}/\pi^2$ for S-S-S-S moderately thick plates ($a/b = 1$) with uniform thickness and elastic rotation support ($K_{x0} = K_{xa} = K_{y0} = K_{yb} = \Gamma_x$).

Γ_x^a	h_0/a	Model sequence					
		1	2	3	4	5	6
0	0.05	1.945	4.834	4.834	7.591	9.515	9.519
		(1.942 ^b)	(4.836)	(4.836)	(7.584)	(9.547)	(9.552)
	0.1	1.856	4.500	4.500	6.829	8.499	8.513
		(1.858 ^b)	(4.510)	(4.510)	(6.843)	(8.539)	(8.554)
	0.15	1.758	4.114	4.114	6.040	7.421	7.445
0.5	0.05	3.541	7.071	7.071	10.227	12.318	12.399
		(3.546 ^b)	(7.091)	(7.091)	(10.253)	(12.386)	(12.468)
	0.1	3.274	6.242	6.242	8.719	10.302	10.421
		(3.282 ^b)	(6.267)	(6.267)	(8.754)	(10.370)	(10.488)
	0.15	2.947	5.361	5.361	7.290	8.465	8.606
1	0.05	3.541	7.072	7.072	10.228	12.320	12.401
		(3.547 ^b)	(7.093)	(7.093)	(10.254)	(12.388)	(12.470)
	0.1	3.278	6.248	6.248	8.727	10.311	10.429
		(3.286 ^b)	(6.274)	(6.274)	(8.762)	(10.378)	(10.497)
	0.15	2.957	5.374	5.374	7.305	8.478	8.620
1.5	0.05	3.542	7.073	7.073	10.229	12.320	12.401
		(3.547 ^b)	(7.093)	(7.093)	(10.254)	(12.388)	(12.471)
	0.1	3.280	6.251	6.251	8.729	10.313	10.432
		(3.287 ^b)	(6.276)	(6.276)	(8.765)	(10.381)	(10.500)
	0.15	2.960	5.379	5.379	7.309	8.482	8.625
2	0.05	3.542	7.073	7.073	10.229	12.321	12.402
		(3.547 ^b)	(7.093)	(7.093)	(10.255)	(12.389)	(12.471)
	0.1	3.280	6.252	6.252	8.730	10.315	10.434
		(3.288 ^b)	(6.277)	(6.277)	(8.766)	(10.382)	(10.501)
	0.15	2.962	5.381	5.381	7.312	8.484	8.628
	0.05	2.650	4.627	4.627	6.175	7.076	7.224
		(2.660 ^b)	(4.652)	(4.652)	(6.210)	(7.128)	(7.277)

^a $\Gamma_x = K_i/K_0$, ($K_0 = 1 \times 10^9$ Nm/rad, $i = x0, xa, y0, yb$).

^bResults in parentheses are taken from FEM.

TABLE 4: Frequency parameters $\Omega = \omega b^2(\rho h/D)^{1/2}/\pi^2$ for moderately thick plates ($a/b = 1$) with uniform thickness and all edge elastic restraints ($K_{x0} = K_{xa} = K_{y0} = K_{yb} = \Gamma_x$ and $k_{x0} = k_{xa} = k_{y0} = k_{yb} = \Gamma_w$).

Γ_x^a	Γ_w^a	h_0/a	Model sequence					
			1	2	3	4	5	6
0.5	0	0.05	0.986	1.967	3.893	3.893	4.850	4.850
			(0.988 ^b)	(1.971)	(3.897)	(3.897)	(4.852)	(4.852)
		0.1	0.916	1.835	3.515	3.522	4.375	4.375
			(0.917 ^b)	(1.837)	(3.519)	(3.528)	(4.341)	(4.341)
		0.15	0.789	1.635	2.983	3.027	3.771	3.771
1	0.5	0.05	3.417	3.417	4.431	5.294	5.461	6.448
			(3.419 ^b)	(3.419)	(4.435)	(5.303)	(5.478)	(6.456)
		0.1	1.593	1.593	2.401	3.849	3.861	4.729
			(1.611 ^b)	(1.611)	(2.408)	(3.852)	(3.866)	(4.767)
		0.15	1.122	1.122	1.913	3.272	3.290	4.038
1.5	1	0.05	4.325	4.325	5.619	6.273	6.571	7.619
			(4.334 ^b)	(4.334)	(5.681)	(6.299)	(6.642)	(7.642)
		0.1	2.013	2.013	2.812	4.088	4.116	4.993
			(2.024 ^b)	(2.024)	(2.827)	(4.122)	(4.187)	(4.994)
		0.15	1.338	1.338	2.108	3.430	3.443	4.196
2	2	0.05	5.231	5.231	6.919	7.630	8.002	9.220
			(5.238 ^b)	(5.238)	(6.931)	(7.637)	(8.011)	(9.241)
		0.1	2.594	2.594	3.439	4.470	4.544	5.428
			(2.597 ^b)	(2.597)	(3.442)	(4.495)	(4.557)	(5.438)
		0.15	1.653	1.653	2.403	3.613	3.632	4.392
		0.05	1.238	1.238	1.949	3.077	3.099	3.751
			(1.244 ^b)	(1.244)	(1.964)	(3.094)	(3.116)	(3.788)

^a $\Gamma_x = K_i/K_0$, ($K_0 = 1 \times 10^9$ Nm/rad, $i = x0, xa, y0, yb$), $\Gamma_w = k_i/k_0$, ($k_0 = 1 \times 10^9$ N/m, $i = x0, xa, y0, yb$).

^bResults in parentheses are taken from FEM.

FIGURE 6: The effect of aspect ratio a/b on the natural frequencies Ω for S-S-S-S boundary condition: (a) uniform thickness and (b) nonuniform thickness.

TABLE 5: Frequency parameter $\Omega = \omega a^2(\rho h_0/D_0)^{1/2}$ for moderately thick plates with linearly varying thickness in different boundary conditions ($h(x, y) = h_0(1 + 0.5x)(1 + 0.5y)$).

Boundary condition	a/b	h_0/a	Model sequence					
			1	2	3	4	5	6
S-S-S-S	1	0.1	27.261 (27.263 ^a)	62.397 (62.410)	62.926 (63.842)	92.423 (92.445)	111.537 (111.867)	111.732 (111.921)
		0.2	23.103 (23.489 ^a)	47.010 (47.209)	47.191 (47.349)	65.018 (65.057)	75.831 (76.605)	75.865 (76.473)
	3/2	0.1	59.338 (59.874 ^a)	99.101 (99.249)	138.798 (138.177)	151.102 (151.362)	166.710 (166.644)	204.855 (205.381)
		0.2	41.175 (41.586 ^a)	63.765 (63.313)	63.765 (63.669)	73.598 (73.661)	83.956 (83.918)	89.084 (89.547)
	1	0.1	4.173 (4.856 ^a)	10.331 (10.680)	27.407 (27.513)	37.826 (37.203)	38.809 (39.135)	64.095 (64.858)
		0.2	3.983 (4.118 ^a)	8.702 (9.306)	21.403 (21.990)	29.443 (29.670)	30.286 (30.602)	46.705 (47.156)
C-F-F-F	3/2	0.1	4.868 (4.887 ^a)	15.488 (15.320)	35.620 (35.175)	63.523 (63.353)	89.969 (89.159)	100.089 (100.783)
		0.2	4.469 (4.944 ^a)	9.763 (9.636)	22.338 (22.087)	42.188 (42.177)	50.104 (50.580)	56.331 (56.559)
	1	0.1	4.117 (4.789 ^a)	24.795 (24.149)	26.216 (26.751)	48.567 (49.324)	68.195 (68.842)	69.667 (70.575)
		0.2	3.820 (4.103 ^a)	20.516 (20.041)	21.924 (22.125)	37.523 (37.221)	48.980 (48.894)	50.327 (50.622)
	3/2	0.1	7.935 (7.856 ^a)	39.454 (40.145)	65.063 (65.162)	92.877 (92.462)	92.877 (93.022)	136.176 (136.336)
		0.2	6.389 (6.406 ^a)	26.149 (26.536)	35.627 (35.626)	49.539 (49.831)	56.844 (57.144)	65.170 (65.981)
C-C-C-C	1	0.1	44.471 (45.102 ^a)	80.299 (80.271)	80.681 (80.656)	109.422 (110.015)	126.871 (126.328)	128.207 (128.888)
		0.2	32.162 (32.240 ^a)	52.781 (52.629)	52.933 (53.166)	69.410 (70.063)	78.474 (79.059)	79.482 (80.266)
	3/2	0.1	82.975 (83.139 ^a)	115.587 (115.838)	151.992 (152.200)	161.579 (162.193)	176.601 (177.058)	212.330 (212.849)
		0.2	48.577 (49.187 ^a)	66.022 (66.259)	83.468 (84.065)	90.830 (91.749)	97.474 (97.827)	98.157 (98.777)
	1	0.1	36.847 (37.901 ^a)	66.664 (68.568)	76.965 (78.356)	101.045 (101.200)	113.594 (113.978)	125.551 (125.651)
		0.2	27.673 (27.897 ^a)	48.469 (48.685)	51.220 (51.491)	67.027 (67.447)	76.358 (77.530)	78.241 (79.573)
S-C-S-C	3/2	0.1	77.378 (78.003 ^a)	106.848 (107.058)	150.079 (150.991)	154.569 (155.171)	173.120 (173.099)	208.391 (208.402)
		0.2	45.895 (45.803 ^a)	64.732 (64.131)	73.393 (73.656)	83.282 (83.801)	90.325 (90.518)	97.089 (97.015)

^aResults in parentheses are taken from FEM.

restrained. That is, $K_{x0} = K_{xa} = K_{y0} = K_{yb} = \Gamma_x$. The six frequency parameters $\Omega = \omega a^2(\rho h_0/D_0)^{1/2}$ are given in Table 6 for several different restraining coefficient values; the finite element method (FEM) results calculated using ABAQUS are also presented as a reference. The second model is a complete square moderately thick plate with all edges elastically restrained. That is, $k_{x0} = k_{xa} = k_{y0} = k_{yb} = \Gamma_w$ and

$K_{x0} = K_{xa} = K_{y0} = K_{yb} = \Gamma_x$. The six frequency parameters $\Omega = \omega a^2(\rho h_0/D_0)^{1/2}$ are given in Table 7 for several different restraining coefficient values; the finite element method (FEM) results are also listed as a reference. It can be clearly seen that the comparison is extremely good, which implies that the current method is able to make correct predictions for the modal characteristics of linearly varying thickness

TABLE 6: Frequency parameter $\Omega = \omega a^2 (\rho h_0 / D_0)^{1/2}$ for S-S-S moderately thick plates ($a/b = 1$) with linearly varying thickness and elastic rotation support ($h(x, y) = h_0(1 + 0.5x)(1 + 0.5y)$, $K_{x0} = K_{xa} = K_{y0} = K_{yb} = \Gamma_x$).

Γ_x^a	h_0/a	Model sequence					
		1	2	3	4	5	6
0	0.05	28.809 (28.897 ^b)	70.080 (70.334)	70.891 (71.129)	108.967 (109.112)	134.884 (135.254)	135.295 (135.772)
	0.1	27.261 (26.606 ^b)	62.398 (61.784)	62.926 (62.303)	92.423 (91.193)	111.540 (111.323)	111.735 (111.648)
	0.15	25.231 (25.204 ^b)	54.192 (55.297)	54.505 (55.595)	77.118 (77.503)	91.228 (91.871)	91.312 (91.941)
	0.2	23.103 (23.947 ^b)	47.010 (47.071)	47.191 (47.208)	65.018 (65.404)	75.832 (76.531)	75.866 (76.589)
	0.5	23.103 (23.210 ^b)	47.010 (47.245)	47.191 (47.358)	65.018 (65.241)	75.832 (76.122)	75.866 (76.345)
0.5	0.1	48.377 (48.524 ^b)	94.650 (94.891)	95.597 (95.992)	135.484 (135.845)	162.635 (167.012)	163.501 (164.024)
	0.15	36.239 (36.647 ^b)	70.939 (71.214)	71.298 (71.687)	99.997 (100.024)	118.584 (119.114)	119.009 (119.457)
	0.2	28.749 (29.142 ^b)	56.937 (57.034)	57.037 (57.544)	79.214 (79.654)	93.009 (93.668)	93.162 (93.754)
	0.05	49.651 (49.755 ^b)	96.759 (97.245)	97.619 (97.964)	138.089 (138.544)	165.368 (165.864)	166.304 (166.878)
	0.1	38.951 (39.021 ^b)	73.735 (74.145)	74.251 (74.664)	102.748 (102.874)	121.079 (121.613)	121.709 (121.817)
1	0.15	30.630 (30.638 ^b)	58.406 (58.684)	58.533 (58.644)	80.432 (80.874)	93.984 (94.121)	94.255 (94.478)
	0.2	25.548 (25.687 ^b)	48.636 (48.788)	48.674 (48.992)	66.165 (66.544)	76.708 (76.824)	76.824 (76.944)
	0.05	50.118 (50.247 ^b)	97.561 (97.877)	98.374 (98.661)	139.096 (139.247)	166.421 (166.875)	167.382 (167.662)
	0.1	40.310 (40.366 ^b)	75.216 (75.542)	75.783 (75.921)	104.229 (104.557)	122.402 (122.874)	123.164 (123.385)
	0.15	31.837 (31.902 ^b)	59.352 (59.451)	59.530 (59.871)	81.245 (81.375)	94.611 (94.784)	94.983 (95.105)
1.5	0.2	26.302 (26.418 ^b)	49.119 (49.354)	49.156 (49.334)	66.528 (66.921)	76.954 (77.010)	77.130 (77.470)
	0.05	50.360 (50.472 ^b)	97.983 (98.005)	98.769 (99.114)	139.631 (140.154)	166.979 (167.188)	167.954 (167.986)
	0.1	41.130 (41.421 ^b)	76.146 (76.642)	76.724 (78.114)	105.165 (105.661)	123.229 (123.278)	124.082 (124.974)
	0.15	32.687 (32.778 ^b)	60.025 (60.244)	60.245 (60.549)	81.832 (82.146)	95.052 (95.114)	95.508 (95.842)
	0.2	26.893 (26.922 ^b)	49.492 (49.874)	49.539 (49.924)	66.815 (66.987)	77.138 (77.367)	77.370 (77.588)

^a $\Gamma_x = K_i/K_0$, ($K_0 = 1 \times 10^9$ Nm/rad, $i = x0, xa, y0, yb$).

^bResults in parentheses are taken from FEM.

TABLE 7: Frequency parameter $\Omega = \omega a^2 (\rho h_0 / D_0)^{1/2}$ for moderately thick plates ($a/b = 1$) with linearly varying thickness and elastic rotation and translation support ($h(x, y) = h_0(1 + 0.5x)(1 + 0.5y)$, $K_{x0} = K_{xa} = K_{y0} = K_{yb} = \Gamma_x$, and $k_{x0} = k_{xa} = k_{y0} = k_{yb} = \Gamma_w$).

Γ_x^a	Γ_w^a	h_0/a	Model sequence					
			1	2	3	4	5	6
0.5	0	0.05	14.751 (14.780 ^b)	14.917 (14.822)	28.232 (29.252)	55.251 (55.476)	55.397 (55.505)	69.512 (70.904)
		0.1	11.811 (11.818 ^b)	11.910 (11.897)	23.857 (24.711)	44.462 (44.376)	45.037 (45.257)	55.453 (56.511)
		0.15	8.487 (8.500 ^b)	8.514 (8.544)	20.011 (20.738)	34.783 (34.404)	36.335 (36.663)	44.970 (45.625)
		0.2	6.073 (6.096 ^b)	6.073 (6.106)	17.431 (18.039)	28.679 (28.129)	30.805 (31.165)	38.010 (38.311)
		0.25	5.125 (5.158 ^b)	5.159 (5.196)	14.811 (15.568)	24.431 (24.004)	26.115 (26.500)	32.600 (33.011)
1	0.5	0.05	21.225 (20.708 ^b)	32.173 (32.097)	32.173 (32.285)	44.381 (45.110)	64.061 (64.182)	65.222 (65.355)
		0.1	8.140 (7.943 ^b)	16.878 (16.838)	16.941 (16.970)	28.100 (28.888)	48.663 (48.700)	49.015 (49.171)
		0.15	4.493 (4.383 ^b)	11.859 (11.867)	11.878 (11.891)	22.429 (23.125)	38.352 (38.177)	39.247 (39.488)
		0.2	2.932 (3.540 ^b)	8.690 (8.791)	8.691 (8.791)	18.919 (16.229)	31.108 (27.520)	32.600 (27.889)
		0.25	2.412 (2.412 ^b)	7.811 (7.811)	7.811 (7.811)	16.229 (16.229)	27.520 (27.520)	27.889 (27.889)
1.5	1	0.05	27.819 (27.146 ^b)	40.919 (41.008)	41.430 (41.434)	54.595 (55.256)	70.792 (70.895)	72.896 (73.051)
		0.1	11.285 (11.012 ^b)	20.068 (20.025)	20.230 (20.270)	31.101 (31.849)	51.046 (51.126)	51.429 (51.569)
		0.15	6.292 (6.139 ^b)	13.912 (13.912)	13.922 (13.934)	24.132 (24.808)	40.500 (40.426)	41.120 (41.316)
		0.2	4.121 (4.021 ^b)	10.321 (10.329)	10.331 (10.357)	20.051 (20.635)	32.796 (32.540)	33.921 (34.178)
		0.25	3.415 (3.380 ^b)	9.172 (9.172)	9.172 (9.172)	18.344 (18.344)	27.520 (27.520)	27.889 (27.889)
2	2	0.05	34.615 (33.806 ^b)	51.972 (52.100)	53.088 (53.157)	68.255 (68.891)	81.344 (81.433)	84.510 (84.761)
		0.1	15.416 (15.045 ^b)	24.659 (24.625)	24.829 (24.888)	35.615 (36.303)	53.879 (53.970)	54.494 (54.626)
		0.15	8.740 (8.529 ^b)	16.420 (16.414)	16.481 (16.499)	26.353 (26.996)	42.417 (42.398)	42.946 (43.114)
		0.2	5.761 (5.622 ^b)	12.127 (12.134)	12.127 (12.148)	21.434 (21.998)	34.286 (34.110)	35.192 (35.415)
		0.25	4.811 (4.811)	10.321 (10.321)	10.321 (10.321)	18.344 (18.344)	27.520 (27.520)	27.889 (27.889)

^a $\Gamma_x = K_i/K_0$, ($K_0 = 1 \times 10^9$ Nm/rad, $i = x0, xa, y0, yb$), $\Gamma_w = k_i/k_0$, ($k_0 = 1 \times 10^9$ N/m, $i = x0, xa, y0, yb$).

^bResults in parentheses are taken from FEM.

moderately thick plate with not only classical boundary conditions but also elastic edge restraints.

As the last case of this subsection, the influence of the gradient α and β on the fundamental frequency parameters for a linearly varying thickness moderately thick plate is investigated. The model is a square moderately thick plate with all edges elastically restrained. That is, $k_{x0} = k_{xa} = k_{y0} = k_{yb} = \Gamma_w = 2$ and $K_{x0} = K_{xa} = K_{y0} = K_{yb} = \Gamma_x = 2$. The six frequency parameters $\Omega = \omega a^2 (\rho h_0 / D_0)^{1/2}$ are given in Table 8 for several different α and β values; the finite element method (FEM) results are also listed as a reference. Again, good agreement can be observed. Through Table 8, it is also found that the frequency parameter increases with increasing gradient parameters.

The above studies are given as linearly varying thickness moderately thick plates with several of boundary condition

and different structure parameters. In the next section, vibration results for the plates subjected to nonlinear variation thickness will be presented.

3.4. Nonlinearly Varying Thickness Moderately Thick Plates with Classical and Elastic Boundary Conditions. In Section 3.3, the linearly varying thickness moderately thick plates were studied. However, in the practical engineering applications, the varying thickness of a moderately thick plate may not always be linear variation in nature. A variety of possible thickness varying cases may be encountered in practice. Therefore, the moderately thick plates with nonlinear variation thickness subjected to general elastic edge restraints are examined in this subsection. For the sake of brevity, the indexes s and t will be chosen as 2 to imitate the nonlinearly varying thickness moderately thick plates structure in this

TABLE 8: Frequency parameter $\Omega = \omega a^2 (\rho h_0 / D_0)^{1/2}$ for elastic support moderately thick plates ($a/b = 1$) with different gradient ($k_{x0} = k_{xa} = k_{y0} = k_{yb} = \Gamma_w = 2$, $K_{x0} = K_{xa} = K_{y0} = K_{yb} = \Gamma_x = 2$).

α, β	h_0/a	Model sequence					
		1	2	3	4	5	6
$\alpha = \beta = 0.1$	0.05	31.275	52.615	52.657	68.954	76.146	79.850
		(31.303 ^a)	(52.664)	(52.675)	(69.033)	(76.174)	(79.948)
	0.1	17.270	25.244	25.250	33.949	46.050	46.625
		(17.271 ^a)	(25.255)	(25.258)	(34.089)	(46.122)	(46.708)
	0.15	10.190	16.266	16.274	24.164	37.327	37.526
		(10.189 ^a)	(16.269)	(16.280)	(24.323)	(37.401)	(37.622)
	0.2	6.780	12.258	12.260	19.685	31.452	31.754
		(6.780 ^a)	(12.265)	(12.270)	(19.851)	(31.514)	(31.864)
	0.05	32.530	52.947	53.141	68.954	77.034	80.589
		(32.577 ^a)	(53.014)	(53.203)	(69.172)	(77.088)	(80.773)
$\alpha = \beta = 0.2$	0.1	16.882	24.979	24.979	34.154	48.057	48.575
		(16.883 ^a)	(24.983)	(25.001)	(34.433)	(48.134)	(48.667)
	0.15	9.804	16.284	16.309	24.683	38.844	39.085
		(9.803 ^a)	(16.283)	(16.322)	(24.980)	(38.909)	(39.191)
	0.2	6.499	12.290	12.295	20.133	32.354	32.770
		(6.498 ^a)	(12.296)	(12.307)	(20.420)	(32.387)	(32.895)
	0.05	33.516	52.845	53.308	68.704	78.203	81.561
		(33.563 ^a)	(52.939)	(53.389)	(69.055)	(78.271)	(81.789)
	0.1	16.419	24.799	24.830	34.527	50.061	50.583
		(16.419 ^a)	(24.790)	(24.869)	(34.947)	(50.143)	(50.688)
$\alpha = \beta = 0.3$	0.15	9.430	16.334	16.378	25.236	40.193	40.506
		(9.429 ^a)	(16.329)	(16.395)	(25.660)	(40.243)	(40.625)
	0.2	6.235	12.288	12.293	20.572	33.102	33.663
		(6.235 ^a)	(12.296)	(12.306)	(20.965)	(33.090)	(33.811)
	0.05	34.215	52.463	53.264	68.430	79.654	82.865
		(34.252 ^a)	(52.575)	(53.346)	(68.921)	(79.732)	(83.113)
	0.1	15.921	24.695	24.793	35.027	52.011	52.571
		(15.921 ^a)	(24.674)	(24.842)	(35.583)	(52.098)	(52.688)
	0.15	9.074	16.388	16.444	25.797	41.379	41.789
		(9.074 ^a)	(16.381)	(16.464)	(26.336)	(41.404)	(41.928)
$\alpha = \beta = 0.4$	0.2	5.990	12.236	12.238	21.002	33.735	34.461
		(5.989 ^a)	(12.249)	(12.250)	(21.487)	(33.653)	(34.641)
	0.05	34.615	51.972	53.088	68.255	81.344	84.510
		(33.806 ^a)	(52.100)	(53.157)	(68.891)	(81.433)	(84.761)
	0.1	15.416	24.659	24.829	35.615	53.879	54.494
		(15.045 ^a)	(24.625)	(24.888)	(36.303)	(53.970)	(54.626)
	0.15	8.740	16.420	16.481	26.353	42.417	42.946
		(8.529 ^a)	(16.414)	(16.499)	(26.996)	(42.398)	(43.114)
	0.2	5.761	12.127	12.127	21.434	34.286	35.192
		(5.622 ^a)	(12.134)	(12.148)	(21.998)	(34.110)	(35.415)

^aResults in parentheses are taken from FEM.

section; that is, $s = t = 2$. Also the thickness varying functions are also be expressed as Fourier cosine series according to (18) and (19).

In order to validate the accuracy and reliability of the proposed method for predicting the vibration behavior of nonlinearly varying thickness moderately thick plates with

arbitrary boundary conditions, the typical classical boundary conditions viewed as the special cases of elastically restrained edges will be considered. The comparison of the first six frequency parameters $\Omega = \omega a^2 (\rho h_0 / D_0)^{1/2}$ for the moderately thick plates with nonlinearly varying thickness is presented in Table 9. The S-S-S-S, C-F-F-F, S-S-F-F, C-C-C-C, and

TABLE 9: Frequency parameter $\Omega = \omega a^2(\rho h_0/D_0)^{1/2}$ for moderately thick plates with linearly varying thickness in different boundary conditions ($h(x, y) = h_0(1 + 0.5x^2)(1 + 0.5y^2)$).

Boundary condition	a/b	h_0/a	Model sequence					
			1	2	3	4	5	6
S-S-S-S	1	0.1	23.462 (23.741 ^a)	54.999 (55.114)	55.573 (55.983)	82.950 (83.146)	100.900 (101.004)	101.187 (101.347)
		0.2	18.606 (18.936 ^a)	42.725 (43.014)	42.971 (43.148)	60.040 (60.247)	71.167 (71.642)	71.761 (71.993)
	3/2	0.1	53.055 (53.467 ^a)	65.251 (65.472)	85.404 (85.984)	106.700 (107.211)	114.880 (115.687)	143.531 (144.007)
		0.2	17.439 (17.874 ^a)	21.531 (21.748)	32.170 (32.651)	44.987 (45.264)	59.438 (60.188)	64.179 (64.879)
	1	0.1	3.658 (3.749 ^a)	9.317 (9.547)	24.135 (24.034)	32.816 (33.627)	35.433 (36.308)	59.179 (60.142)
		0.2	3.520 (3.611 ^a)	7.986 (8.176)	19.591 (20.075)	26.959 (27.125)	27.491 (27.625)	43.882 (43.984)
C-F-F-F	3/2	0.1	3.901 (3.930 ^a)	9.963 (10.095)	28.984 (29.381)	69.187 (69.578)	73.686 (73.994)	89.360 (90.242)
		0.2	3.292 (3.386 ^a)	4.439 (4.498)	13.385 (14.012)	23.947 (24.068)	40.850 (41.221)	44.758 (45.142)
	1	0.1	3.557 (3.649 ^a)	21.268 (21.441)	22.762 (22.796)	43.524 (43.916)	60.253 (60.622)	62.429 (62.630)
		0.2	3.171 (3.489 ^a)	17.822 (18.006)	19.512 (20.430)	33.616 (33.815)	45.042 (45.247)	46.833 (49.004)
	3/2	0.1	6.738 (6.957 ^a)	31.658 (31.882)	52.071 (52.518)	75.625 (75.823)	86.604 (86.939)	94.903 (95.610)
		0.2	3.653 (6.807 ^a)	12.930 (13.160)	20.955 (21.312)	26.517 (26.864)	44.097 (44.422)	48.058 (48.873)
C-C-C-C	1	0.1	41.020 (41.405 ^a)	75.196 (75.452)	75.614 (75.882)	104.142 (104.711)	119.683 (119.637)	120.949 (121.137)
		0.2	31.237 (32.010 ^a)	51.686 (52.960)	51.868 (52.145)	68.411 (69.335)	76.868 (77.481)	77.864 (78.099)
	3/2	0.1	95.823 (95.364 ^a)	128.727 (128.932)	163.657 (163.588)	173.751 (172.350)	190.890 (190.350)	223.595 (222.948)
		0.2	53.306 (53.587 ^a)	71.372 (71.759)	89.384 (90.103)	102.404 (102.503)	104.660 (105.174)	120.908 (121.275)
	1	0.1	33.715 (34.148 ^a)	60.298 (60.786)	71.446 (71.828)	94.187 (94.515)	103.802 (104.263)	117.928 (119.326)
		0.2	26.143 (26.789 ^a)	45.499 (46.622)	49.458 (50.680)	64.253 (64.329)	72.941 (73.474)	76.191 (76.843)
S-C-S-C	3/2	0.1	73.228 (73.761 ^a)	115.845 (116.121)	137.445 (137.788)	148.166 (148.530)	168.342 (168.698)	182.542 (182.921)
		0.2	41.302 (41.304 ^a)	53.648 (53.757)	65.630 (65.842)	72.202 (72.794)	81.632 (82.033)	88.105 (88.474)

^aResults in parentheses are taken from FEM.

S-C-S-C boundary conditions are performed in the comparison. The results adequately demonstrated the great accuracy of the modified Fourier method.

We now turn to elastically restrained moderately thick plates. The first one involves an S-S-S-S moderately thick square plate with a uniform rotational restraint along each edge; that is, $K_{x0} = K_{xa} = K_{y0} = K_{yb} = \Gamma_x$. The calculated

frequency parameters $\Omega = \omega a^2(\rho h_0/D_0)^{1/2}$ are given in Table 10 together with the FEM results. Since the elastically restrained plates with nonlinear variation thickness are rarely investigated, the FEM results are used as the reference. A good agreement is observed between the current and FEM results. The second example concerns a moderately thick square plate elastically supported along all edges. The stiffness of the linear

TABLE 10: Frequency parameter $\Omega = \omega a^2 (\rho h_0 / D_0)^{1/2}$ for S-S-S-S moderately thick plates ($a/b = 1$) with linearly varying thickness and elastic rotation support ($h(x, y) = h_0(1 + 0.5x^2)(1 + 0.5y^2)$, $K_{x0} = K_{xa} = K_{y0} = K_{yb} = \Gamma_x$).

Γ_x^a	h_0/a	Model sequence					
		1	2	3	4	5	6
0	0.05	23.462	54.999	55.573	82.950	100.900	101.187
		(23.741 ^b)	(55.114)	(55.983)	(83.146)	(101.004)	(101.347)
	0.1	18.606	42.725	42.971	60.040	71.167	71.761
		(18.936 ^b)	(43.014)	(43.148)	(60.247)	(71.642)	(71.993)
	0.15	53.055	65.251	85.404	106.700	114.880	143.531
0.5	0.05	(53.467 ^b)	(65.472)	(85.984)	(107.211)	(115.687)	(144.007)
		17.439	21.531	32.170	44.987	59.438	64.179
	0.1	(17.874 ^b)	(21.748)	(32.651)	(45.264)	(60.188)	(64.879)
		6.904	13.374	13.514	19.400	22.904	23.056
	0.15	(9.909 ^b)	(13.548)	(13.651)	(19.564)	(23.088)	(23.379)
1	0.05	5.316	10.237	10.314	14.499	17.155	17.264
		(5.374 ^b)	(10.448)	(10.651)	(14.764)	(17.188)	(17.879)
	0.1	4.136	8.201	8.220	11.474	13.578	13.671
		(4.174 ^b)	(8.348)	(8.651)	(11.764)	(13.688)	(13.879)
	0.15	3.449	6.915	6.924	9.549	11.225	11.327
1.5	0.05	(3.474 ^b)	(6.948)	(6.951)	(9.764)	(11.388)	(11.579)
		7.040	13.617	13.745	19.726	23.233	23.402
	0.1	(7.074 ^b)	(13.748)	(13.851)	(19.964)	(23.488)	(23.879)
		5.712	10.696	10.793	14.991	17.594	17.744
	0.15	(5.774 ^b)	(10.748)	(10.851)	(15.054)	(17.788)	(17.879)
2	0.05	4.489	8.507	8.540	11.759	13.803	13.922
		(4.514 ^b)	(8.548)	(8.611)	(11.864)	(13.988)	(14.119)
	0.1	3.677	7.084	7.090	9.692	11.330	11.445
		(3.684 ^b)	(7.148)	(7.191)	(9.764)	(11.588)	(11.879)
	0.15	7.089	13.708	13.828	19.847	23.355	23.530
2.5	0.05	(7.124 ^b)	(13.748)	(13.951)	(19.967)	(23.588)	(23.879)
		5.895	10.923	11.024	15.239	17.813	17.988
	0.1	(5.874 ^b)	(10.748)	(11.651)	(15.264)	(17.818)	(17.979)
		4.697	8.694	8.737	11.938	13.790	14.083
	0.15	(4.774 ^b)	(8.748)	(8.951)	(11.864)	(13.688)	(14.879)
3	0.05	3.836	7.201	7.210	9.792	11.403	11.530
		(3.874 ^b)	(7.748)	(7.651)	(9.864)	(11.588)	(11.879)
	0.1	7.114	13.754	13.871	19.911	23.418	23.597
		(7.174 ^b)	(12.848)	(13.851)	(19.864)	(23.588)	(23.879)
	0.15	6.001	11.060	11.159	15.391	17.944	18.136
3.5	0.05	(6.004 ^b)	(11.148)	(11.151)	(15.464)	(17.988)	(18.379)
		4.836	8.821	8.872	12.061	14.037	14.189
	0.1	(4.874 ^b)	(8.848)	(8.951)	(12.164)	(14.188)	(14.879)
		3.954	7.289	7.301	9.871	11.457	11.594
	0.15	(3.974 ^b)	(7.348)	(7.651)	(9.964)	(11.488)	(11.679)

^a $\Gamma_x = K_i/K_0$, ($K_0 = 1 \times 10^9$ Nm/rad, $i = x0, xa, y0, yb$).

^bResults in parentheses are taken from FEM.

and rotational restraints is set equal to $k_{x0} = k_{xa} = k_{y0} = k_{yb} = \Gamma_w$ and $K_{x0} = K_{xa} = K_{y0} = K_{yb} = \Gamma_x$, respectively. The six frequency parameters $\Omega = \omega a^2 (\rho h_0 / D_0)^{1/2}$ are shown in Table 11 for several different restraining coefficient values. It can also be noticed that the Fourier series method is able

to predict the modal characteristics of nonlinearly varying thickness moderately thick plate with not only classical boundary conditions but also elastic edge restraints correctly.

Finally, the influence of the gradient on the fundamental frequency parameters of a nonlinearly varying thickness

TABLE 11: Frequency parameter $\Omega = \omega a^2 (\rho h_0 / D_0)^{1/2}$ for moderately thick plates ($a/b = 1$) with linearly varying thickness and elastic rotation and translation support ($h(x, y) = h_0(1 + 0.5x^2)(1 + 0.5y^2)$, $K_{x0} = K_{xa} = K_{y0} = K_{yb} = \Gamma_x$, and $k_{x0} = k_{xa} = k_{y0} = k_{yb} = \Gamma_w$).

Γ_x^a	Γ_w^a	h_0/a	Model sequence					
			1	2	3	4	5	6
0.5	0	0.05	12.926	13.041	25.090	47.935	47.943	61.761
			(12.931 ^b)	(13.053)	25.099	(47.947)	(47.952)	(61.788)
		0.1	10.900	10.988	21.545	39.980	40.147	50.322
			(10.904 ^b)	(10.992)	(21.557)	(39.984)	(40.166)	(50.334)
		0.15	8.245	8.298	18.081	31.503	32.617	40.950
			(8.247 ^b)	(8.230)	(18.092)	(31.508)	(32.622)	(40.955)
		0.2	6.106	6.124	15.639	25.631	27.601	34.458
			(6.110 ^b)	(6.125)	(15.642)	(25.637)	(27.617)	(34.462)
		0.05	21.575	31.615	31.644	42.573	57.889	58.607
			(21.580 ^b)	(31.617)	(31.644)	(42.581)	(57.893)	(58.610)
1	0.5	0.1	8.451	16.024	16.210	25.969	43.839	43.867
			(8.451 ^b)	(16.025)	(16.217)	(25.972)	(43.844)	(43.870)
		0.15	4.682	11.479	11.480	20.642	35.172	35.598
			(4.684 ^b)	(11.480)	(11.481)	(20.644)	(35.178)	(35.604)
		0.2	3.059	8.621	8.648	17.320	28.487	29.638
			(3.062 ^b)	(8.619)	(8.644)	(17.327)	(28.492)	(29.641)
		0.05	27.713	40.203	40.978	52.951	65.352	66.815
			(27.714 ^b)	(40.207)	(40.988)	(52.972)	(65.368)	(66.822)
		0.1	11.667	19.322	19.615	29.122	46.115	46.183
			(11.674 ^b)	(19.327)	(19.619)	(29.137)	(46.120)	(46.188)
1.5	1	0.15	6.544	13.416	13.481	22.405	37.238	37.430
			(6.547 ^b)	(13.420)	(13.488)	(22.410)	(37.241)	(37.434)
		0.2	4.295	10.175	10.190	18.543	30.330	31.056
			(4.296 ^b)	(10.177)	(10.194)	(18.544)	(30.336)	(31.060)
		0.05	33.453	50.854	52.088	66.311	76.610	78.947
			(33.455 ^b)	(50.852)	(50.097)	(66.321)	(76.627)	(78.971)
		0.1	15.832	24.130	24.372	33.924	49.118	49.362
			(15.830 ^b)	(24.127)	(24.374)	(33.931)	(49.147)	(49.382)
		0.15	9.060	15.956	16.099	24.754	39.134	39.249
			(9.064 ^b)	(15.961)	(16.102)	(24.755)	(39.138)	(39.452)
2	2	0.2	5.992	11.955	11.971	20.034	31.908	32.392
			(5.991 ^b)	(11.957)	(11.978)	(20.042)	(31.910)	(32.398)

^a $\Gamma_x = K_i/K_0$, ($K_0 = 1 \times 10^9$ Nm/rad, $i = x0, xa, y0, yb$), $\Gamma_w = k_i/k_0$, ($k_0 = 1 \times 10^9$ N/m, $i = x0, xa, y0, yb$).

^bResults in parentheses are taken from FEM.

moderately thick plate is investigated. The plate is elastically restrained, in which the stiffness of the boundary springs is taken as $k_{x0} = k_{xa} = k_{y0} = k_{yb} = \Gamma_w = 2$ and $K_{x0} = K_{xa} = K_{y0} = K_{yb} = \Gamma_x = 2$, respectively. The six frequency parameters $\Omega = \omega a^2 (\rho h_0 / D_0)^{1/2}$ are presented in Table 12 for several different slop values. It can be seen that the fundamental frequency parameters will decrease with the increase of the parameter h_0/a .

In the above examples, it has been demonstrated that the presented method can be universally applied to nonlinearly varying thickness moderately thick plates with several boundary conditions and different structure parameters. New results are obtained for plates with nonlinear variation thickness in both directions subjected to general elastic

boundary restraints, which may be used for benchmarking of researchers in the field. In addition, it is interesting to see that the nature frequency decreases with the increase of the index for thickness function.

4. Conclusions

In this paper, a modified Fourier method has been presented to study the free vibration behaviors of moderately thick rectangular plates with variable thickness and arbitrary boundary conditions. The first-order shear deformation plate theory is adopted to formulate the theoretical model. The displacements and rotation components of the plate, regardless of boundary conditions, are invariantly expressed as the superposition of a 2D Fourier cosine series and four

TABLE 12: Frequency parameter $\Omega = \omega a^2 (\rho h_0 / D_0)^{1/2}$ for elastic support moderately thick plates ($a/b = 1$) with different gradient ($k_{x0} = k_{xa} = k_{y0} = k_{yb} = \Gamma_w = 2, K_{x0} = K_{xa} = K_{y0} = K_{yb} = \Gamma_x = 2$).

α, β	h_0/a	Model sequence					
		1	2	3	4	5	6
$\alpha = \beta = 0.1$	0.05	30.187	51.102	51.141	67.066	73.954	77.412
		(30.188 ^a)	(51.105)	(51.144)	(67.068)	(73.961)	(77.477)
	0.1	16.995	24.719	24.724	33.071	44.343	44.819
		(16.996 ^a)	(24.725)	(24.729)	(33.084)	(44.347)	(44.822)
	0.15	10.077	15.872	15.883	23.443	35.937	36.065
		(10.081 ^a)	(15.877)	(15.892)	(23.447)	(35.941)	(36.084)
	0.2	6.713	11.953	11.958	19.086	30.372	30.604
		(6.713 ^a)	(11.957)	(11.965)	(19.092)	(30.378)	(30.612)
$\alpha = \beta = 0.2$	0.05	31.206	51.436	51.614	67.058	74.398	77.595
		(31.208 ^a)	(51.438)	(51.619)	(67.064)	(74.402)	(77.596)
	0.1	16.786	24.503	24.508	33.144	45.583	45.923
		(16.790 ^a)	(24.510)	(24.513)	(33.147)	(45.587)	(45.932)
	0.15	9.821	15.865	15.906	23.766	36.897	36.990
		(9.824 ^a)	(15.870)	(15.907)	(23.768)	(36.894)	(36.997)
	0.2	6.523	11.978	11.992	19.356	30.912	31.162
		(6.524 ^a)	(11.981)	(11.997)	(19.362)	(30.916)	(31.174)
$\alpha = \beta = 0.3$	0.05	32.097	51.462	51.900	66.832	74.991	77.863
		(32.098 ^a)	(51.466)	(51.907)	(66.849)	(74.996)	(77.872)
	0.1	16.507	24.332	24.383	33.330	46.806	47.074
		(16.511 ^a)	(24.327)	(24.380)	(33.339)	(46.810)	(47.086)
	0.15	9.562	15.885	15.964	24.103	37.748	37.828
		(9.562 ^a)	(15.892)	(15.977)	(21.105)	(37.751)	(37.832)
	0.2	6.338	11.992	12.013	19.603	31.338	31.634
		(6.340 ^a)	(11.994)	(12.011)	(19.612)	(31.344)	(31.639)
$\alpha = \beta = 0.4$	0.05	32.851	51.239	52.046	66.547	75.735	78.303
		(33.455 ^a)	(50.852)	(50.097)	(66.321)	(76.627)	(78.971)
	0.1	16.182	24.209	24.343	33.598	47.990	48.230
		(33.455 ^a)	(50.852)	(50.097)	(66.321)	(76.627)	(78.971)
	0.15	9.307	15.919	16.034	24.436	38.491	38.579
		(33.455 ^a)	(50.852)	(50.097)	(66.321)	(76.627)	(78.971)
	0.2	6.161	11.986	12.008	19.828	31.666	32.040
		(33.455 ^a)	(50.852)	(50.097)	(66.321)	(76.627)	(78.971)
$\alpha = \beta = 0.5$	0.05	33.453	50.854	52.088	66.311	76.610	78.947
		(33.455 ^a)	(50.852)	(50.097)	(66.321)	(76.627)	(78.971)
	0.1	15.832	24.130	24.372	33.924	49.118	49.362
		(15.830 ^a)	(24.127)	(24.374)	(33.931)	(49.147)	(49.382)
	0.15	9.060	15.956	16.099	24.754	39.134	39.249
		(9.064 ^a)	(15.961)	(16.102)	(24.755)	(39.138)	(39.452)
	0.2	5.992	11.955	11.971	20.034	31.908	32.392
		(5.991 ^a)	(11.957)	(11.978)	(20.042)	(31.910)	(32.398)

^aResults in parentheses are taken from FEM.

supplementary functions in the form of the product of a polynomial function and a single cosine series expansion to ensure and accelerate the convergence of the solution. At each edge of the plate, the general restraint conditions are implemented by introducing one group of linear springs and two groups of rotational springs, which are continuously distributed and determined by the stiffness of these springs.

Instead of seeking a solution in strong forms in the previous studies, all the Fourier coefficients will be treated equally and independently as the generalized coordinates and solved directly from the Rayleigh-Ritz technique. The change of the boundary conditions can be easily achieved by only varying the stiffness of the three sets of boundary springs along all edges of the rectangular plates without involving any change

to the solution procedure. The convergence of the present solution is examined and the excellent accuracy is validated by comparison with existing results published in the literature and FEM data. Excellent agreements are obtained from these comparisons. The proposed method provides a unified means for extracting the modal parameters and predicting the vibration behaviors of moderately thick plates with variable thickness variation functions and arbitrary elastic edge restraints. A variety of free vibration results for moderately thick rectangular plates with different thickness variation functions and boundary conditions are presented. New results for free vibration of moderately thick rectangular plates with various thickness variation functions and edge conditions are presented, which may be used for benchmarking of researchers in the field.

Appendices

A. Representative Calculation for Stiffness and Mass Matrices

To illuminate the particular expression of the mass and stiffness matrixes clearly and tersely, four new variables are defined as $s = m * (N + 1) + n + 1$, $t = m' * (N + 1) + n' + 1$, $p = (l - 1) * (M + 1) + m' + 1$, and $q = (l - 1) * (N + 1) + n' + 1$. The first row elements of $\mathbf{K}$ and $\mathbf{M}$ are given below:

$$\begin{aligned} & \{\mathbf{K}_{1-1}\}_{s,t} \\ &= \frac{1}{2} \int_0^a \int_0^b [D(x, y) \\ & \quad \times \{2\lambda_{am}\lambda_{am'} \sin \lambda_{am}x \sin \lambda_{am'}x \cos \lambda_{bn}y \\ & \quad \times \cos \lambda_{bn'}y + (1 - \mu) \lambda_{bn}\lambda_{bn'} \cos \lambda_{am}x \\ & \quad \times \cos \lambda_{am'}x \sin \lambda_{bn}y \sin \lambda_{bn'}y\} \\ & \quad + \kappa Gh(x, y) \cos \lambda_{am}x \cos \lambda_{am'}x \cos \lambda_{bn} \\ & \quad \times y \cos \lambda_{bn'}y dx dy] dx dy \\ &+ \int_0^a \left\{ K_{xy0} \cos \lambda_{am}x \cos \lambda_{am'}x \right. \\ & \quad \left. + (-1)^{m+m'} K_{xyb} \cos \lambda_{am}x \cos \lambda_{am'}x \right\} dx \\ &+ \int_0^b \left\{ K_{x0} \cos \lambda_{bn}y \cos \lambda_{bn'}y \right. \\ & \quad \left. + (-1)^{n+n'} K_{xa} \cos \lambda_{bn}y \cos \lambda_{bn'}y \right\} dy, \\ & \{\mathbf{K}_{1-2}\}_{s,p} \\ &= \frac{1}{2} \int_0^a \int_0^b \left[D(x, y) \left\{ 2\lambda_{am}\lambda_{am'} \sin \lambda_{am}x \sin \lambda_{am'}x \right. \right. \\ & \quad \times \sum_{l=1}^2 \zeta_b^l(y) \cos \lambda_{bn'}y \\ & \quad - (1 - \mu) \lambda_{bn'} \cos \lambda_{am}x \cos \lambda_{am'}x \\ & \quad \left. \times \sum_{l=1}^2 \zeta_b^{l'}(y) \sin \lambda_{bn'}y \right\} + \kappa Gh(x, y) \\ & \quad \times \cos \lambda_{am}x \sin \lambda_{bn}y - 2(1 - \mu)_{bn} \\ & \quad \times \lambda_{am'} \sin \lambda_{am'}x \cos \lambda_{bn'}y \cos \lambda_{am}x \\ & \quad \times \sin \lambda_{bn}y + (1 - \mu) \lambda_{am}\lambda_{bn'} \cos \lambda_{am'}x \\ & \quad \times \sin \lambda_{bn'}y \sin \lambda_{am}x \cos \lambda_{bn}y \} dx dy, \end{aligned}$$

$$\begin{aligned} & \times \left(\cos \lambda_{am'}x \cos \lambda_{bn'}y \right. \\ & \quad \left. \times \sum_{l=1}^2 \zeta_b^l(y) \cos \lambda_{am}x \right) dx dy \\ &+ \int_0^a \left\{ K_{xy0} \cos \lambda_{am}x \cos \lambda_{am'}x \sum_{l=1}^2 \zeta_b^l(0) \right. \\ & \quad \left. + (-1)^{n'} K_{xyb} \cos \lambda_{am}x \cos \lambda_{am'}x \sum_{l=1}^2 \zeta_b^l(b) \right\} dx \\ &+ \int_0^b \left\{ K_{x0} \sum_{l=1}^2 \zeta_b^l(y) \cos \lambda_{bn'}y \right. \\ & \quad \left. + (-1)^{m+m'} K_{xa} \sum_{l=1}^2 \zeta_b^l(y) \cos \lambda_{bn'}y \right\} dy, \\ & \{\mathbf{K}_{1-3}\}_{s,q} \\ &= \frac{1}{2} \int_0^a \int_0^b \left[D(x, y) \left\{ -2\lambda_{am'} \sin \lambda_{am'}x \cos \lambda_{bn'}y \right. \right. \\ & \quad \times \sum_{l=1}^2 \zeta_a^{l'}(x) \cos \lambda_{bn}y \\ & \quad + (1 - \mu) \lambda_{bn}\lambda_{bn'} \cos \lambda_{am'}x \\ & \quad \times \sin \lambda_{bn'}y \sum_{l=1}^2 \zeta_a^l(x) \sin \lambda_{bn}y \} \\ & \quad + \kappa Gh(x, y) \cos \lambda_{am'}x \cos \lambda_{bn'}y \\ & \quad \times \sum_{l=1}^2 \zeta_a^l(x) \cos \lambda_{bn}y \left. \right] dx dy \\ &+ \int_0^a \left\{ K_{xy0} \cos \lambda_{am'}x \sum_{l=1}^2 \zeta_a^l(x) \right. \\ & \quad \left. + (-1)^{n+n'} K_{xyb} \cos \lambda_{am'}x \sum_{l=1}^2 \zeta_a^l(x) \right\} dx \\ &+ \int_0^b \left\{ K_{x0} \sum_{l=1}^2 \zeta_a^l(0) \cos \lambda_{bn'}y \right. \\ & \quad \left. + (-1)^{m+m'} K_{xa} \sum_{l=1}^2 \zeta_a^l(0) \cos \lambda_{bn'}y \right\} dy, \end{aligned}$$

$$\begin{aligned} & \{\mathbf{K}_{1-4}\}_{s,t} \\ &= \frac{1}{2} \int_0^a \int_0^b D(x, y) \{ 2\lambda_{bn}\lambda_{am'} \sin \lambda_{am'}x \cos \lambda_{bn'}y \\ & \quad \times \cos \lambda_{am}x \sin \lambda_{bn}y - 2(1 - \mu)_{bn} \\ & \quad \times \lambda_{am'} \sin \lambda_{am'}x \cos \lambda_{bn'}y \cos \lambda_{am}x \\ & \quad \times \sin \lambda_{bn}y + (1 - \mu) \lambda_{am}\lambda_{bn'} \cos \lambda_{am'}x \\ & \quad \times \sin \lambda_{bn'}y \sin \lambda_{am}x \cos \lambda_{bn}y \} dx dy, \end{aligned}$$

$$\begin{aligned}
& \{\mathbf{K}_{1-5}\}_{s,p} \\
&= \frac{1}{2} \int_0^a \int_0^b D(x, y) \left\{ -2\lambda_{am'} \sin \lambda_{am'} x \cos \lambda_{bn'} y \right. \\
&\quad \times \sum_{l=1}^2 \zeta_b^{l'}(y) \cos \lambda_{am} x \\
&\quad + 2(1-\mu) \lambda_{am'} \sin \lambda_{am'} x \cos \lambda_{bn'} y \\
&\quad \times \sum_{l=1}^2 \zeta_b^{l'}(y) \cos \lambda_{am} x \\
&\quad + (1-\mu) \lambda_{am} \lambda_{bn'} \cos \lambda_{am'} x \sin \lambda_{bn'} y \\
&\quad \left. \times \sum_{l=1}^2 \zeta_b^l(y) \sin \lambda_{am} x \right\} dx dy,
\end{aligned}$$

$$\begin{aligned}
& \{\mathbf{K}_{1-6}\}_{s,q} \\
&= \frac{1}{2} \int_0^a \int_0^b D(x, y) \left\{ \lambda_{bn} \lambda_{am'} \sin \lambda_{am'} x \cos \lambda_{an'} y \right. \\
&\quad \times \sum_{l=1}^2 \zeta_a^l(x) \sin \lambda_{bn'} y \\
&\quad - 2(1-\mu) \lambda_{bn} \lambda_{am'} \sin \lambda_{am'} x \cos \lambda_{an'} y \\
&\quad \times \sum_{l=1}^2 \zeta_a^l(x) \sin \lambda_{bn'} y \\
&\quad + \lambda_{bn'} \cos \lambda_{am'} x \sin \lambda_{bn'} y \\
&\quad \left. \times \sum_{l=1}^2 \zeta_a^{l'}(x) \cos \lambda_{bn} y \right\} dx dy,
\end{aligned}$$

$$\begin{aligned}
& \{\mathbf{K}_{1-7}\}_{s,t} \\
&= -\frac{1}{2} \int_0^a \int_0^b \kappa Gh(x, y) \lambda_{am} \cos \lambda_{am'} x \cos \lambda_{bn'} y \\
&\quad \times \sin \lambda_{am} x \cos \lambda_{bn} y dx dy,
\end{aligned}$$

$$\begin{aligned}
& \{\mathbf{K}_{1-8}\}_{s,p} \\
&= -\frac{1}{2} \int_0^a \int_0^b \kappa Gh(x, y) \lambda_{am'} \cos \lambda_{am'} x \cos \lambda_{bn'} y \\
&\quad \times \sum_{l=1}^2 \zeta_b^l(y) \sin \lambda_{am'} x dx dy,
\end{aligned}$$

$$\begin{aligned}
& \{\mathbf{K}_{1-9}\}_{s,q} \\
&= \frac{1}{2} \int_0^a \int_0^b \kappa Gh(x, y) \cos \lambda_{am'} x \cos \lambda_{bn'} y \\
&\quad \times \sum_{l=1}^2 \zeta_a^{l'}(x) \cos \lambda_{bn} y dx dy,
\end{aligned}$$

$$\begin{aligned}
& \{\mathbf{M}_{1-1}\}_{s,t} \\
&= \frac{1}{24} \rho \int_0^a \int_0^b h^3(x, y) \cos \lambda_{am} x \cos \lambda_{bn} y \\
&\quad \times \cos \lambda_{am'} x \cos \lambda_{bn'} y dx dy,
\end{aligned}$$

$$\begin{aligned}
& \{\mathbf{M}_{1-2}\}_{s,p} \\
&= \frac{1}{24} \rho \int_0^a \int_0^b h^3(x, y) \sum_{l=1}^2 \zeta_b^l(y) \cos \lambda_{am} x \\
&\quad \times \cos \lambda_{am'} x \cos \lambda_{bn'} y dx dy,
\end{aligned}$$

$$\begin{aligned}
& \{\mathbf{M}_{1-3}\}_{s,q} \\
&= \frac{1}{24} \rho \int_0^a \int_0^b h^3(x, y) \sum_{l=1}^2 \zeta_a^l(x) \cos \lambda_{bn} y \\
&\quad \times \cos \lambda_{am'} x \cos \lambda_{bn'} y dx dy,
\end{aligned}$$

$$\begin{aligned}
& \{\mathbf{M}_{1-4}\}_{s,t} = \{\mathbf{M}_{1-5}\}_{s,p} = \{\mathbf{M}_{1-6}\}_{s,q} \\
&= \{\mathbf{M}_{1-7}\}_{s,t} = \{\mathbf{M}_{1-8}\}_{s,p} = \{\mathbf{M}_{1-9}\}_{s,q} = \mathbf{0}.
\end{aligned}$$

(A.1)

B. Useful Integrals for the Moderately Thick Rectangular Plates with Variable Thickness and Arbitrary Boundary Conditions

Consider

$$\int_0^L \cos \lambda_m x \cos \lambda_n x dx = \begin{cases} 0, & m \neq n, \\ \frac{L}{2}, & m = n, \\ L, & m = n = 0, \end{cases}$$

$$\begin{aligned}
& \int_0^L \sin \lambda_m x \cos \lambda_n x dx \\
&= \begin{cases} -\frac{(-1 + (-1)^{m+n}) L m}{(m^2 - n^2) \pi}, & m \neq n, \\ 0, & m = n, \end{cases}
\end{aligned}$$

$$\int_0^L \sin \lambda_m x \sin \lambda_n x dx = \begin{cases} 0, & m \neq n, \\ \frac{L}{2}, & m = n, \\ 0, & m = n = 0 \end{cases}$$

$$\begin{aligned}
& \int_0^L \cos \lambda_m x \zeta_a^{l'}(x) dx \\
&= \begin{cases} \frac{4L^2}{3\pi^2}, & l = 1, m = 0, \\ -\frac{4L^2(4m^2 - 3)}{(9 - 40m^2 + 16m^4)\pi^2}, & l = 1, m \neq 0, \\ -\frac{4L^2}{3\pi^2}, & l = 2, m = 0, \\ \frac{4L^2(4m^2 - 3)(-1)^m}{(9 - 40m^2 + 16m^4)\pi^2}, & l = 2, m \neq 0, \end{cases}
\end{aligned}$$

$$\int_0^L \sin \lambda_m x \zeta_a^l(x) dx = \begin{cases} 0, & l = 1, m = 0, \\ \frac{16L^2 m (-1)^m}{(9 - 40m^2 + 16m^4) \pi^2}, & l = 1, m \neq 0, \\ 0, & l = 2, m = 0, \\ \frac{16mL^2}{(9 - 40m^2 + 16m^4) \pi^2}, & l = 2, m \neq 0, \end{cases}$$

$$\int_0^L \cos \lambda_m x \zeta_a^{l'}(x) dx = \begin{cases} 0, & l = 1, m = 0, \\ \frac{16Lm^2 (-1)^m}{(9 - 40m^2 + 16m^4) \pi}, & l = 1, m \neq 0, \\ 0, & l = 2, m = 0, \\ \frac{16Lm^2}{(9 - 40m^2 + 16m^4) \pi}, & l = 2, m \neq 0, \end{cases}$$

$$\int_0^L \sin \lambda_m x \zeta_a^{l'}(x) dx = \begin{cases} 0, & l = 1, m = 0, \\ \frac{4Lm(-3 + 4m^2)}{(9 - 40m^2 + 16m^4) \pi}, & l = 1, m \neq 0, \\ 0, & l = 2, m = 0, \\ Lm \left(\frac{1}{\pi - 4m^2 \pi} + \frac{3}{9\pi - 4m^2 \pi} \right) (-1)^m, & l = 2, m \neq 0, \end{cases}$$

$$\int_0^L \zeta_a^{l_1}(x) \zeta_a^{l_2}(x) dx = \begin{cases} \frac{L^3}{4\pi^2}, & l_1 = 1, l_2 = 1, \\ -\frac{2L^3}{3\pi^3}, & l_1 = 1, l_2 = 2, \\ -\frac{2L^3}{3\pi^3}, & l_1 = 2, l_2 = 1, \\ \frac{L^3}{4\pi^2}, & l_1 = 2, l_2 = 2, \end{cases}$$

$$\int_0^L \zeta_a^{l_1}(x) \zeta_a^{l_2'}(x) dx = \begin{cases} 0, & l_1 = 1, l_2 = 1, \\ -\frac{L^2}{8\pi}, & l_1 = 1, l_2 = 2, \\ \frac{L^2}{8\pi}, & l_1 = 2, l_2 = 1, \\ 0, & l_1 = 2, l_2 = 2, \end{cases}$$

$$\int_0^L \zeta_a^{l_1'}(x) \zeta_a^{l_2'}(x) dx = \begin{cases} \frac{5L}{16}, & l_1 = 1, l_2 = 1, \\ -\frac{L}{2\pi}, & l_1 = 1, l_2 = 2, \\ -\frac{L}{2\pi}, & l_1 = 2, l_2 = 1, \\ \frac{5L}{16}, & l_1 = 2, l_2 = 2, \end{cases}$$

$$\int_0^L x^n \cos \lambda_m x dx = \begin{cases} \frac{L^{1+n}}{1+n}, & m = 0, \\ \left(a^{1+n} \text{HypergeometricPFQ} \right. \\ \quad \times \left[\{1/2 + n/2\}, \{1/2, 3/2 + n/2\}, \right. \\ \quad \left. \left. - (1/4) m^2 \pi^2 \right] \right) \times (1+n)^{-1}, & m > 0. \end{cases} \quad (\text{B.1})$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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