Research Article

Confinement of Vibrations in Variable-Geometry Nonlinear Flexible Beam

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In this paper, we propose a novel strategy for controlling a flexible nonlinear beam with the confinement of vibrations. We focus principally on design issues related to the passive control of the beam by proper selection of its geometrical and physical parameters. Due to large deflections within the regions where the vibrations are to be confined, we admit a nonlinear model that describes with precision the beam dynamics. In order to design a set of physical and geometrical parameters of the beam, we first formulate an inverse eigenvalue problem. To this end, we linearize the beam model and determine the linearly assumed modes that guarantee vibration confinement in selected spatial zones and satisfy the boundary conditions of the beam to be controlled. The approximation of the physical and geometrical parameters is based on the orthogonality of the assumed linear mode shapes. To validate the strategy, we input the resulting parameters into the nonlinear integral-partial differential equation that describes the beam dynamics. The nonlinear frequency response curves of the beam are approximated using the differential quadrature method and the finite difference method. We confirm that using the linear model, the strategy of vibration confinement remains valid for the nonlinear beam.

1. Introduction

Vibration is one of the major problems that influence the performance of flexible structures. Vibration is a natural phenomenon that is unavoidable whatever its size may be, including conventional systems, such as aircraft wings, robot manipulators, blades in turning engines, crank mechanisms, and nonconventional systems that include large space structures, arm-type positioning mechanisms of magnetic disk drives, and microbeams in microelectromechanical systems. In certain cases, vibration excites unwanted resonances characterized by intolerable amplitudes. Because of the need for controlling structural vibrations and satisfying the increasing demand on security, accuracy, and long-life of these structures, researches focused on synthesizing control strategies, which are classified into three types: active [1, 2], passive [3, 4], and hybrid [5, 6]. Allaei [7] showed that vibration confinement is a superior control issue over the conventional control in isolating the sensitive parts of a structure. It has the potential to confine the vibrational energy, to reduce

the control effort, and to optimize the required sensors and actuators. Choura et al. [8] proposed a design methodology for vibration confinement in nonhomogeneous rods. They established conditions for selecting the rod's material and geometrical properties by constructing positive Lyapunov functions whose derivative with respect to the space variable is required to be negative. Baccouch et al. [3] and Gafsi et al. [4] used the orthogonality conditions of mode shapes, for approximating the physical and geometrical parameters of an inhomogeneous beam for the purpose of confining the vibratory motion in prespecified parts of its spatial domain.

Structural regions, where vibrations are to be confined, experience large amplitudes [9]. These structures must be described by nonlinear models, since linear models fail to depict their dynamical behavior. Nonlinear models are characterized by natural nonlinear phenomena, such as multiple solutions, jumps, frequency entrainments, natural frequency shifts, and modal interactions resulting in energy exchanges among modes.

FIGURE 1: Flexible beam.

The aim of this paper is to develop a passive control strategy for the vibration confinement in a flexible nonlinear beam via the inverse eigenvalue problem. This strategy consists of determining the geometric and physical parameters of the structure to yield a desired set mode shapes and associated natural frequencies. Here, we consider a geometric nonlinearity due to large deflections of the beam.

2. Problem Formulation and Objectives

We consider a flexible beam composed of m regions that are sensitive to vibrations. These regions are characterized by the spatial subdomains D_{si} ($i = 1, 2, \dots, m$), as shown in Figure 1.

The principal objective of this study is to find sets of spatially varying geometric and physical parameters that reduce the amplitudes of vibration in its sensitive regions while confining the vibrational energy in the remainder of regions. In general, vibration confinement yields an increase of vibration amplitudes in the less sensitive parts of the beam, and thus, a nonlinear model must be considered. For this, we adopt the dynamic model for beams given by [10] based on the nonlinear 2D Euler-Bernoulli beam theory (Figure 2). They included three-dimensional stress effects (due to the out-of-plane and in-plane warpings) and geometric nonlinearities as well as anisotropy and initial curvatures, which result in linear elastic couplings. The dynamic behavior is described by

$$\begin{aligned}
 & \rho A \ddot{v} + c \dot{v} - j_3 \ddot{v}'' + (\widehat{EI} \widehat{v}'')'' \\
 &= \left[\widehat{EA} \left(\widehat{u}' \widehat{v}' - \widehat{u}'' \widehat{v} + \frac{1}{2} \widehat{v}'^3 \right) \right]' \\
 &+ \left\{ \left[\widehat{EI} (\widehat{u}' \widehat{v}')' \right]' (1 - \widehat{u}') + (\widehat{EI} \widehat{v}'')' (\widehat{u}' - \widehat{u}''^2 + \widehat{v}'^2) \right. \\
 &\quad \left. - \left[\widehat{EI} \left(\widehat{u}'' \widehat{v}' - \frac{1}{3} \widehat{v}'^3 \right) \right]' \right\}' + q_2, \\
 & \widehat{\rho} \widehat{A} \ddot{u} - (\widehat{EA} \widehat{u}')' = \left[\widehat{EA} \widehat{v}''^2 \left(\frac{1}{2} - \widehat{u}' \right) \right]' \\
 &\quad + \left\{ v' \left[\widehat{EI} (\widehat{v}' - \widehat{u}' \widehat{v}')' \right]' - 2 \widehat{u}' \widehat{v}' (\widehat{EI} \widehat{v}'')' \right\}', \tag{1}
 \end{aligned}$$

where $\widehat{E}(\widehat{x})$ is the Young modulus, $\widehat{\rho}(\widehat{x})$ is the mass density, $\widehat{I}(\widehat{x})$ is the second moment of area, $\widehat{A}(\widehat{x})$ is the cross section area, $\widehat{c}$ is the damping coefficient, j_3 is the rotary inertia, and q_2 is the external excitation. The prime and dot denote,

FIGURE 2: The nonlinear Euler-Bernoulli beam theory: undeformed coordinate system xy and the deformed coordinate system $\xi\eta$.

respectively, the spatial and time derivatives. We assume that the longitudinal deflection $\widehat{u}$ is mainly induced by the transverse deformation $\widehat{v}$ [10]; that is,

$$\widehat{u}'' = -\left(\frac{1}{2} \widehat{v}'^2\right)' + O(\widehat{v}'^2). \tag{2}$$

Integrating (2) with respect to $\widehat{x}$, we obtain

$$\widehat{u}' = -\frac{1}{2} \widehat{v}'^2 + c_1(t), \tag{3}$$

where c_1 is determined by applying the boundary conditions associated with $\widehat{u}$.

3. Inverse Eigenvalue Problem

In this work, the basic idea of the proposed strategy for vibration confinement consists of altering the mode shapes and/or natural frequencies to maintain at lower levels the vibration amplitudes in the sensitive regions of the structure and allow the less sensitive regions to vibrate at relatively higher level amplitudes. Therefore, the main objective of this study is to devise a methodology for approximating a set of physical and geometrical parameters that produce vibration confinement in nonlinear structures. In this paper, we develop the methodology for controlling vibrations of beams described by the nonlinear equations (1). The strategy of vibration confinement applied to nonlinear beams consists of linearizing (1) and neglecting the axial deformation.

For convenience we define the following nondimensional variables and parameters:

$$\begin{aligned}
 E(\xi) &= \frac{\widehat{E}(x)}{E_0}, & \rho(\xi) &= \frac{\widehat{\rho}(x)}{\rho_0}, & I(\xi) &= \frac{\widehat{I}(x)}{I_0}, \\
 A(\xi) &= \frac{\widehat{A}(x)}{A_0}, & \xi &= \frac{x}{L}, & u &= \frac{\widehat{u}}{L}, & v &= \frac{\widehat{v}}{L}, \\
 \tau &= \frac{t}{L^2} \sqrt{\frac{E_0 I_0}{\rho_0 A_0}},
 \end{aligned} \tag{4}$$

$$p_1(\xi) = E(\xi) I(\xi), \quad p_2(\xi) = \rho(\xi) A(\xi),$$

$$p_3(\xi) = E(\xi) A(\xi),$$

where E_0 , ρ_0 , I_0 , and A_0 are the values of $\hat{E}$, $\hat{\rho}$, $\hat{I}$, and $\hat{A}$ at $x = 0$, respectively.

Therefore, the nondimensional linear model describing the beam flexure is

$$\rho A \ddot{v} + c \dot{v} + (EI v'')'' = q_2. \quad (5)$$

We now apply the strategy of vibration confinement for linear structures based on the orthogonality of linear mode shapes [4]. This strategy outputs a set of physical and geometrical parameters for the confinement of vibrations in desired regions of the structure to be controlled. Consequently, these parameters are substituted in (1) to examine the dynamical behavior of the nonlinear model that experience relatively large deflections due to confining the vibration energy around the less sensitive regions.

Without loss of generality, we consider the case of a spatially varying parameter beam clamped at both ends with vibration confinement in the middle. We consider the case of assumed modes that are constructed by premultiplying the confining function $\Gamma(\xi)$ by the modes $\Psi_i(\xi)$ ($i = 1, 2, \dots, n$) associated with the spatially invariant beam subjected to the same boundary conditions; that is,

$$\Psi_i(\xi) = \Gamma(\xi) \psi_i(\xi), \quad (6)$$

where $\psi_i(x) = C_1 \sin(\beta_i \xi) + C_2 \cos(\beta_i \xi) + C_3 \sinh(\beta_i \xi) + C_4 \cosh(\beta_i \xi)$ ($i = 1, 2, \dots, n$). The constants C_j ($j = 1, 2, \dots, 4$) and β_i ($i = 1, 2, \dots, n$) can be determined by using the boundary conditions. In order to confine the vibration in the middle of the beam, we consider, for instance, the confining function in the following Gaussian distribution $\Gamma(\xi) = e^{-\alpha \xi(1-\xi)}$, where L is the total length of the beam. This type of function gives the possibility to set the location and the amplitude of the peak of the required confinement. The interest here is to examine the effect of vibration confinement on the nonlinear behavior of the beam. In particular, we study the influence of the confinement parameter α on the frequency responses of the nonlinear beam. For this, we select a set of confinement parameters given by $-2, -1, -0.5, 0, 0.5, 1$, and 2 . Figure 3 displays the first four assumed mode shapes of the beam for the different values of α .

We now apply the inverse eigenvalue problem to each of the values of α to determine the beam spatially varying geometry. To this end, $p_1(\xi)$ and $p_2(\xi)$ are written as linear combinations of simple polynomials [4]:

$$p_1(\xi) = a_0 + \sum_{k=1}^p a_k \xi^k, \quad p_2(\xi) = b_0 + \sum_{l=1}^q b_l \xi^l. \quad (7)$$

To obtain accurate results, the inverse eigenvalue problem is numerically solved using the first 5 assumed modes ($n = 5$) [4]. Figures 4 and 5 show the resulting nondimensional stiffness $p_1(\xi)$ and mass functions $p_2(\xi)$.

With reference to Nayfeh and Pai [10], the expression of $c_1(t)$ for the clamped-clamped beam is given by

$$c_1 = \frac{1}{2} \int_0^1 v'^2 d\xi. \quad (8)$$

Therefore,

$$u' = -\frac{1}{2} v'^2 + \frac{1}{2} \int_0^1 v'^2 d\xi. \quad (9)$$

Using (9), we reduce (1) into one equation in v ; the equation of motion in nondimensional form is given by

$$\begin{aligned} & p_2 \ddot{v} + c \dot{v} - j_3 \ddot{v}'' + (p_1 v'')'' \\ &= \left[p_3 v' \left(\frac{1}{2} \int_0^1 v'^2 d\xi - \frac{1}{4} \left(\int_0^1 v'^2 d\xi - v'^2 \right)^2 \right) \right]' \\ &+ \left\{ \left[p_1 \left(\frac{v'}{2} \left(\int_0^1 v'^2 d\xi - v'^2 \right) \right) \right]' \right. \\ &\quad \times \left(1 - \frac{1}{4} \left(\int_0^1 v'^2 d\xi - v'^2 \right) \right) \\ &\quad \left. + (p_1 v'')' \left(\frac{1}{2} \left(\int_0^1 v'^2 d\xi - v'^2 \right) \right. \right. \\ &\quad \left. \left. - \frac{1}{4} \left(\int_0^1 v'^2 d\xi - v'^2 \right)^2 + v'^2 \right) \right. \\ &\quad \left. - \left[p_1 \left(\frac{1}{4} \left(\int_0^1 v'^2 d\xi - v'^2 \right)^2 v' - \frac{1}{3} v'^3 \right) \right]' \right\} \\ &+ \frac{L^3}{E_0 I_0} q_2 \end{aligned} \quad (10)$$

with $q_2(\xi, \tau) = f_0 \cos(\Omega \tau) \delta(\xi - 1/2)$.

In order to examine the nonlinear behavior of the beam, we propose to discretize (10) using an efficient numerical technique for variable cross section beams [11, 12]. To this end, we use the differential quadrature method (DQM) to transform the integral-partial differential equation into a set of ordinary differential equations and the finite difference method (FDM) to compute a limit-cycle solution for the nonlinear beam model.

The DQM is used to solve the space dependent partial differential equation by transforming it into ordinary differential equations describing the motion of the beam with respect to time at n preselected grid points

$$\xi_i = \frac{1}{2} \left[1 - \cos \left(\frac{i-1}{n-1} \pi \right) \right]. \quad (11)$$

Following Najjar et al. [11], the derivatives of the deflection with respect to space variable are expressed as a weighted linear sum of the deflection at all grid points; that is,

$$\left[\frac{\partial^r v(\xi, t)}{\partial \xi^r} \right]_{\xi=\xi_i} = \sum_{j=1}^n A_{i,j}^{(r)} v_j(t). \quad (12)$$

FIGURE 3: First four mode shapes of the linear clamped-clamped beam for different values of α .

The integral terms are discretized using the Newton-Cotes formula at the same grid points:

$$\int_0^1 v'^2 d\xi = - \int_0^1 v v'' d\xi = - \sum_{i=1}^n \sum_{j=1}^n C_i v_i A_{i,j}^{(2)} v_j \quad (13)$$

where $C_i = \int_0^1 \left(\prod_{j=1; j \neq i}^n \frac{\xi - \xi_j}{\xi_i - \xi_j} \right) d\xi$.

Using the boundary conditions, neglecting the rotary inertia, and keeping terms only to the third order, we end up with the following $n - 4$ ODEs:

$$p_2(\xi) \ddot{v}_i + c \dot{v}_i + p_1''(\xi) \sum_{j=1}^n A_{i,j}^{(2)} v_j + 2p_1'(\xi) \sum_{j=1}^n A_{i,j}^{(3)} v_j + p_1(\xi) \sum_{j=1}^n A_{i,j}^{(4)} v_j$$

$$= -2p_1'(\xi) \left(\sum_{j=1}^n A_{i,j}^{(3)} v_j \right) \left(\sum_{j=1}^n \sum_{k=1}^n C_j A_{j,k}^{(2)} v_j v_k \right) + 2p_1(\xi) \left(\sum_{j=1}^n A_{i,j}^{(2)} v_j \right)^3 - \frac{1}{2} p_3'(\xi) \left(\sum_{j=1}^n A_{i,j}^{(1)} v_j \right) \times \left(\sum_{j=1}^n \sum_{k=1}^n C_j A_{j,k}^{(2)} v_j v_k \right) + p_3(\xi) \left(\sum_{j=1}^n A_{i,j}^{(1)} v_j \right)^2 \sum_{j=1}^n A_{i,j}^{(2)} v_j - p_1''(\xi) \sum_{j=1}^n A_{i,j}^{(2)} v_j \left(\sum_{j=1}^n \sum_{k=1}^n C_j A_{j,k}^{(2)} v_j v_k \right) + p_1''(\xi) \left(\sum_{j=1}^n A_{i,j}^{(1)} v_j \right)^2 \sum_{j=1}^n A_{i,j}^{(2)} v_j$$

FIGURE 4: Variation of the beam stiffness for different values of α . FIGURE 5: Variation of the beam mass for different values of α .

$$\begin{aligned}
& -\frac{1}{2}p_3(\xi) \sum_{j=1}^n A_{i,j}^{(2)} v_j \left(\sum_{j=1}^n \sum_{k=1}^n C_j A_{j,k}^{(2)} v_j v_k \right) \\
& + p_1(\xi) \left(\sum_{j=1}^n A_{i,j}^{(1)} v_j \right)^2 \left(\sum_{j=1}^n A_{i,j}^{(4)} v_j \right) \\
& - p_1(\xi) \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n C_j A_{j,k}^{(2)} A_{i,l}^{(4)} v_k v_j v_l \\
& + 8p_1(\xi) \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n A_{i,j}^{(1)} A_{i,l}^{(2)} A_{i,k}^{(3)} v_j v_k v_l \\
& + 6p_1'(\xi) \left(\sum_{j=1}^n A_{i,j}^{(1)} v_j \right) \left(\sum_{j=1}^n A_{i,j}^{(2)} v_j \right)^2 \\
& + 2p_1'(\xi) \left(\sum_{j=1}^n A_{i,j}^{(1)} v_j \right)^2 \left(\sum_{j=1}^n A_{i,j}^{(3)} v_j \right) + q_2
\end{aligned}$$

$i = 3, \dots, n-2.$

(14)

To obtain the limit-cycle solutions associated with the ODE system (14), we have to solve the set of equations obtained by the DQM. Using the symmetry of the problem, we end up with $(n-3)/2$ ODEs describing the motion of the system. The limit-cycle solutions are obtained by assuming that the periodic orbits have the same frequency (Ω) than the excitation. The time is normalized to 1 then discretized along one period to 100 time steps. The FDM is now applied to discretize the time. In addition we enforce the condition that the first and last solution along the orbit are equal;

FIGURE 6: Frequency response convergence of a nonlinear beam $\alpha = -2$.

this will satisfy the periodicity condition of the limit-cycle solution. The final algebraic nonlinear system is solved using a Newton-Raphson technique [12].

Before proceeding with simulating the frequency response of the nonlinear beam for different α , we test the convergence of the DQM-FDM discretization scheme. For this, we take $\alpha = -2$, $f_0 = 1000$, and $c = 60$. The resulting frequency response curves with different DQM grid points and fixed FDM grid points are shown in Figure 6. We note that the frequency response curves for $n = 19, 21, 23$ are comparable. For computational reasons, the number of 19

FIGURE 7: Frequency response of a nonlinear uniform beam $\alpha = 0$.

grid points is adopted for the rest of the simulations. For $n = 19$, the vibration amplitude is small when the forcing frequency is away from the first resonant frequency ($\omega_1 = 24.658$) of the linear spatially varying beam. However, when the forcing frequency comes closer to ($\omega_1 = 24.658$), relatively higher amplitudes occur while the frequency curve tilts to the right, indicating a hardening behavior of the beam.

We now investigate the influence of α on the nonlinear behavior of the spatially varying beam. Structural nonlinearities become significant when the spatially varying beam experiences large deflections. This nonlinearity is likely to introduce multitude of phenomena, such as parametric resonance, multivalued responses and jumps, and secondary resonances [13]. Figure 7 displays the frequency response curves associated with the linear and nonlinear models of a uniform beam ($\alpha = 0$). We observe that both models depict similar amplitudes of vibration away from the fundamental frequency ($\omega_1 = 22.431$). As the frequency approaches ($\omega_1 = 22.431$) either from left or right, the frequency response curves veer from each other. The nonlinear model introduces two jumps in the frequency response curves at two cyclic-fold bifurcation points leading to hysteresis. Figure 8 displays the maximum deflection of the midpoint $v_{\max}$ for different values of α as the excitation frequency Ω is varied near the corresponding fundamental frequencies (24.568, 23.058, 22.757, 22.431, 24.193, 25.145, and 40.937).

In order to compare the resulting frequency response curves at different values of α , we maintain the forcing frequency interval length at 12. We present all frequency response curves by shifting their linear natural frequencies to zero. We note that, as α is more negative, the maximum amplitude increases and the frequency response curve widens in the neighborhood of the first linear natural frequency and bends more to the right. To quantify these observations,

FIGURE 8: Comparison between the frequency responses for different values of α .

TABLE 1: Variation of the inverse fundamental frequency ω_0 and the bifurcation points Ω_{b1} and Ω_{b2} with respect to α .

α	ω_0	Ω_{b1}	Ω_{b2}	$\Delta\Omega_b$	Max amplitude
-2.0	24.658	26.84	31.01	4.17	0.009
-1.0	23.058	24.8	29.18	4.38	0.01
-0.5	22.757	24.2	28.27	4.07	0.01
0.0	22.431	23.66	27.3	3.64	0.00975
+0.5	24.193	25.2	27.7	2.5	0.009
+1.0	25.145	25.85	27.88	2.03	0.0084
+2.0	40.937	41.25	41.35	0.1	0.0052

Table 1 provides for each value of α the frequencies Ω_{b1} and Ω_{b2} corresponding to the two bifurcation points, their differences, and the maximum amplitude.

We conclude that the vibrations are more confined for more negative values of α , which lead to more concentrated mode shapes in the middle. For lower values of the force amplitude f_0 , the linear and nonlinear frequency responses are nearly similar (comparable amplitudes and the nonlinear frequency curve bends slightly to the right).

4. Conclusions

In this study, we addressed the issue of vibration confinement in a nonlinear flexible beam. In particular, we considered the design of geometrical parameters of a beam whose dynamics is described by a nonlinear integral-partial differential equation. The design of these parameters was based on the linear dynamics associated with the nonlinear beam, and thus, the design of linear structures developed by Gafsi et al. [4] was adopted to approximate the geometry of the beam. The resulting parameters were then inputted into the nonlinear

integral-partial differential equation. In order to approximate the nonlinear frequency response curves of the beam as function of the confinement parameter, we discretized the nonlinear equation in space and time using DQM and FDM, respectively. In all simulations, we considered vibration confinement in the middle of the nonlinear beam. We confirmed that the strategy of vibration confinement and suppression remains valid for the nonlinear beam. We also concluded that having higher amplitudes on a larger frequency interval in conjunction with significant level of vibration confinement on a smaller region of the spatial domain presents a viable design for energy harvesting.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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