Research Article

The Analysis of Curved Beam Using B-Spline Wavelet on Interval Finite Element Method

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A B-spline wavelet on interval (BSWI) finite element is developed for curved beams, and the static and free vibration behaviors of curved beam (arch) are investigated in this paper. Instead of the traditional polynomial interpolation, scaling functions at a certain scale have been adopted to form the shape functions and construct wavelet-based elements. Different from the process of the direct wavelet addition in the other wavelet numerical methods, the element displacement field represented by the coefficients of wavelets expansions is transformed from wavelet space to physical space by aid of the corresponding transformation matrix. Furthermore, compared with the commonly used Daubechies wavelet, BSWI has explicit expressions and excellent approximation properties, which guarantee satisfactory results. Numerical examples are performed to demonstrate the accuracy and efficiency with respect to previously published formulations for curved beams.

1. Introduction

Curved beams, which are also called arches in some fields, are well established due to their capacity of transferring loads through the combined action of bending and stretching. They are more efficient than straight beams. However, compared with the analysis of straight beams, the analysis of curved beams is more complex due to the presence of bending stretching coupling. Besides, the effects of shear deformation and rotatory inertia also increase the complexity. Static analysis neglecting these effects can lead to inaccuracies especially when the ratio of length to thickness is small. It is the same with the free vibration analysis, which results in erroneous frequencies and mode shapes for higher modes [1]. Thus, it is difficult to solve this problem exactly due to the aforementioned reasons. Although the energy methods such as Castigliano’s theorem can be used to obtain the deflections, they are only useful in solving some simple problems [2]. The Rayleigh-Ritz is also an alternative method for the analysis of arches. However, its accuracy depends on the selection of displacement function and the chosen displacement function must satisfy the boundary condition, which turns out to restrict the application of the Rayleigh-Ritz method seriously. As an improvement of the Rayleigh-Ritz method, the finite element method (FEM) is widely used now in solving complex boundary problems.

In the finite element analysis of curved structures, the use of curved beam element is an efficient alternative to the use of large number of straight beam element to approximate the geometric shape of arches. However, many kinds of successful shape functions used in straight beam result in slow convergence or poor performance when they are applied to arches [3]. In the early studies [4, 5], there was a main view that it is the ignorance of the explicit rigid body motion that results in the slow convergence and poor performance of elements. Meck [6] showed that the unsatisfactory performance of curved elements is not due to the neglect of rigid body motions but due to the neglect of coupling required between normal and tangential displacements to satisfy the condition of inextensibility. Then he suggested the use of an independent interpolation for normal and tangential displacements to get a good result. Yamamoto and Ohtsubo [7] verified that cubic and quartic are the suitable orders of shape functions for curved beam element.
2. Theoretical Development for Curved Beam

2.1. The Geometry of Curved Beam. The development of a shear deformable curved beam theory is done along the same line as the development for straight beam (Timoshenko beam). The main differences are that the development of the curved beam is performed in a natural coordinate system and there exists a coupling between normal and tangential displacements. These differences make curved beam require three functions (normal, tangential, and rotational) that are coupled in the differential equations rather than two functions (tangential and rotational) for Timoshenko's straight beam. Figure 1 depicts the geometry of a curved beam having a general cross-section of area $A$ and moment of inertia $I$ about the area's centroid and curvature of $R$. In Figure 1, $h$ indicates the thickness of beam, $L$ is the span of beam, and $u$, $w$, and $\theta$ present the normal, tangential, and rotational displacements, respectively. We define a natural coordinate system as one in which the $x$-coordinate is coincident with the centroidal curved axis and $y$-coordinate is coincident with the principal axes of the cross-section. Similar to Timoshenko's beam, the centroidal axis is the line about which the cross-sections are rotating during bending and therefore represents a zero stress point at each cross-section and is known as the neutral axis. Two parameters are used to depict arches in this study according to the classifications given by [25] the following.

(i) Slenderness ratio ($R/h$):

(1) thick arch ($R/h < 40$);
(2) moderately thick arch ($R/h = 40$);
(3) thin arch ($R/h > 40$).

(ii) Subtended angle ($\alpha$ (deg.)):

(1) shallow arch ($\alpha < 40$);
(2) moderately deep arch ($\alpha = 40$);
(3) deep arch ($40 < \alpha < 180$);
(4) very deep arches ($\alpha > 180$).

Considering the conclusions given by [6, 7], the objective of this study is to present a BSWI (4-order 3-scale) Timoshenko curved beam element for arch analysis. The scaling functions of BSWI wavelets are selected because the scaling functions have excellent analyzing and computing capabilities. The BSWI curved beam element formulation is derived by using the Hamilton principle with the first-order shear deformation theory. The coupling between normal and tangential displacements is considered, and the normal displacement, tangential displacements, and rotation are interpolated by BSWI scaling function, respectively, according to the first-order shear deformation theory. At last, various numerical examples are performed to demonstrate the accuracy and efficiency of the present method with respect to previously published formulations for curved beam.
2.2. Energy Functional of Curved Beam. The strain vector 
\[ \mathbf{\varepsilon} = [\varepsilon_0, \kappa, \gamma_0]^T \] of the curved beam is determined using the definitions for strain in a natural coordinate system, and the nonzero strain components are obtained from the generalized shell theory [10, 11]:

\[ \varepsilon_0 = \frac{1}{R} \frac{\partial u}{\partial \alpha} - \frac{v}{R}, \quad \kappa = \frac{1}{R} \frac{\partial \theta}{\partial \alpha}, \quad \gamma_0 = \frac{u}{R} + \frac{1}{R} \frac{\partial v}{\partial \alpha} - \theta. \] (1)

For the sake of simplicity, (1) is depicted in the polar coordinate system as

\[ \varepsilon_0 = \frac{1}{R} \frac{\partial u}{\partial \alpha} - \frac{v}{R}, \quad \kappa = \frac{1}{R} \frac{\partial \theta}{\partial \alpha}, \quad \gamma_0 = \frac{u}{R} + \frac{1}{R} \frac{\partial v}{\partial \alpha} - \theta, \] (2)

where the physical means of \( \alpha \) can be found in Figure 1. Using a vector form to rewrite (2), the relations between displacement and strain are

\[ \varepsilon = \mathbf{B}_e \mathbf{d}, \] (3)

where

\[ \mathbf{B}_e = \begin{bmatrix} \frac{1}{R} \frac{\partial}{\partial \alpha} & 0 & 0 \\ 0 & \frac{1}{R} \frac{\partial}{\partial \alpha} & 0 \\ \frac{1}{R} & \frac{1}{R} \frac{\partial}{\partial \alpha} & -1 \end{bmatrix}, \quad \mathbf{d} = [u, v, \theta]^T. \] (4)

The strain energy is given by

\[ U = \frac{1}{2} \int_{\mathcal{V}} \varepsilon^T \mathbf{D} \varepsilon \, d\mathcal{V}. \] (5)

By making use of (3), the strain energy of curved beam is obtained:

\[ U = \frac{R}{2} \int_0^{\alpha_e} (\mathbf{B}_e \mathbf{d})^T \mathbf{D} \mathbf{B}_e \mathbf{d} \, d\alpha, \] (6)

where \( \mathbf{D} = \begin{bmatrix} \frac{E A}{E I} & \frac{G A}{E I} \\ \frac{G A}{E I} & \frac{G I}{G A} \end{bmatrix} \), \( E \) and \( G \) are Young's and shear moduli, \( k \) is the shear modifying factor, and \( \alpha_e \) is used to depict ending angle of arch.

Similar to the process of obtaining the strain energy, the kinetic energy of curved beam is obtained by

\[ T = \frac{1}{2} \int_{\mathcal{V}} \rho \left[ \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial v}{\partial t} \right)^2 + \left( \frac{\partial \theta}{\partial t} \right)^2 \right] \, d\mathcal{V} \]

\[ = \frac{R}{2} \int_0^{\alpha_e} \frac{\rho A}{\rho I} \left[ \frac{\partial \mathbf{d}}{\partial t} \right]^T \mathbf{B}_e \frac{\partial \mathbf{d}}{\partial t} \, d\alpha, \] (7)

where \( t \) depicts time and \( \rho \) is the mass density of beam material. Also, the work of the external forces is given by

\[ W = R \int_0^{\alpha_e} \mathbf{d}^T \mathbf{W} \, d\alpha, \quad \mathbf{W} = [f \quad q \quad m]^T, \] (8)

where \( f, q, \) and \( m \) are the distributed axial force, radial force, and moment along the length of beam, respectively. By aid of (6)–(8), the total energy of curved beam is formulated as follows:

\[ \Pi = U - T - W. \] (9)

Usually, the kinetic energy part is neglected in static analysis, and part of the work from the external forces is neglected when free vibration analysis is needed. Based on (9) and Hamilton’s principle, the corresponding terms of BSWI curved beam element static and free vibration analysis are obtained in Section 4.

3. Scaling Functions on the Interval \([0,1]\)

B-spline on interval \([0,1]\) is given by Quak et al. [26, 27]. Since there should be at least one inner wavelet on the interval \([0,1]\), the following condition must be satisfied:

\[ 2^j \geq 2m - 1, \] (10)

where \( m \) and \( j \) are the order and scale of BSWI, respectively. According to the 0 scale \( m \)th order B-spline functions and the corresponding wavelets given by Goswami et al. [27], the \( j \) scale \( m \)th order BSWI, simply denoted as BSWI \( m_j \), scaling functions \( \phi_{m,k}^j(\xi) \) can be evaluated by the following formulas:

\[ \phi_{m,k}^j(\xi) = \begin{cases} \phi_{m,k}^j(2^{j-1}\xi), k = -m + 1, \ldots, -1 & \text{(0 boundary scaling functions)} \\ \phi_{m,k}^j(1 - 2^{j-1}\xi), k = 2^j - m + 1, \ldots, 2^j - 1 & \text{(1 boundary scaling functions)} \\ \phi_{m,0}^j(2^{j-1}\xi - 2^{j-2}), k = 0, \ldots, 2^j - m & \text{(inner scaling functions)} \end{cases} \] (11)

Therefore, the scaling functions on the interval \([0,1]\) can be written in the vector form as follows:

\[ \Phi = [\phi_{m,-m+1}^j(\xi) \quad \phi_{m,-m+2}^j(\xi) \quad \ldots \quad \phi_{m,2m-1}^j(\xi)], \] (12)

where \( \xi \) belongs to the interval \([0,1]\). Figure 2 presents the BSWI43, which is used in this study as the shape function of curved beam element.

4. Finite Element Formulation Using B-Spline on Interval

Using Hamilton’s principle, the following equations of motions can be derived:

\[ \delta \Pi = \int_{t_1}^{t_2} (\delta U - \delta T - \delta W) \, dt = 0. \] (13a)

For static analysis,

\[ \delta \Pi = \int_{t_1}^{t_2} (\delta U - \delta W) \, dt = 0. \] (13b)

For free vibration analysis,

\[ \delta \Pi = \int_{t_1}^{t_2} (\delta U - \delta T) \, dt = 0. \] (13c)
For the finite element formulation, the normal displacement, tangential displacements, and rotation should be interpolated by BSWI scaling functions, respectively, to follow the first-order shear deformation theory. A displacement field assumption as shown in (14) is made firstly:

\[ u = \Phi^T \mathbf{u}, \quad v = \Phi^T \mathbf{v}, \quad \theta = \Phi^T \mathbf{\theta}, \quad (14) \]

where \( \mathbf{T} = [\Phi^T(\xi_1) \, \Phi^T(\xi_2) \, \ldots \, \Phi^T(\xi_{m+1})]^T \) is the BSWI element transform matrix. The element displacement field represented by the coefficients of wavelets expansions is transformed from wavelet space to physical space by aid of this transform matrix. In order to give a clear expression, we use some necessary notations here:

\[
\begin{align*}
\chi_{0,0} &= \alpha_E \int_0^1 T^T \Phi^T \Phi^T \Phi^T d\xi, \\
\chi_{0,1} &= \int_0^1 T^T \frac{d\Phi^T}{d\xi} \Phi^T d\xi, \\
\chi_{1,0} &= \int_0^1 T^T \Phi^T \frac{d\Phi}{d\xi} \Phi^T d\xi, \\
\chi_{1,1} &= \frac{1}{\alpha_E} \int_0^1 T^T \Phi^T \Phi^T \frac{d\Phi}{d\xi} \Phi^T d\xi, \\
\chi_{1,2} &= \frac{1}{\alpha_E^2} \int_0^1 T^T \frac{d\Phi^T}{d\xi} \Phi^T \Phi^T \frac{d\Phi}{d\xi} \Phi^T d\xi, \\
\chi_{2,1} &= \frac{1}{\alpha_E^2} \int_0^1 T^T \frac{d\Phi^T}{d\xi^2} \Phi^T \Phi^T \frac{d\Phi}{d\xi} \Phi^T d\xi, \\
\chi_{2,2} &= \frac{1}{\alpha_E^2} \int_0^1 T^T \frac{d\Phi^T}{d\xi^2} \Phi^T \frac{d\Phi}{d\xi} \Phi^T \Phi^T \frac{d\Phi}{d\xi} \Phi^T d\xi.
\end{align*}
\]

Substituting (14) with (13b), a finite element formulation of curved beam static analysis can be obtained:

\[
\mathbf{K} \Delta = \mathbf{P},
\]

\[
\mathbf{K} = \frac{1}{R} \int_0^\alpha_e \mathbf{B}_e^T \mathbf{D}_e \mathbf{B}_e d\alpha,
\]

\[
\mathbf{P} = R \int_0^\alpha_e (\Phi \mathbf{T})^T [f \\ q] d\alpha,
\]

where \( \Delta = [u \ v \ \theta]^T \) is the vector formulation of element displacement, \( \mathbf{K} \) is the stiffness matrix, and \( \mathbf{P} \) is the force vector. The details of the stiffness matrix are

\[
\mathbf{K} = \frac{1}{R} \times
\begin{bmatrix}
E_A \chi_{1,1}^{1,1} + k G A \chi_{1,0}^{0,0} & -E_A \chi_{1,1}^{1,0} + k G A \chi_{1,1}^{0,1} & -k G A \chi_{1,1}^{0,0} \\
-E_A \chi_{1,1}^{1,0} + k G A \chi_{1,1}^{0,1} & E_A \chi_{1,1}^{1,1} + k G A \chi_{1,1}^{0,1} & -k G A \chi_{1,1}^{0,1} \\
-k G A \chi_{1,1}^{0,0} & -k G A \chi_{1,1}^{0,1} & E I \chi_{1,1}^{1,1} + R^2 k G A \chi_{1,1}^{0,0}
\end{bmatrix}.
\]

Substituting (14) with (13c), a finite element formulation of arch free vibration analysis is obtained as a generalized eigenproblem:

\[
\left( \mathbf{K} - \omega^2 \mathbf{M} \right) \mathbf{X} = 0,
\]

where \( \omega \) is the natural frequency and \( \mathbf{X} \) is the mode shape of arches. The mass matrix \( \mathbf{M} \) is defined as

\[
\mathbf{M} = R \int_0^\alpha_e (\Phi \mathbf{T})^T \begin{bmatrix} \rho A & \rho A & \rho I \end{bmatrix} \Phi \mathbf{T} d\alpha.
\]

5. Numerical Examples

The validity and efficiency of the element formulated in the previous section are verified through numerical examples in this part, and static and dynamic numerical examples are given, respectively.

5.1. Static Analysis

5.1.1. A Quarter Circular Cantilever Arch. The model consists of a quarter circular cantilever arch, where the origin of coordinate is at the fixed end, as shown in Figure 3. The quarter ring is subjected to a radial tip load. This moderately deep arch is idealized with one BSWI element and analyzed for a wide range of thick to thin beams by changing the slenderness ratio of \( R/h \). The exact solution for this problem is derived using Castiglione’s energy theorem as [28]

\[
\begin{align*}
u_C &= \frac{P R^3}{2 E I} - \frac{P R}{2 K G A} - \frac{P R^3}{2 E A}, \\
\theta_C &= \frac{\pi P R^3}{4 E I} + \frac{\pi P R}{4 K G A} + \frac{\pi P R}{4 E A}, \\
\end{align*}
\]
In Table 1, MFE method [3] and the solution given by Lee and Sin [28] are used to contrast with the present method. In order to compare them with the exact solutions given by (20a)–(20c), we transform the data into the form of a ratio between numerical result and exact solution. It is noteworthy that the presence of BSWI is analogous to MFE, and the accuracy of $V$ and $\theta$ is better than MFE in a wide range of thick to thin beams, even in the thick condition. Compared with the solution given by [28], the present method also shows its superiority.

5.1.2. A Pinched Ring. Babu and Prathap [29] considered that the pinched ring is the best example to demonstrate the behavior of the elements in a deep arch configuration. Figure 4 depicts a pinched ring, on which the same radial loads are applied at the top and bottom of the ring. The physical model is idealized with one BSWI element and analyzed for a wide range of thick to thin beams by changing the slenderness ratio of $R/h$. The exact solution of points A and B is derived by means of Castigliano’s energy theorem as [28]

\[
v_{AC} = \frac{PR^3 \pi^2 - 8}{4\pi} + \frac{\pi PR}{4kGA} + \frac{\pi PR}{4EA},
\]

\[
v_{BC} = -\left(\frac{PR^3}{EI} - \frac{4 - \pi}{2\pi} + \frac{PR}{2kGA} - \frac{PR}{2EA}\right).
\]

Numerical results calculated by BSWI are shown in Table 2, and, as a reference, the solution given by Lee and Sin [28] is also shown there. It can be seen that the present method has a satisfying accuracy in a wide range of slenderness ratio. Although there are some slight accuracy losses with tangential displacement of point B when slenderness ratio is big, it is still in an acceptable confine. It should be noticed that the present method shows a good presence for the calculation of tangential displacement of point A.

5.1.3. A Nearly Straight Cantilever Curved Beam. In order to validate the reducibility of the present curved beam element to straight beam configuration, a numerical example as shown in Figure 5 is given. This arch has a very large radius and short span subjected to a concentrated load. The properties are $R = 2000$ m, $L = 100$ m, $h = 10$ m, $P = 1$ N, $E = 1 \times 10^6$ Pa, Poisson’s ratio $\mu = 0.3$, and width of beam $b = 50$ m. It should be mentioned that the properties presented above are defined in SI (System International), but all closed-form unit systems are suitable in this example; they will not influence the solved values. This structure is modeled using one BSWI element. For this type of thin beam ($R/h > 40$), the exact solutions of tip displacement $v$ for the three kinds of load conditions shown in Figure 5 are given by [3]

\[
v_a = \frac{PL^3}{3EI} \left(1 + 6I(1 + \mu) \frac{1}{kL^2A}\right),
\]

\[
v_b = \frac{qL^4}{8EI} \left(1 + 8I(1 + \mu) \frac{1}{kL^2A}\right),
\]

\[
v_c = \frac{qL^4}{30EI} \left(1 + 10I(1 + \mu) \frac{1}{kL^2A}\right).
\]

Equations (23)–(25) correspond to Figures 5(a)–5(c), respectively. Table 3 presents the comparison of numerical results and the exact solution in which the relative errors are all less than 1%. Thus, the present method has a good reducibility to straight beam configuration.

5.2. Free Vibration Analysis

5.2.1. A Hinged Arch. Figure 6 gives the geometry of a hinged arch which is widely analyzed in previously published works. The following properties of arch are employed for the computation: $R = 12$ in, $h = 0.25$ in, $A = 0.1563$ in$^2$, $I = 8.138 \times 10^{-4}$ in$^4$, $k = 8.497$, $E = 3.04 \times 10^7$ psi, $\mu = 0.3$, and $\rho = 0.02763$ slugs ft/in$^4$.

In order to test the accuracy of the present method for the change of subtended angle, the whole results for shallow arch, moderately deep arch, deep arch, and very deep arches, as classified in [25], are given in Table 4. With a various subtended angle $\alpha$ ranging from 10 to 350 degrees, Table 4 shows the comparisons of fundamental natural frequency.
Table 1: Comparisons of the present solutions of the one quarter ring with Castigliano's energy solution and other methods.

<table>
<thead>
<tr>
<th>Slenderness ratio (R/h)</th>
<th>MFE [3]</th>
<th>Lee and Sin [28]</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>u/uC</td>
<td>v/vC</td>
<td>u/uC</td>
</tr>
<tr>
<td>5</td>
<td>1.0124</td>
<td>0.9996</td>
<td>1.0149</td>
</tr>
<tr>
<td>10</td>
<td>1.0036</td>
<td>1.0000</td>
<td>1.0035</td>
</tr>
<tr>
<td>20</td>
<td>1.0010</td>
<td>1.0000</td>
<td>1.0007</td>
</tr>
<tr>
<td>100</td>
<td>1.0001</td>
<td>1.0000</td>
<td>0.9986</td>
</tr>
<tr>
<td>200</td>
<td>1.0001</td>
<td>1.0000</td>
<td>0.9997</td>
</tr>
<tr>
<td>1000</td>
<td>1.0001</td>
<td>1.0000</td>
<td>0.9998</td>
</tr>
</tbody>
</table>

Table 2: Comparisons of the present solutions of the pinched ring with Castigliano's energy solution and the solution given by [28].

<table>
<thead>
<tr>
<th>Slenderness ratio (R/h)</th>
<th>Lee and Sin [28]</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>v_d/v_{AC}</td>
<td>v_d/v_{BC}</td>
</tr>
<tr>
<td>2.5</td>
<td>0.98730</td>
<td>0.98533</td>
</tr>
<tr>
<td>5</td>
<td>0.99575</td>
<td>0.99556</td>
</tr>
<tr>
<td>10</td>
<td>0.99834</td>
<td>0.99828</td>
</tr>
<tr>
<td>20</td>
<td>0.99904</td>
<td>0.99897</td>
</tr>
<tr>
<td>100</td>
<td>0.99923</td>
<td>0.99918</td>
</tr>
<tr>
<td>200</td>
<td>0.99927</td>
<td>0.99924</td>
</tr>
<tr>
<td>1000</td>
<td>0.99925</td>
<td>0.99922</td>
</tr>
</tbody>
</table>

Table 3: Numerical results for the tip displacement (m) of a nearly straight cantilever curved beam.

<table>
<thead>
<tr>
<th>Load</th>
<th>Present (x10^{-3})</th>
<th>Exact solution (x10^{-3})</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concentrated</td>
<td>1.338</td>
<td>1.344</td>
<td>0.44%</td>
</tr>
<tr>
<td>Uniform distributed</td>
<td>50.255</td>
<td>50.520</td>
<td>0.53%</td>
</tr>
<tr>
<td>Linear distributed</td>
<td>13.419</td>
<td>13.507</td>
<td>0.65%</td>
</tr>
</tbody>
</table>

5.2.2 A Thin Circular Ring. The geometry of a circular ring has been given in Figure 4. Different from the problem considered for a pinched ring mentioned above, there exists no external force for the free vibration of circular ring. Timoshenko [32] gave the exact solution of the 4th order natural vibration of a thin ring:

\[ \omega_i = \sqrt{\frac{EI^2 (1 - i^2)^2}{\rho A R^3 (1 + i^2)}}. \]  

Considering the classifications given by [25], we use the slenderness ratio \( R/h > 40 \) to depict thin rings. Thus the slenderness ratio ranging from 50 to 1000 is studied in Table 5. The other properties for the model are \( R = 0.3048 \) m, \( E = 1.31 \times 10^{11} \) Pa, \( \mu = 0.3 \), and \( \rho = 1741 \) kg/m³. Two BSWI elements are used to idealize the model. Table 5 presents the first four order mode frequencies for ring, including the rigid body mode. The numerical results obtained using BSWI agree well with the exact solutions.

5.2.3 A 90° Arch with Different Boundaries. In this example, free vibrations of a 90° arch with different boundaries are analyzed to test the adaptability of BSWI element. The dimensionless frequency of the arch is defined as \( C_i = \omega a^2 R^2 \sqrt{\rho A/EI} \). The main property of arch is \( R/r, \) where \( r \) is the radius of gyration \( \sqrt{I/A} \). Arches (\( R/r = 100 \)) with different boundaries are studied, and the numerical results are presented in Table 6. The results show that the present method has a good agreement with the references.

5.2.4 A Three-Span Clamped Arch. This example is a curved beam with three subspans of equal span angle and different radii of curvature are depicted in Figure 8. The total span angle is 180° and each subspan angle is 60°. In this example, three BSWI elements are used to idealize this model. In contrast, the dimensionless frequency of the arch (\( R/r = 100 \)) is selected as \( C_i = \omega a R^2 \sqrt{\rho A/EI} \), which is slightly different from the \( C_i \) used in Section 5.2.3. The problem presented here has also been solved by applying the finite element method (FEM) [9, 33] and the wave analysis method [34]. In Table 7, the dimensionless frequencies obtained from the present approach are compared with those of [9, 33, 34]. It is noticed that the natural frequencies obtained from the BSWI agree well with these solutions, especially with [34], which
Table 4: Fundamental frequency (rad/s) of a hinged arch with various subtended angles.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5845.78</td>
<td>5841.74</td>
<td>5852.32</td>
<td>5874.30</td>
<td>5881.64</td>
</tr>
<tr>
<td>20</td>
<td>2836.20</td>
<td>2827.63</td>
<td>2829.66</td>
<td>2823.10</td>
<td>2829.13</td>
</tr>
<tr>
<td>30</td>
<td>2370.01</td>
<td>2339.82</td>
<td>2373.23</td>
<td>2345.20</td>
<td>2348.11</td>
</tr>
<tr>
<td>60</td>
<td>564.05</td>
<td>560.25</td>
<td>567.71</td>
<td>561.20</td>
<td>560.62</td>
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<tr>
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<td>229.66</td>
<td>232.94</td>
<td>230.40</td>
<td>229.69</td>
</tr>
<tr>
<td>120</td>
<td>115.82</td>
<td>115.64</td>
<td>117.50</td>
<td>116.30</td>
<td>115.64</td>
</tr>
<tr>
<td>150</td>
<td>64.52</td>
<td>64.43</td>
<td>76.24</td>
<td>64.93</td>
<td>64.42</td>
</tr>
<tr>
<td>180</td>
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<td>37.86</td>
<td>38.71</td>
<td>38.24</td>
<td>37.85</td>
</tr>
<tr>
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<td>22.77</td>
<td>23.42</td>
<td>23.05</td>
<td>22.76</td>
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<tr>
<td>270</td>
<td>7.94</td>
<td>7.92</td>
<td>8.39</td>
<td>8.06</td>
<td>7.92</td>
</tr>
<tr>
<td>300</td>
<td>4.20</td>
<td>4.18</td>
<td>4.65</td>
<td>4.27</td>
<td>4.19</td>
</tr>
<tr>
<td>330</td>
<td>1.70</td>
<td>1.69</td>
<td>2.28</td>
<td>1.73</td>
<td>1.69</td>
</tr>
<tr>
<td>350</td>
<td>0.50</td>
<td>0.49</td>
<td>1.38</td>
<td>0.50</td>
<td>0.50</td>
</tr>
</tbody>
</table>

6. Conclusions

A B-spline wavelet on interval curved beam element is constructed in this paper. This method gives satisfactory results for static and free vibration behaviors of arches with varied curvatures, thicknesses, and boundaries. The reason for getting acceptable results can be attributed to the fact that the present element is developed in generalized shell theory, which is adapted to obtain the couple of normal, tangential, and rotational displacement. Another reason for getting acceptable results can be attributed to the numerical properties of B-spline wavelet on interval. By means of the numerical examples, the accuracy and efficiency of the present element are validated. It can be seen that the proposed method can obtain good results for static and free vibration analysis. The methodology and results presented here can help in understanding the more complicated behavior of the curved shell element.
Table 5: Frequencies (rad/s) of a thin circular ring with various slenderness ratios.

<table>
<thead>
<tr>
<th>Slenderness ratio (R/h)</th>
<th>Method</th>
<th>Rigid body</th>
<th>Mode 1</th>
<th>Mode 2</th>
<th>Mode 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Present</td>
<td>0.021</td>
<td>70.158</td>
<td>198.397</td>
<td>380.314</td>
</tr>
<tr>
<td></td>
<td>Timoshenko [32]</td>
<td>0</td>
<td>70.169</td>
<td>198.469</td>
<td>380.546</td>
</tr>
<tr>
<td>50</td>
<td>Present</td>
<td>0.021</td>
<td>35.085</td>
<td>99.234</td>
<td>190.301</td>
</tr>
<tr>
<td></td>
<td>Timoshenko [32]</td>
<td>0</td>
<td>35.085</td>
<td>99.234</td>
<td>190.273</td>
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<tr>
<td>100</td>
<td>Present</td>
<td>0.021</td>
<td>17.545</td>
<td>49.640</td>
<td>95.240</td>
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<tr>
<td></td>
<td>Timoshenko [32]</td>
<td>0</td>
<td>17.542</td>
<td>49.617</td>
<td>95.137</td>
</tr>
<tr>
<td>200</td>
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<td>7.023</td>
<td>19.899</td>
<td>38.229</td>
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<tr>
<td></td>
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<td>0</td>
<td>7.017</td>
<td>19.847</td>
<td>38.055</td>
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<tr>
<td>500</td>
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<td>0.020</td>
<td>4.394</td>
<td>12.474</td>
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<tr>
<td></td>
<td>Timoshenko [32]</td>
<td>0</td>
<td>4.386</td>
<td>12.404</td>
<td>23.784</td>
</tr>
<tr>
<td>800</td>
<td>Present</td>
<td>0.019</td>
<td>3.518</td>
<td>10.002</td>
<td>19.212</td>
</tr>
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</table>

Table 6: First and second lowest dimensionless frequencies of 90° arch with different boundaries.

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>Method</th>
<th>Mode 1</th>
<th>Mode 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Raveendranath et al. [3]</td>
<td>33.83</td>
<td>78.72</td>
</tr>
<tr>
<td>Hinged-hinged</td>
<td>Krishnan and Suresh [9]</td>
<td>33.93</td>
<td>79.42</td>
</tr>
<tr>
<td></td>
<td>Sabir et al. [14]</td>
<td>33.79</td>
<td>79.02</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>33.87</td>
<td>78.95</td>
</tr>
<tr>
<td>Clamped-clamped</td>
<td>Raveendranath et al. [3]</td>
<td>55.34</td>
<td>102.28</td>
</tr>
<tr>
<td></td>
<td>Krishnan and Suresh [9]</td>
<td>55.82</td>
<td>104.28</td>
</tr>
<tr>
<td></td>
<td>Sabir et al. [14]</td>
<td>55.45</td>
<td>103.59</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>55.54</td>
<td>103.00</td>
</tr>
<tr>
<td>Clamped-hinged</td>
<td>Raveendranath et al. [3]</td>
<td>43.81</td>
<td>90.60</td>
</tr>
<tr>
<td></td>
<td>Krishnan and Suresh [9]</td>
<td>44.05</td>
<td>91.82</td>
</tr>
<tr>
<td></td>
<td>Sabir et al. [14]</td>
<td>43.81</td>
<td>91.29</td>
</tr>
<tr>
<td></td>
<td>Present</td>
<td>43.91</td>
<td>91.00</td>
</tr>
</tbody>
</table>

Table 7: Dimensionless frequencies of the three-span clamped arch.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Krishnan and Suresh [9]</th>
<th>Maurizi et al. [33]</th>
<th>Kang et al. [34]</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.701</td>
<td>2.680</td>
<td>2.6833155</td>
<td>2.682</td>
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<td>2</td>
<td>4.828</td>
<td>4.824</td>
<td>4.8337572</td>
<td>4.828</td>
</tr>
</tbody>
</table>

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References


Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.


