Research Article

Dynamic Analysis of a Timoshenko Beam Subjected to an Accelerating Mass Using Spectral Element Method

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This paper presents formulations for a Timoshenko beam subjected to an accelerating mass using spectral element method in time domain (TSEM). Vertical displacement and bending rotation of the beam were interpolated by Lagrange polynomials supported on the Gauss-Lobatto-Legendre (GLL) points. By using GLL integration rule, the mass matrix was diagonal and the dynamic responses can be obtained efficiently and accurately. The results were compared with those obtained in the literature to verify the correctness. The variation of the vibration frequencies of the Timoshenko and moving mass system was researched. The effects of inertial force, centrifugal force, Coriolis force, and tangential force on a Timoshenko beam subjected to an accelerating mass were investigated.

1. Introduction

Dynamic response of structures subjected to a moving force or moving mass is an important issue in engineering problems. For example, the trains have experienced great advances characterized by increasingly higher speeds and weights of vehicles. As a result, the dynamic response, as well as stresses, can be significantly higher than that before or static loads. The problem arose from the observations is a structure subjected to moving masses. Many researchers studied these problems and many studies are presented in the literatures. For examples, references [1–3] assuming the moving load to be a moving force have given some analytical solutions. Abu-Hilal [4] studied the dynamic response of a double Euler-Bernoulli beam due to a moving constant load. Fryba [5] extensively analyzed the solution of moving loads on structures. Rao [6] gave a detailed analysis of the vibration of a beam excited by a moving oscillator using a perturbation method. Zibdeh and Juma [7] presented the dynamic response of a rotating beam subjected to a random moving load using analytical and numerical methods. Akin and Mofid [8] investigated the dynamic behavior of Bernoulli-Euler beams carrying a moving mass with different boundary conditions using analytical-numerical method, and achievements by other researchers are presented in literatures [9–14].

The studies mentioned are based on Bernoulli-Euler beam, while the moment of inertia and shear deformation should be taken into account when ratio of the height to span is large. References [15–20] studied the dynamic response of Timoshenko beams subject to moving force. Ross [21] studied the problem of a viscoelastic Timoshenko beam subjected to a step-loading using the Laplace transform method. Katz et al. [22] solved the dynamic response of a rotating shaft subjected to a moving load with constant velocity using the modal analysis method and an integral transformation method. Yavari et al. [23] determined the dynamic behavior of Timoshenko beams due to a moving mass using the discrete element technique (DET). Lee [24] obtained the dynamic responses of a Timoshenko beam subjected to a moving mass using the AMM.

Two different kinds of spectral element method (SEM) have been developed to analyse various problems of engineering, namely, spectral element method in frequency domain (FSEM) and TSEM. The FSEM proposed by Doyle [25] is accurate and suited for simple 1D or 2D problems [26–28], while 3D problems or complex geometry are analyzed difficultly by this method and it does not work for nonlinear problems. TSEM is first used by Patera [29] to fluid dynamics; it has characteristics of spectral analysis and finite element method, thus, it is highly accurate and suited for complex

geometry. This method has been successfully applied to many problems, that is, wave, fluid, seismology and acoustics [30–35].

In this paper, a Timoshenko beam spectral element has been developed on the basis of Legendre polynomials-based spectral finite element. The beam is discretized into a very small number of elements with ten degrees of freedom each. The equations of motion in matrix form for a Timoshenko beam due to a moving mass are derived by using the Hamilton principle. The shape functions of the vertical displacement and the bending rotation for a Timoshenko beam element are formulated by employing Lagrange interpolation supported on the GLL points. The element mass matrix, stiffness matrix, and damping matrix are obtained by GLL integration rule. The effects of inertial forces are considered by the added mass, stiffness, and damping matrix. By assembling element matrices and element nodal vectors, respectively, the global equations of motion for a Timoshenko beam subjected to a moving mass are obtained. The vibration frequencies of the Timoshenko and a moving mass system were analyzed; in addition, the effects of the various forces considering the inertia of the mass will be investigated in detail. The presented formulations can be applied to solve the eigenvalue problem and dynamic responses for a Timoshenko beam with various boundary conditions by direct integration using generalized- ω_1 method.

2. Theory and Formulation

A uniform Timoshenko beam acted on by a moving mass m with a variable speed $v(t)$ and acceleration $a(t)$ along the beam is shown in Figure 1. A coordinate system is assumed to be fixed in the inertial frame with the x -axis parallel to the undeformed longitudinal axis of the beam and the y -axis pointing vertically downward in reverse direction as the gravitational acceleration g .

According to the Timoshenko's beam theory, the deformed beam can be described by the rotation of the across-area and the shear deformation [36]. The relation of the all rotation, section rotation, and shear strain is as follows:

$$\frac{\partial y}{\partial x} = \varphi + \gamma, \quad (1)$$

where φ and γ are, respectively, the bending rotation and shear strain.

As the shear strain $\gamma \neq 0$, the curvature of the Timoshenko beam is different from that of the Euler-Bernoulli beam and it can be expressed as follows:

$$R = -\frac{d\varphi}{dx}, \quad (2)$$

where R is the curvature of the Timoshenko beam.

The Timoshenko beam is discretized into a very small number of spectral elements with equal length, l each. Every beam element consists of five nodes; each node has two degrees of freedom, that is, vertical displacement y and section rotation φ . The nodes are not uniformly distributed

on the element. These nodes, denoted by $\xi_j \in [-1, 1]$, $j = 1, \dots, k$, are the GLL points which are the (k) roots of

$$(1 - \xi^2) L'_{k-1}(\xi) = 0, \quad (3)$$

where $L'_{k-1}(\xi)$ is the derivative of the Legendre polynomial of degree $(k - 1)$.

Figure 2 shows the n -th beam element on which the moving mass applies, at time t . The position $x \in [0, l]$ of the arbitrary point in the element can be transferred to the local coordinate along the axis of the beam element $\xi \in [-1, 1]$ by

$$\xi = \frac{2}{l}x - 1, \quad (4)$$

where l is the length of the element. The vertical displacement y and section rotation φ of an arbitrary point on the beam element can be expressed as follows:

$$y = [N_1 \ N_2 \ N_3 \ N_4 \ N_5] \begin{Bmatrix} y_1^n \\ y_2^n \\ y_3^n \\ y_4^n \\ y_5^n \end{Bmatrix} = [N] \{y^n\}, \quad (5)$$

$$\varphi = [N_1 \ N_2 \ N_3 \ N_4 \ N_5] \begin{Bmatrix} \varphi_1^n \\ \varphi_2^n \\ \varphi_3^n \\ \varphi_4^n \\ \varphi_5^n \end{Bmatrix} = [N] \{\varphi^n\},$$

where $\{y^n\}$ and $\{\varphi^n\}$ are, respectively, the element nodal displacement and section rotation vector. The shape functions N_i which are the orthogonal shape functions that denote Lagrange interpolation supported on the k GLL points can be expressed as follows:

$$N_i(\xi) = \prod_{j=1, j \neq i}^k \frac{\xi - \xi_j}{\xi_i - \xi_j}, \quad (6)$$

where k is the number of nodes. The equation of motion for the Timoshenko beam and moving mass system can be obtained by using the Hamilton principle, expressed as follows:

$$\delta \int_{t_1}^{t_2} L dt = 0, \quad (7)$$

where L is Lagrange function; it can be represented by

$$L = T + U + W, \quad (8)$$

where T , U , and W are, respectively, the kinetic energy, potential energy, and work done by external force. δ is the

FIGURE 1: Schematic figure of a Timoshenko beam and a moving mass system.

FIGURE 2: The n th Timoshenko beam spectral element with an accelerating mass.

variation operator; t_1 and t_2 are any instance. The kinetic energy of the Timoshenko beam element is given by

$$\begin{aligned} T &= \frac{1}{2} \rho A \int_0^l \left(\frac{\partial y(x, t)}{\partial t} \right)^2 d\xi + \frac{1}{2} \rho I \int_0^l \left(\frac{\partial \varphi(x, t)}{\partial t} \right)^2 d\xi \\ &= \frac{1}{2} \rho A \int_{-1}^1 \dot{y}(\xi, t)^2 \frac{l}{2} d\xi + \frac{1}{2} \rho I \int_{-1}^1 \dot{\varphi}(\xi, t)^2 \frac{l}{2} d\xi, \end{aligned} \quad (9)$$

where ρ is the density of the beam, A is the cross-sectional area, I is the second order moment of area, and the superposed dot stands for differentiation with respect to time. The potential energy of the Timoshenko beam element is expressed as follows:

$$\begin{aligned} U &= \frac{1}{2} \int_0^l E I R^2 dx + \frac{1}{2} \int_0^l \frac{G A}{\kappa} \left(\frac{\partial y}{\partial x} - \varphi \right)^2 dx \\ &= \frac{1}{2} \int_{-1}^1 \frac{l}{2} E I \left(\frac{2}{l} \right)^2 \varphi'(\xi)^2 d\xi \\ &\quad + \frac{1}{2} \int_{-1}^1 \frac{l}{2} \frac{G A}{\kappa} \frac{2}{l} (y'(\xi) - \varphi)^2 d\xi, \end{aligned} \quad (10)$$

where E is the Young's modulus, G is the shear modulus, and κ is the shear coefficient depending on the shape of the cross-section; the prime symbol ($'$) denotes differentiation with respect to ξ .

When the moving mass is located at $x = x_c$, the virtual work done by the moving force is represented by

$$\begin{aligned} \delta W &= m \int_0^l \left(g - \frac{\partial^2 y}{\partial t^2} + 2v \frac{\partial^2 y}{\partial x \partial t} + v^2 \frac{\partial^2 y}{\partial x^2} + a \frac{\partial y}{\partial x} \right) \\ &\quad \times \delta y(\xi) \delta(x - x_c) dx \\ &= m \int_{-1}^1 \frac{l}{2} \left(g - \ddot{y}(\xi) + 2v \frac{2}{l} \dot{y}'(\xi) \right. \\ &\quad \left. + v^2 \left(\frac{2}{l} \right)^2 y''(\xi) + a \frac{2}{l} y'(\xi) \right) \\ &\quad \times \delta y(\xi) \frac{\delta(\xi + 1 - (2/l)x_c)}{l/2} d\xi, \end{aligned} \quad (11)$$

where g is the gravitational acceleration. $\delta(\xi + 1 - (2/l)x_c)$ represents the Dirac delta function at $\xi = (2/l)x_c - 1$. By introducing (5), the expressions of the differentiations with respect to local coordinate ξ and time t for y and φ in (9)–(11) can be written as

$$\ddot{\mathbf{y}} = [\mathbf{N}] \{\ddot{\mathbf{y}}^n\}, \quad (12a)$$

$$\ddot{\boldsymbol{\varphi}} = [\mathbf{N}] \{\ddot{\boldsymbol{\varphi}}^n\}, \quad (12b)$$

$$\mathbf{y}' = [\mathbf{N}'] \{\mathbf{y}^n\}, \quad (12c)$$

$$\boldsymbol{\varphi}' = [\mathbf{N}'] \{\boldsymbol{\varphi}^n\}, \quad (12d)$$

$$\mathbf{y}'' = [\mathbf{N}''] \{\mathbf{y}^n\}, \quad (12e)$$

$$\dot{\mathbf{y}}' = [\mathbf{N}'] \{\dot{\mathbf{y}}^n\}. \quad (12f)$$

By substituting (12a)–(12f) into (8) and (8) into (7), respectively, and using the Hamilton principle, the equation

of motion for a Timoshenko beam element that acted on by a moving mass in matrix form can be expressed as

$$\mathbf{M}^e \ddot{\mathbf{q}}^e + \mathbf{C}^e \dot{\mathbf{q}}^e + \mathbf{K}^e \mathbf{q}^e = \mathbf{F}^e, \quad (13)$$

where $\mathbf{M}^e$, $\mathbf{C}^e$, and $\mathbf{K}^e$ are the mass, damping, and stiffness matrices of the Timoshenko beam element, respectively, $\mathbf{F}^e$ is the equivalent nodal force vector of the element, and $\ddot{\mathbf{q}}^e$, $\dot{\mathbf{q}}^e$, and $\mathbf{q}^e$ denote the nodal acceleration, velocity, and displacement vector of the element, respectively.

Where

$$\mathbf{q} = \begin{Bmatrix} \mathbf{y}^e \\ \boldsymbol{\varphi}^e \end{Bmatrix} \quad (14)$$

$\mathbf{M}^e$, $\mathbf{C}^e$, and $\mathbf{K}^e$ can be expressed as

$$\mathbf{M}^e = \mathbf{M}_T^e + \mathbf{M}_f^e, \quad (15a)$$

$$\mathbf{C}^e = \mathbf{C}_f^e, \quad (15b)$$

$$\mathbf{K}^e = \mathbf{K}_b^e + \mathbf{K}_s^e + \mathbf{K}_{f1}^e + \mathbf{K}_{f2}^e, \quad (15c)$$

where $\mathbf{M}_f^e$, $\mathbf{C}_f^e$, $\mathbf{K}_{f1}^e$, and $\mathbf{K}_{f2}^e$ are, respectively, added mass, damping, and stiffness matrix coming from the moving load; the matrices $\mathbf{M}_T^e$, $\mathbf{M}_f^e$, $\mathbf{C}_f^e$, $\mathbf{K}_b^e$, $\mathbf{K}_s^e$, $\mathbf{K}_{f1}^e$, $\mathbf{K}_{f2}^e$, and $\mathbf{F}^e$ are matrices defined as

$$\mathbf{M}_T^e = \int_{-1}^1 \mathbf{N}^T \begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix} \mathbf{N} d\xi, \quad (16a)$$

$$\mathbf{M}_f^e = m \int_{-1}^1 \mathbf{N}^T \mathbf{N} \delta \left(\xi + 1 - \frac{2}{l} x_c \right) d\xi, \quad (16b)$$

$$\mathbf{C}_f^e = 2m\nu \int_{-1}^1 \mathbf{N}^T \mathbf{N}' \delta \left(\xi + 1 - \frac{2}{l} x_c \right) d\xi, \quad (16c)$$

$$\mathbf{K}_b^e = EI \int_{-1}^1 \mathbf{B}_b^T \mathbf{B}_b d\xi, \quad (16d)$$

$$\mathbf{K}_s^e = \frac{GA}{k} \int_{-1}^1 \mathbf{B}_s^T \mathbf{B}_s d\xi, \quad (16e)$$

$$\mathbf{K}_{f1}^e = m\nu^2 \int_{-1}^1 \mathbf{N}^T \mathbf{N}'' \delta \left(\xi + 1 - \frac{2}{l} x_c \right) d\xi, \quad (16f)$$

$$\mathbf{K}_{f2}^e = ma \int_0^1 \mathbf{N}^T \mathbf{N}' \delta \left(\xi + 1 - \frac{2}{l} x_c \right) d\xi, \quad (16g)$$

$$\mathbf{F}^e = \int_{-1}^1 mg \mathbf{N}^T \delta \left(\xi + 1 - \frac{2}{l} x_c \right) d\xi \quad (16h)$$

in which

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 \end{bmatrix}, \quad (17a)$$

$$\mathbf{N}' = \frac{2}{l} \begin{bmatrix} N_1' & 0 & N_2' & 0 & N_3' & 0 & N_4' & 0 & N_5' & 0 \\ 0 & N_1' & 0 & N_2' & 0 & N_3' & 0 & N_4' & 0 & N_5' \end{bmatrix}, \quad (17b)$$

$$\mathbf{N}'' = \left(\frac{2}{l} \right)^2 \times \begin{bmatrix} N_1'' & 0 & N_2'' & 0 & N_3'' & 0 & N_4'' & 0 & N_5'' & 0 \\ 0 & N_1'' & 0 & N_2'' & 0 & N_3'' & 0 & N_4'' & 0 & N_5'' \end{bmatrix}, \quad (17c)$$

$$\mathbf{B}_b = \begin{bmatrix} 0 & -\frac{2}{l} N_1' & 0 & -\frac{2}{l} N_2' & 0 & -\frac{2}{l} N_3' & 0 & -\frac{2}{l} N_4' & 0 & -\frac{2}{l} N_5' \end{bmatrix} \quad (17d)$$

$$\mathbf{B}_s = \begin{bmatrix} -\frac{2}{l} N_1' & -N_1' & -\frac{2}{l} N_2' & -N_2' & -\frac{2}{l} N_3' & -N_3' \\ -\frac{2}{l} N_4' & -N_4' & -\frac{2}{l} N_5' & -N_5' \end{bmatrix}, \quad (17e)$$

where the superscript T denotes the transpose. The matrices $\mathbf{M}_T^e$, $\mathbf{K}_b^e$, and $\mathbf{K}_s^e$ should be calculated numerically by GLL integral which is different from Gauss integral; the integral nodes are the GLL nodes which contain the boundary nodes. It can be written as follows:

$$\mathbf{M}_T^e = \int_{-1}^1 \mathbf{N}^T \begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix} \mathbf{N} d\xi = \sum_{i=1}^n \omega_i \mathbf{N}^T(\xi_i) \begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix} \mathbf{N}(\xi_i), \quad (18a)$$

$$\mathbf{K}_b^e = EI \int_{-1}^1 \mathbf{B}_b^T \mathbf{B}_b d\xi = EI \sum_{i=1}^n \omega_i \mathbf{B}_b^T(\xi_i) \mathbf{B}_b(\xi_i), \quad (18b)$$

$$\mathbf{K}_s^e = \frac{GA}{k} \int_{-1}^1 \mathbf{B}_s^T \mathbf{B}_s d\xi = \frac{GA}{k} \sum_{i=1}^n \omega_i \mathbf{B}_s^T(\xi_i) \mathbf{B}_s(\xi_i), \quad (18c)$$

where EI and GA are the bending stiffness and shear stiffness, respectively; ρA and ρI are the inertia properties coefficient; weights ω_i are determined numerically by

$$\omega_i = \frac{2}{k(k+1)} \cdot \frac{1}{[L_k(x_i)]^2}, \quad i = 0, 1, \dots, k. \quad (19)$$

By assembling element matrices and element nodal vectors, respectively, the global equations of motion for a Timoshenko beam subjected to a moving mass can be obtained as follows:

$$\mathbf{M} \ddot{\mathbf{q}} + \mathbf{C} \dot{\mathbf{q}} + \mathbf{K} \mathbf{q} = \mathbf{F}, \quad (20)$$

where the matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are the global mass, damping, and stiffness matrices, respectively, of the Timoshenko beam, the vectors $\ddot{\mathbf{q}}$, $\dot{\mathbf{q}}$, and $\mathbf{q}$ are the global nodal acceleration,

velocity, and displacement vectors, respectively, and the vector $\mathbf{F}$ is the equivalent nodal force. If the added matrices of the mass and the equivalent force vector are zero, undamped natural frequencies and vibration node-shape of the beam are obtained from homogenous solution of (20). For this case, (20) reduces to

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}. \quad (21)$$

The solution of (21) can be expressed as

$$\mathbf{q} = \mathbf{A}e^{j\omega t}, \quad (22a)$$

$$\ddot{\mathbf{q}} = -\omega^2 \mathbf{A}e^{j\omega t}. \quad (22b)$$

Introducing (22a) and (22b) into (21) gives

$$(\mathbf{K} - \omega^2 \mathbf{M}) \Phi e^{j\omega t} = \mathbf{0}. \quad (23)$$

As $\Phi e^{j\omega t} \neq 0$, the solution of (23) exists only if the determinant of $(\mathbf{K} - \omega^2 \mathbf{M})$ is zero.

Consider the following:

$$\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0. \quad (24)$$

By solving (24) and (23), the frequencies and the corresponding vectors of the dynamic system can be obtained.

In order to obtain the dynamic response of the Timoshenko beam and the moving mass system, the generalized- α time integration method is used. It has more stability and lower numerical dissipation and numerical dispersion. In the generalized- α method, (20) is transformed to

$$\mathbf{K}^* = a_0 \mathbf{M} (1 - \alpha_m) + a_1 \mathbf{C} (1 - \alpha_f) + \mathbf{K} (1 - \alpha_f), \quad (25)$$

$$\begin{aligned} \mathbf{F}^* = & \mathbf{F} \left[(1 - \alpha_f) t_{i+1} + \alpha_f t_i \right] - \alpha_f \mathbf{K} \mathbf{q}_i \\ & + \mathbf{C} \left[a_1 (1 - \alpha_f) \mathbf{u}_i + \alpha_4 (1 - \alpha_f) \dot{\mathbf{u}}_i \right. \\ & \quad \left. + a_5 (1 - \alpha_f) \ddot{\mathbf{u}}_i - \alpha_f \dot{\mathbf{u}}_i \right] \\ & + \mathbf{M} \left[a_0 (1 - \alpha_m) \mathbf{q}_i + a_2 (1 - \alpha_m) \dot{\mathbf{q}}_i \right. \\ & \quad \left. + a_3 (1 - \alpha_m) \ddot{\mathbf{q}}_i - \alpha_m \ddot{\mathbf{q}}_i \right], \end{aligned} \quad (26)$$

where $a_0, a_1, a_2, a_3, a_4, a_5, a_6$, and a_7 are, respectively, the same as that in Newmark- β method; that is,

$$\begin{aligned} a_0 &= \frac{1}{\beta \Delta t^2}, & a_1 &= \frac{\gamma}{\beta \Delta t}, \\ a_2 &= \frac{1}{\beta \Delta t}, & a_3 &= \frac{1}{2\beta} - 1 \\ a_4 &= \frac{\gamma}{\beta} - 1, & a_5 &= \frac{\Delta t}{2} \left(\frac{\gamma}{\beta} - 2 \right), \\ a_6 &= \Delta t (1 - \gamma), & a_7 &= \gamma \Delta t, \end{aligned} \quad (27)$$

where $\alpha_m, \alpha_f, \beta$, and γ are the parameters which can control the precision and stability of the algorithm. In this study, the

following values are used: $\alpha_f = \alpha_m = 1/2$, $\beta = 1/4$, and $\gamma = 1/2$; the details of the generalized- α method can be referred to in [37].

The updated equations for the displacement velocity and acceleration vectors are expressed as

$$\mathbf{K}^* \mathbf{q}_{i+1} = \mathbf{F}^*,$$

$$\dot{\mathbf{q}}_{i+1} = a_0 (\mathbf{q}_{i+1} - \mathbf{q}_i) - a_2 \dot{\mathbf{q}}_i - a_3 \ddot{\mathbf{q}}_i, \quad (28)$$

$$\ddot{\mathbf{q}}_{i+1} = \dot{\mathbf{q}}_i + a_6 \ddot{\mathbf{q}}_i + a_7 \ddot{\mathbf{q}}_{i+1}.$$

3. Numerical Calculations and Analysis

To illustrate this method and verify its effectiveness and correctness, examples 1 and 2 are solved; the solutions obtained by the proposed method are compared with those obtained by other methods (including exact analytical solutions). Example 3 is solved to present the variation of the vibration frequencies of the system during the mass moves on the beam. Example 4 is solved to present the effects of inertial force, centrifugal force, Coriolis force, and tangential force on a Timoshenko beam subjected to an accelerating mass.

3.1. Example 1. Three types of prismatic Timoshenko beams considering the natural vibration problems are solved by the present method. The parameters of the beams are the same with different boundaries, simple-simple, clamped-free, and clamped-simple. The properties of the beam are modulus of elasticity $E = 2.1 \times 10^{11}$ N/m², modulus of rigidity $G = 3E/8$, density $\rho = 7850$ kg/m³, beam length $L = 0.4$ m, the cross section in the form of a rectangle with width $b = 0.02$ m and height $h = 0.08$ m, and cross-sectional shape factor $\kappa = 2/3$.

To test the convergence and validity of the method, the first nine-order natural frequencies obtained by presented method are compared with the analytical solution presented by Huang [38]. Fifth order of the spectral element is deployed to solve the problem, and the numbers of the beam spectral elements are from 1 to 16. The natural frequencies determined by the proposed method and the exact value calculated on the basis of [38] and Ruta [39] are shown in Table 1 to Table 3.

Tables 1, 2, and 3 show that the first nine frequencies of the Timoshenko beam are convergent by using 16 five-order Timoshenko beam spectral elements. As the excellent characteristic of the SEM that the mass matrix is diagonal because of the choice of Lagrange interpolation function supported on the GLL points in conjunction with the GLL integration rule, the values of the natural frequencies are equal to those of the exact.

As the proposed method has the advantage of the finite elements and the spectral method, it can analyse free vibration problem with various boundary conditions and good accuracy.

3.2. Example 2. For the purpose of verification, a Timoshenko beam and a moving mass system neglecting the damping effect with boundary condition simple-simple were considered. The beam is discretized to 16 elements by five-order Timoshenko beam spectral element. The dynamic

TABLE 1: Free vibration frequencies for simple-simple Timoshenko beam.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9
$n = 4$	6838.8338	23190.888	43444.615	64925.614	86738.326	108481.00	111981.29	120647.23	129920.78
$n = 8$	6838.8336	23190.827	43443.498	64939.222	86711.057	108431.78	111981.29	120647.23	130004.36
$n = 12$	6838.8336	23190.827	43443.493	64939.186	86710.905	108431.36	111981.29	120647.23	130003.65
$n = 16$	6838.8336	23190.827	43443.493	64939.185	86710.899	108431.34	111981.29	120647.23	130003.61
Exact [38]	6838.8336	23190.827	43443.493	64939.185	86710.889	108431.34	111981.29	120647.23	130003.61

TABLE 2: Free vibration frequencies for clamped-free Timoshenko beam.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9
$n = 4$	2529.4927	13279.912	31045.083	50829.910	71550.753	91995.694	110969.19	119289.42	131460.96
$n = 8$	2529.4927	13279.905	31044.792	50825.847	71565.121	91995.100	110976.56	119244.54	131607.53
$n = 12$	2529.4927	13279.905	31044.791	50825.835	71565.050	91994.834	110976.00	119244.58	131606.55
$n = 16$	2529.4927	13279.905	31044.791	50825.843	71565.047	91994.824	110975.98	119244.57	131606.52
Exact [38]	2529.4927	13279.905	31044.791	50825.843	71565.047	91994.824	110975.98	119244.57	131606.52

TABLE 3: Free vibration frequencies for clamped-simple Timoshenko beam.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9
$n = 4$	9741.9478	26150.351	45547.079	66197.391	87399.691	108645.85	114294.62	128761.33	131483.10
$n = 8$	9741.9469	26150.251	45545.516	66212.036	87376.814	108601.58	114295.46	128739.40	131611.56
$n = 12$	9741.9469	26150.251	45545.510	66211.995	87376.651	108601.16	114295.44	128739.41	131610.66
$n = 16$	9741.9469	26150.251	45545.510	66211.993	87376.644	108601.14	114295.44	128739.40	131610.63
Reference [39]	9741.9469	26150.251	45545.510	66211.993	87376.644	108601.14	114295.44	128739.40	131610.63

FIGURE 3: Normalized displacement under the equivalent moving loads with the initial speed given by $\tau = 0.11$. $\beta = 0.03$. (a) $\lambda = 1$, (b) $\lambda = 0$, and (c) $\lambda = -0.05$.

FIGURE 4: The first two normalized natural frequencies of simple-simple Timoshenko beam subjected to a moving mass with different velocities.

response of the system is obtained by using generalized- α method with $\alpha_f = \alpha_m = 1/2$, $\beta = 1/4$, and $\gamma = 1/2$. The same parameters and material properties of the Timoshenko beam as defined in Lee [24] are used in the numerical simulation, that is, $L = 1$ m, $E = 2.07 \times 10^{11}$ N/m², $G = 7.76 \times 10^{10}$ N/m², $\kappa = 0.9$, and $\rho = 7700$ kg/m³. The cross-sectional area A of the beam is computed from the radius of gyration r_0 defined by a nondimensional parameter (Rayleigh's coefficient) $\beta = r_0\pi/L$. The prescribed travelling speed of the moving mass, v , is similarly defined by a nondimensional parameter given by $\tau = v/v_{cr}$, where $v_{cr} = (\pi/L)\sqrt{EI/\rho A}$. The acceleration of the moving mass a is similarly defined by a nondimensional parameter $\lambda = amL/EI$.

The vertical displacement of the beam is normalized by y_{st} , defining normalized displacement $U = y/y_{st}$, where $y_{st} = MgL^3/48EI$ is the beam static displacement at mid-point when a load Mg is applied at the same point for an Euler-Bernoulli beam.

Figure 3 shows the normalized displacements under the moving mass by $\lambda = 1$, $\lambda = 0$, and $\lambda = -0.05$, respectively. The numerical results for the displacements agreed very much with the reported results by AMM using ten-term assumed functions [24].

3.3. Example 3. With only the equivalent force vector that is zero, the natural frequencies of the beam and moving masses system are obtained. For this case, we rewrite (20) as

$$\mathbf{A}\dot{\mathbf{p}} + \mathbf{B}\mathbf{p} = \mathbf{0}, \quad (29)$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{bmatrix}, \quad (30)$$

$$\mathbf{p} = \begin{bmatrix} \dot{\mathbf{q}} \\ \mathbf{q} \end{bmatrix}$$

And the eigenvalue problem relatives to (25) can be expressed as

$$\begin{aligned} \mathbf{A}\mathbf{u} &= \lambda\mathbf{B}\mathbf{u}, \\ \mathbf{v}^T\mathbf{A} &= \lambda\mathbf{v}^T\mathbf{B}, \end{aligned} \quad (31)$$

where λ is the vibration frequency and $\mathbf{u}$ and $\mathbf{v}$ are the right and left vector of the system, respectively.

The properties and parameters of the beam are the same as Example 1. Figures 4, 5, 6, 7, 8, and 9 show the first two normalized frequencies of the same Timoshenko beam subjected to a moving mass with different boundary conditions. It has a constant mass with $m = 10$ kg and different initial speed of the mass with $v = 5, 10, 20$ m/s in Figure 4 to Figure 6, while in Figure 7 to Figure 9 the initial speed is the same with $v = 10$ m/s and the masses are $m = 2, 5$ and 10 kg, respectively. the acceleration is $a = 0$ m/s². The vibration frequencies ω_s normalized by the natural frequencies ω of the beam with same boundary are denoted by f defined by $f = \omega_s/\omega$.

Figure 4 to Figure 9 indicate the natural frequencies of the beam and a moving mass system that are modified periodicity during the mass moving on the beam, and the number of the periods is equal to the order of the frequencies; it also shows that frequencies are not more than those of the Timoshenko

FIGURE 5: The first two normalized natural frequencies of clamped-free Timoshenko beam subjected to a moving mass with different velocities.

FIGURE 6: The first two normalized natural frequencies of clamped-simple Timoshenko beam subjected to a moving mass with different velocities.

beam. It can also be seen in Figure 3 to Figure 9 that the frequencies are not sensitive to the velocity of the mass, while they can be sensitive to the mass, as the velocity affects the added damping and stiffness matrices, and the added stiffness matrix is smaller than that of the beam with bending stiffness EI and shear stiffness GA , while the added mass affected with the inertia of the mass is corresponding to the inertia property coefficient ρA of the mass matrix in the motion equations.

3.4. Example 4. To analyse the effect of the different forces considering the inertia of the moving mass, the Timoshenko beam and a moving mass system are introduced with the same parameters as Example 2. The inertial force, Coriolis force, centrifugal force, and tangential force at the moving mass point are, respectively, shown in Figure 10.

Motion equations show that all the forces are the important members of the loads considering the effect of the

FIGURE 7: The first two normalized natural frequencies of simple-simple Timoshenko beam subjected to a moving mass with different masses.

FIGURE 8: The first two normalized natural frequencies of clamped-simple Timoshenko beam subjected to a moving mass with different masses.

moving mass and cannot be neglected. Figure 10 shows that the tangential force is small than the others. The effect of the loads on the dynamic response of the Timoshenko beam is shown in Figure 11.

It shows that the displacements with the tangential force neglected agreed with the displacement without neglected any forces, because the property of the Timoshenko beam is that the ratio of the height to span is large; therefore,

the deformation of the beam is smaller, and $y'(\xi)$ is very small. The Coriolis force is the important role of the loads. From the formulations of the motion equations, the effect of Coriolis force is appearing in the damping matrix; hence, with Coriolis force neglected, the displacements are larger than those without any neglected forces. Thus, the effect of the Coriolis force should be considered. The effect of Centrifugal force and inertia force are appearing in the stiff and mass

FIGURE 9: The first two normalized natural frequencies of clamped-simple Timoshenko beam subjected to a moving mass with different masses.

FIGURE 10: Normalized force under the equivalent moving loads with the initial speed given by $\tau = 0.5$, $\beta = 0.03$. (a) $\lambda = 1$, (b) $\lambda = 0$, and (c) $\lambda = -1$. Dashed line: inertia force; dotted dashed line: Coriolis force; dotted line: centrifugal force; solid line: tangential force.

matrix; therefore, with Centrifugal force and inertia being neglected, both the amplitude and the phase are different from those without any neglected forces.

4. Conclusion

The equation of motion in matrix form has been formulated for the dynamic response of a Timoshenko beam subjected to a moving mass by using TSEM. The inertia effect of the

moving mass can easily be taken into account by assembling added mass, damping, and stiffness matrices to the global mass, damping, and stiffness matrices. The TSEM has the advantages of the spectral method and finite element method; the degrees of freedom can be less with the characteristic of high accuracy; the eigenvalue and dynamic problem can be obtained efficiently with the accurate diagonal mass matrix.

The variation of the vibration frequencies of the Timoshenko and a moving mass system was obtained. Numerical

FIGURE 11: Normalized Displacement under the equivalent moving loads with the initial speed given by $\tau = 0.5$, $\beta = 0.03$. (a) $\lambda = 1$, (b) $\lambda = 0$, and (c) $\lambda = -1$. Solid line: without neglected; dashed line: with inertia force neglected; dotted dashed line: with Coriolis force neglected; dotted line: with centrifugal force neglected; bold dotted line: with tangential force neglected.

results for the Timoshenko beam and a moving mass system indicate that the inertia effect of the moving load cannot be neglected, the Coriolis, inertia, and centrifugal forces take a more important role than tangential force in the moving mass system, Coriolis, inertia, centrifugal, and tangential forces take the role of the damping, mass, and stiffness matrices, respectively.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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