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Research Article

A New Approach for Symmetry Preserving Partial Eigenstructure Assignment of Undamped Vibrating Systems

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A new approach for the partial eigenvalue and eigenstructure assignment of undamped vibrating systems is developed. This approach deals with the constant output feedback control with the collocated actuator and sensor configuration, and the output matrix is also considered as a design parameter. It only needs those few eigenpairs to be assigned as well as mass and stiffness matrices of the open-loop vibration system and is easy to implement. In addition, this approach preserves symmetry of the systems. Numerical example demonstrates the effectiveness and accuracy of the proposed approach.

1. Introduction

Eigenstructure assignment techniques based on active feedback control have attracted much attention and have been widely used for vibration suppression during the past three decades. The extensive research results can be found in review articles [1, 2] and references therein. As the dynamics of a structural system is naturally described by a set of second-order differential equations and control theory and estimation techniques are established for first-order realization of the systems, the majority of previous researches are made via transferring second-order equations to first-order configuration (see, e.g., [3]). In the past ten years, in order not to increase the dimension of the equations and to preserve the symmetry and sparsity of the structural matrices, many authors proposed their works based directly on the second-order equation models [4–12]. On the other hand, it is needed to change only few undesirable eigenvalues and/or corresponding eigenvectors which are purposefully assigned to desired values, and it is desirable to keep all other eigenpairs unchanged. So some methods have been proposed to implement the partial eigenvalue or eigenstructure assignment [13–18]. For these methods, the process of applying a control, whether state feedback or output feedback, usually produces a closed-loop matrix which is no longer symmetric. However, it is sometimes necessary for the closed-loop system to satisfy the reciprocity principle of the structure. For systems with this requirement one restriction is that the available controls may need to be symmetric, as indicated in [19].

In this respect, Elhay [19] used the symmetric rank-one matrix modification and derived an explicit solution for a symmetry preserving partial eigenvalue assignment method for the generalised eigenvalue problem. But the method is difficult to implement in feedback control. Ram [20] solved the eigenvalue assignment problem for the vibrating rod. More recently, Liu and Li [21] suggested a method for the symmetry preserving partial eigenvalue assignment of undamped structural systems. They adapted the method proposed in [22] to the requirement of the symmetry preserving. Their results involve complex mathematical calculation. In this paper we propose a new approach for the symmetry preserving eigenstructure assignment of undamped vibrating systems, which is also applicable to the partial eigenvalue assignment. This approach is concerned with the constant output feedback control with the collocated actuator and sensor configuration and uses a partial eigenstructure modification formulation for the incremental mass and stiffness matrices to be satisfied.

This formulation was recently obtained by Zhang et al. [18]. Our method proposed in this paper is easy to implement and the calculation procedure is relatively simple and clear.

The rest of the paper is organized as follows. Section 2 presents some preliminaries and the problem description. Section 3 gives our approach. Section 4 provides a numerical example to show the accuracy and effectiveness of the proposed approach. Finally, conclusions are drawn in Section 5.

2. Preliminaries and Problem Statement

2.1. Preliminaries. Consider an n-degree-of-freedom undamped vibration system that is modelled by the following set of second-order ordinary differential equations:

$$\mathbf{M}_0\ddot{\mathbf{q}}(t) + \mathbf{K}_0\mathbf{q}(t) = \mathbf{f}(t), \tag{1}$$

where $\mathbf{q}(t) \in R^n$ is displacement vector, $\mathbf{f}(t) \in R^n$ is the vector of external forces, and $\mathbf{M}_0, \mathbf{K}_0 \in R^{n\times n}$ are constant mass and stiffness matrices, respectively. In general, $\mathbf{M}_0$ is symmetric and positive definite, and $\mathbf{K}_0$ is symmetric and positive semidefinite; that is, $\mathbf{M}_0 = \mathbf{M}_0^T > 0$, $\mathbf{K}_0 = \mathbf{K}_0^T \geq 0$.

It is well known that if $\mathbf{q}(t) = \mathbf{x}e^{j\omega t}$ is a fundamental solution of (1), then the natural frequency ω and the mode shape vector $\mathbf{x}$ must satisfy the following generalised eigenvalue equation:

$$(\mathbf{K}_0 - \lambda_i\mathbf{M}_0)\,\mathbf{x}_i = 0, \quad i = 1, 2, \ldots, n, \tag{2}$$

where $\lambda_i = \omega_i^2$ is the square of the ith natural frequency ω_i, called the ith eigenvalue, and $\mathbf{x}_i$ is the corresponding ith eigenvector. Equation (2) can be written in a compact representation as follows:

$$\mathbf{K}_0\mathbf{X} = \mathbf{M}_0\mathbf{X}\boldsymbol{\Lambda}, \tag{3}$$

where $\boldsymbol{\Lambda} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ make up the complete eigenstructure of system (1) and $\mathbf{X}$ satisfies the mass-normalised condition $\mathbf{X}^T\mathbf{M}_0\mathbf{X} = \mathbf{I}_n$.

Suppose that the system described by (1) is modified by the incremental matrices $\Delta\mathbf{M} \in R^{n\times n}$ and $\Delta\mathbf{K} \in R^{n\times n}$. Then the motion of the modified system is governed by

$$(\mathbf{M}_0 + \Delta\mathbf{M})\,\ddot{\mathbf{q}}(t) + (\mathbf{K}_0 + \Delta\mathbf{K})\,\mathbf{q}(t) = \mathbf{f}(t) \tag{4}$$

and it satisfies the following eigenmatrix equation:

$$(\mathbf{K}_0 + \Delta\mathbf{K})\,\mathbf{Y} = (\mathbf{M}_0 + \Delta\mathbf{M})\,\mathbf{Y}\boldsymbol{\Sigma}, \tag{5}$$

where $\boldsymbol{\Sigma} = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_n)$ and $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n)$ are the complete eigenstructure of modified system (4).

In [18] a necessary and sufficient condition was proposed for the incremental mass and stiffness matrices that modify some eigenvalues or eigenpairs while keeping other eigenpairs unchanged, which is shown in the following:

$$\begin{aligned}\Delta\mathbf{K}\left(\mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^T\right) - \Delta\mathbf{M}\left(\mathbf{M}_0^{-1}\mathbf{K}_0\mathbf{M}_0^{-1} - \mathbf{X}_1\boldsymbol{\Lambda}_1\mathbf{X}_1^T\right) \\ = 0,\end{aligned} \tag{6}$$

where $\boldsymbol{\Lambda}_1$ and $\mathbf{X}_1$ are submatrices of $\boldsymbol{\Lambda}$ and $\mathbf{X}$ and are composed of eigenvalues and eigenvectors to be modified in system (1), respectively. It implies that if $\Delta\mathbf{M}$ and $\Delta\mathbf{K}$ satisfy (6), the following equation then holds:

$$(\mathbf{K}_0 + \Delta\mathbf{K})\,\mathbf{X}_2 = (\mathbf{M}_0 + \Delta\mathbf{M})\,\mathbf{X}_2\boldsymbol{\Lambda}_2, \tag{7}$$

where $\boldsymbol{\Lambda}_2$ and $\mathbf{X}_2$ are submatrices of $\boldsymbol{\Lambda}$ and $\mathbf{X}$ and are composed of unchanged eigenvalues and eigenvectors of system (1). Here $\boldsymbol{\Lambda} = \operatorname{diag}(\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2)$ and $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$. Equation (7) means that $\boldsymbol{\Lambda}_2$ and $\mathbf{X}_2$ are also eigenpairs of the modified system. When only the stiffness modification $\Delta\mathbf{K}$ is concerned, (6) reduces to

$$\Delta\mathbf{K}\left(\mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^T\right) = 0 \tag{8}$$

which is crucial to address the partial eigenvalue or eigenstructure assignment problem by static output feedback control in this paper.

2.2. Problem Statement. When considering the feedback control, we set $\mathbf{f}(t) = -\mathbf{B}\mathbf{u}(t)$ in (1), where $\mathbf{B} \in R^{n\times m}$ is full rank constant control matrix and $\mathbf{u}(t) \in R^{m\times 1}$ the control vector. If the real constant static output feedback

$$\mathbf{u}(t) = \mathbf{G}\mathbf{v}(t) \tag{9}$$

is applied to system (1), where $\mathbf{G} \in R^{m\times r}$ is an output feedback gain matrix to be determined and $\mathbf{v}(t) = \mathbf{C}\mathbf{q}(t) \in R^r$ is the output or measurement vector, where $\mathbf{C} \in R^{r\times n}$ is a real constant output matrix, then the closed-loop system becomes

$$\mathbf{M}_0\ddot{\mathbf{q}}(t) + (\mathbf{K}_0 + \mathbf{BGC})\,\mathbf{q}(t) = 0 \tag{10}$$

which is supposed to have desired eigenvalues or eigenpairs.

In this paper, we consider $\mathbf{B}$ and $\mathbf{C}$ as design variables as well. Moreover, a special situation, $\mathbf{B} = \mathbf{C}^T$, is concerned. Namely, it means the use of collocated actuator and sensor pairs, and the number of output measurements r is equal to the number of inputs m. The problem of partial eigenvalue or eigenstructure assignment is then formulated as follows: assuming that system (1) with $\mathbf{f}(t) = -\mathbf{B}\mathbf{u}(t)$ and $\mathbf{v}(t) = \mathbf{B}^T\mathbf{q}(t)$ is controllable and observable, given matrices $\mathbf{M}_0$ and $\mathbf{K}_0$, the subset $\{\lambda_i\}_{i=1}^p$ of the open-loop eigenvalues $\{\lambda_i\}_{i=1}^n$, and the corresponding eigenvector set $\{\mathbf{x}_i\}_{i=1}^p$ $(p < n)$, and given a set $\{\mu_i\}_{i=1}^p$ and the suitable vector set $\{\mathbf{y}_i\}_{i=1}^p$, find the control matrix $\mathbf{B}$ and the feedback gain matrix $\mathbf{G}$ such that the closed-loop system

$$\mathbf{M}_0\ddot{\mathbf{q}}(t) + \left(\mathbf{K}_0 + \mathbf{BGB}^T\right)\mathbf{q}(t) = 0 \tag{11}$$

has eigenvalues $\{\mu_i\}_{i=1}^p$ and $\{\lambda_i\}_{i=p+1}^n$ (i.e., the partial eigenvalue assignment) or has eigenvalues $\{\mu_i\}_{i=1}^p$, $\{\lambda_i\}_{i=p+1}^n$ and the corresponding eigenvectors $\{\mathbf{y}_i\}_{i=1}^p$, $\{\mathbf{x}_i\}_{i=p+1}^n$ (i.e., the partial eigenstructure assignment). Additionally, the coefficient matrices of the closed-loop system are of the symmetry preserving; that is, $\mathbf{K}_0 + \mathbf{BGB}^T$ is a symmetric matrix. Note that $\{\lambda_i\}_{i=p+1}^n$ and $\{\mathbf{x}_i\}_{i=p+1}^n$ denote unassigned and unchanged eigenvalues and eigenvectors of open-loop system (1).

For convenience, the following partitions and notation are used:

$\mathbf{\Lambda}_1 = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$, whose diagonal elements are the open-loop eigenvalues to be altered.

$\mathbf{X}_1 = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p)$, whose columns are the open-loop eigenvectors to be altered.

$\mathbf{\Lambda}_2 = \operatorname{diag}(\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_n)$, whose diagonal elements are the open-loop eigenvalues that did not take part in the assignment.

$\mathbf{X}_2 = (\mathbf{x}_{p+1}, \mathbf{x}_{p+2}, \ldots, \mathbf{x}_n)$, whose columns are the open-loop eigenvectors that did not take part in the assignment.

$\mathbf{\Sigma}_1 = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_p)$, whose diagonal elements are the assigned closed-loop eigenvalues, corresponding to $\mathbf{\Lambda}_1$.

$\mathbf{Y}_1 = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p)$, whose columns are the assigned closed-loop eigenvectors, corresponding to $\mathbf{X}_1$.

3. Problem Solution

Let $\Delta\mathbf{K} = \mathbf{B}\mathbf{G}\mathbf{B}^T$; (8) can be rewritten as

$$\mathbf{B}\mathbf{G}\mathbf{B}^T \left(\mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^T\right) = 0. \tag{12a}$$

Because $\mathbf{B}$ is assumed to be of full column rank, (12a) implies that

$$\mathbf{G}\mathbf{B}^T \left(\mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^T\right) = 0. \tag{12b}$$

Transposing both sides of (12a), it has

$$\left(\mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^T\right)\mathbf{B}\mathbf{G}^T = 0. \tag{12c}$$

It is apparent that if $\mathbf{B}$ satisfies the following matrix equation, then (12c) (i.e., (12a)) holds:

$$\left(\mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^T\right)\mathbf{B} = 0 \tag{13}$$

which means that $\mathbf{B}$ belongs to the right null space of the matrix $(\mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^T)$ (denoted by $N(\mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^T)$). Importantly, (13) can be used to determine $\mathbf{B}$; that is, $\mathbf{B}$ can be chosen to be composed of the basis vectors of $N(\mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^T)$. As $\operatorname{rank}(\mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^T) = n - p$, then $\operatorname{rank}\mathbf{B} = p$. It implies that the column order of $\mathbf{B}$ is constrained to be $m \le p$ under the assumption of $\mathbf{B}$ being full column rank. It should be noted that the proposed approach to the determination of $\mathbf{B}$ from basis vectors of $N(\mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^T)$ is also applicable to the situation of $\mathbf{B}$ not to be full column rank and $m > p$.

Now the partial eigenvalue or eigenstructure assignment problem involves solving two matrix equations as follows:

$$\begin{aligned} &\left(\mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^T\right)\mathbf{B} = 0, \\ &\left(\mathbf{K}_0 + \mathbf{B}\mathbf{G}\mathbf{B}^T\right)\mathbf{Y}_1 = \mathbf{M}_0\mathbf{Y}_1\mathbf{\Sigma}_1. \end{aligned} \tag{14}$$

To obtain $\mathbf{G}$, we now turn to discuss the second matrix equation of (14). After rearranging it, we have

$$\mathbf{B}\mathbf{G}\mathbf{B}^T\mathbf{Y}_1 = \mathbf{M}_0\mathbf{Y}_1\mathbf{\Sigma}_1 - \mathbf{K}_0\mathbf{Y}_1. \tag{15}$$

Equation (15) is of the form $\mathbf{AZE} = \mathbf{F}$, where $\mathbf{A}$ and $\mathbf{E}$ are given matrices of appropriate dimensions and the matrix $\mathbf{Z}$ needs to be determined. The necessary and sufficient condition for the existence of solutions on this type of matrix equation is $\mathbf{A}\mathbf{A}^+\mathbf{F}\mathbf{E}^+\mathbf{E} = \mathbf{F}$ [23], where the superscript $+$ denotes the Moore-Penrose inverse of a matrix. Supposing here that (15) has solutions, in what follows, we present a symmetric solution for $\mathbf{G}$. Premultiplying (15) by $\mathbf{Y}_1^T$, (15) can be rewritten as

$$\mathbf{Y}_1^T\mathbf{B}\mathbf{G}\mathbf{B}^T\mathbf{Y}_1 = \mathbf{Y}_1^T\mathbf{M}_0\mathbf{Y}_1\mathbf{\Sigma}_1 - \mathbf{Y}_1^T\mathbf{K}_0\mathbf{Y}_1. \tag{16}$$

However, the symmetry of the solution $\mathbf{G}$ from (16) is not guaranteed yet. We will now show the symmetry of $\mathbf{G}$ by imposing the mass normalisation condition; that is, $\mathbf{Y}_1$ will be such that

$$\mathbf{Y}_1^T\mathbf{M}_0\mathbf{Y}_1 = \mathbf{D}, \tag{17}$$

where $\mathbf{D}$ is a diagonal matrix. First note that $\mathbf{\Sigma}_1$ is a diagonal matrix. Then, using (17) and noting that $\mathbf{Y}_1^T\mathbf{K}_0\mathbf{Y}_1$ is a symmetric matrix, we see that the right-hand side of (16) is symmetric. Thus, the left-hand side matrix $\mathbf{Y}_1^T\mathbf{B}\mathbf{G}\mathbf{B}^T\mathbf{Y}_1$ of (16) is also symmetric, implying that $\mathbf{G}$ is symmetric.

The minimal norm solution of (16) is given as follows [23]:

$$\mathbf{G} = \left(\left(\mathbf{B}^T\mathbf{Y}_1\right)^+\right)^T \left(\mathbf{Y}_1^T\mathbf{M}_0\mathbf{Y}_1\mathbf{\Sigma}_1 - \mathbf{Y}_1^T\mathbf{K}_0\mathbf{Y}_1\right)\left(\mathbf{B}^T\mathbf{Y}_1\right)^+, \tag{18}$$

where $(\mathbf{B}^T\mathbf{Y}_1)^+$ is the Moore-Penrose inverse of the matrix $\mathbf{B}^T\mathbf{Y}_1$ which is a $p \times p$ matrix with a lower order.

As is known, not any given vectors $\mathbf{Y}_1$ can be assigned to closed-loop system (11). In order that the second matrix equation of (14) (or (15)) holds for a symmetric matrix $\mathbf{G}$ and the partial eigenstructure assignment involved in this paper can be achieved, an additional requirement for $\mathbf{Y}_1$ must be satisfied. Because $\mathbf{Y}_1^T\mathbf{M}_0\mathbf{X}_2 = 0$ based on the orthogonality properties of the generalised eigenvalue problem for the real symmetric matrices, $\mathbf{Y}_1$ must be of the form $\mathbf{Y}_1 = \mathbf{X}_1\mathbf{L}$ for some $p \times p$ nonsingular matrix $\mathbf{L}$. Furthermore, $\mathbf{L}$ can be written as $\mathbf{L} = \mathbf{VT}$, where $\mathbf{V}$ is nonsingular diagonal matrix and $\mathbf{T}$ is orthogonal [24]. It is worthwhile noting that this condition of $\mathbf{Y}_1$ is seemingly rather strict; but, in practice, this condition is almost always satisfied [25].

Based on the discussion above, we now state the proposed approach of the partial eigenvalue or eigenstructure assignment in this paper as follows.

Step 1. Compute the orthonormal basis vectors of $N(\mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^T)$; use the obtained basis vectors to form the control matrix $\mathbf{B}$.

Step 2. Check the given eigenvector matrix $\mathbf{Y}_1$ to be sure that (17) holds; otherwise, update the matrix $\mathbf{Y}_1$ by computing the singular value decomposition of $\mathbf{Y}_1^T\mathbf{M}_0\mathbf{Y}_1 = \mathbf{PSQ}^T$, and $\mathbf{Y}_1$ should be substituted by $\mathbf{Y}_1\mathbf{P}^T$. When only the partial eigenvalue assignment is required, it is appropriate to set $\mathbf{Y}_1$ to be $\mathbf{X}_1$.

Step 3. Check the compatibility condition of (15). If it is satisfied, Step 4 gives an exact solution of the feedback gain matrix $\mathbf{G}$; otherwise, it gives just a least-squares solution.

Step 4. Compute the feedback gain matrix $\mathbf{G}$ from (18).

4. A Numerical Example

In this section, a numerical example in [21] is used to illustrate our approach. For an undamped vibrating system, it has the stiffness and mass matrices as follows:

$$\mathbf{K}_0 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$
$$\mathbf{M}_0 = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{2}{3} \end{pmatrix}. \quad (19)$$

Its eigenvalues are $\{0.0, 0.3564, 1.5403, 3.8816, 7.6127, 11.3551\}$. Two eigenvalues $\{0.3564, 1.5403\}$ among them are to be reassigned to $\{0.75, 1.85\}$, which is also the assignment in [21]. We have $\mathbf{\Lambda}_1 = \operatorname{diag}(0.3564, 1.5403)$ and $\mathbf{\Sigma}_1 = \operatorname{diag}(0.75, 1.85)$. The corresponding eigenvectors of $\mathbf{\Lambda}_1$ are

$$\mathbf{X}_1 = \begin{pmatrix} -0.6293 & 0.6821 \\ -0.5235 & 0.2641 \\ -0.2415 & -0.4776 \\ 0.1217 & -0.6340 \\ 0.4439 & -0.0133 \\ 0.6169 & 0.6236 \end{pmatrix}. \quad (20)$$

Case 1. Only the partial eigenvalue assignment is required. Here $m = r = p = 2$. Let $\mathbf{Y}_1 = \mathbf{X}_1$; using our approach, we obtain the control matrix

$$\mathbf{B} = \begin{pmatrix} -1.4264 & 0.6197 \\ -1.7167 & 0.4642 \\ 0.5051 & -0.9061 \\ 1.6380 & -1.1779 \\ 1.0000 & 0 \\ 0 & 1.0000 \end{pmatrix}. \quad (21)$$

Now we can inspect the controllability and observability conditions with respect to $\mathbf{B}$ and $\mathbf{\Lambda}_1$ and present the following results:

$$\operatorname{rank}\left[\mathbf{B}, \mathbf{K}_0 - \lambda_i \mathbf{M}_0\right] = 6,$$
$$\operatorname{rank} = \begin{bmatrix} \mathbf{B}^T \\ \mathbf{K}_0 - \lambda_i \mathbf{M}_0 \end{bmatrix} = 6, \quad (22)$$
$$(i = 2, 3).$$

Furthermore, it is believed that (15) has solutions after we check its solvability condition. Thus we have the output feedback gain matrix and the symmetric matrix $\mathbf{BGB}^T$ as follows:

$$\mathbf{G} = \begin{pmatrix} 0.0692 & 0.0787 \\ 0.0787 & 0.1457 \end{pmatrix},$$
$$\mathbf{BGB}^T = \begin{pmatrix} 0.0575 & 0.0754 & -0.0053 & -0.0558 & -0.0499 & -0.0220 \\ 0.0754 & 0.1098 & 0.0196 & -0.0552 & -0.0822 & -0.0675 \\ -0.0053 & 0.0196 & 0.0652 & 0.0491 & -0.0364 & -0.0922 \\ -0.0558 & -0.0552 & 0.0491 & 0.0840 & 0.0206 & -0.0427 \\ -0.0499 & -0.0822 & -0.0364 & 0.0206 & 0.0692 & 0.0787 \\ -0.0220 & -0.0675 & -0.0922 & -0.0427 & 0.0787 & 0.1457 \end{pmatrix}. \quad (23)$$

The closed-loop eigenvalues are $\{0.0000, 0.7500, 1.8500, 3.8816, 7.6127, 11.3551\}$, and $\|\mathbf{M}_0\mathbf{X}_2\mathbf{\Lambda}_2 - (\mathbf{K}_0 + \mathbf{BGB}^T)\mathbf{X}_2\|_F = 4.9665e-015$. It can be seen that our approach can accurately solve the partial eigenvalue assignment problem. Interestingly, the symmetric matrix $\mathbf{BGB}^T$ obtained here is the same as that in [21].

Case 2. The partial eigenstructure assignment: $\mathbf{\Lambda}_1$, $\mathbf{X}_1$, and $\mathbf{\Sigma}_1$ are the same as the above; let the assigned closed-loop eigenvectors $\mathbf{Y}_1$ be

$$\mathbf{Y}_1 = \begin{pmatrix} -0.0432 & 0.6133 \\ -0.3053 & 0.3582 \\ -0.6734 & -0.1351 \\ -0.4376 & -0.3701 \\ 0.4461 & -0.1982 \\ 1.1902 & 0.0463 \end{pmatrix} \quad (24)$$

which satisfies mass normalisation condition (17). Thus we obtain the following:

$$\mathbf{G} = \begin{pmatrix} 0.1562 & 0.0845 \\ 0.0845 & -0.0425 \end{pmatrix},$$
$$\mathbf{BGB}^T = \begin{pmatrix} 0.1521 & 0.2244 & 0.0470 & -0.1061 & -0.1704 & -0.1469 \\ 0.2244 & 0.3164 & 0.0337 & -0.1808 & -0.2289 & -0.1648 \\ 0.0470 & 0.0337 & -0.0724 & -0.0918 & 0.0023 & 0.0812 \\ -0.1061 & -0.1808 & -0.0918 & 0.0340 & 0.1563 & 0.1884 \\ -0.1704 & -0.2289 & 0.0023 & 0.1563 & 0.1562 & 0.0845 \\ -0.1469 & -0.1648 & 0.0812 & 0.1884 & 0.0845 & -0.0425 \end{pmatrix} \quad (25)$$

and $\mathbf{B}$ is the same as that in Case 1. $\boldsymbol{\Sigma}_1$ and $\mathbf{Y}_1$ are accurately assigned in the closed-loop system (for the sake of saving space, they are not listed here). Additionally, we can compute $\|\mathbf{M}_0\mathbf{Y}_1\boldsymbol{\Sigma}_1-(\mathbf{K}_0+\mathbf{BGB}^T)\mathbf{Y}_1\|_F = 1.2071e-015$ and $\|\mathbf{M}_0\mathbf{X}_2\boldsymbol{\Lambda}_2-(\mathbf{K}_0 + \mathbf{BGB}^T)\mathbf{X}_2\|_F = 5.1580e - 015$. This also means that our approach can accurately solve the partial eigenstructure assignment problem, which is a distinctive advantage with respect to the approach proposed in [21].

5. Conclusions

Based on a partial eigenstructure modification formulation for the incremental mass and stiffness matrices to be satisfied, an approach is successfully developed to assign the partial eigenvalue and eigenstructure of undamped vibrating systems. It can preserve symmetry of the closed-loop system's matrices and its calculation steps are simple and clear.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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