Research Article

Vibration and Stability of Variable Cross Section Thin-Walled Composite Shafts with Transverse Shear

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A dynamic model of composite shaft with variable cross section is presented. Free vibration equations of the variable cross section thin-walled composite shaft considering the effect of shear deformation are established based on a refined variational asymptotic method and Hamilton's principle. The numerical results calculated by Galerkin method are analyzed to indicate the effects of ply angle, taper ratio, and transverse shear deformation on the first natural frequency and critical rotating speed. The results are compared with those obtained by using finite element package ANSYS and available in the literature using other models.

1. Introduction

The shaft is an extremely important part in various mechanical transmissions. The shafts suffer not only static load such as bending moment, torque, and axial force which usually leads to its combined deformation, but also dynamic load caused by effect of inertia force and gyroscope which produces alternating stress and vibrations due to its rotation. Composite material has been widely utilized in the shaft structural design due to its various advantages such as lightweight, high strength, resistance to fatigue, and designability. Compared with metal shafts, composite shafts have outstanding advantages such as lighter, anti-vibration and high temperature resistant, which can reduce the friction loss, improve the fatigue life, and save running costs. Research on dynamic characteristics of anisotropic composite shaft is a focus of composite material structural dynamics. One of the studies of the laminated composite beams considering cross section warping is done by Librescu et al. [1–6]. In their works, some theories are proposed and applied to analyze the dynamic characteristics of uniform cross section, variable cross section, and containing intelligent material beams. The effects of transverse shear deformation, torsion warping, and

the wall thickness have been considered in their theory. However, the warping caused by stretch-bending and torsion-bending coupled deformations is not included. A dynamic model of variable cross section laminated composite shafts based on the theory of Timoshenko beam is presented by Kim et al. [7–10]. The proposed model is of high precision for thick wall shaft due to using the shear factor calculated by the ratio of inner and outer diameter. Another high precision model of thin long beams considering cross section warping caused by extension-bending and torsion-bending deformation is presented by Armanios et al. [11–13]. A displacement function of that beam is proposed based on variational asymptotic method and vibration differential equations are derived and solved by using exact method. However, the proposed function does not consider the effect of shear deformation. The further works are done by Yongsheng et al. by using a refined variational asymptotic method. The theory and model involve the free vibration and stability of conventional composite shaft [14] and the composite shaft with internal damping. In these studies, the effect of transverse shear deformation has been incorporated in theoretical formulation. In this paper, a new dynamic model for the analysis of rotating variable cross section composite shaft is presented. The equations of motion

FIGURE 1: Coordinate systems and geometry of thin-walled rotating shaft.

are derived using a refined variational asymptotic method and Hamilton's principle. The free vibration characteristics and stability of variable cross section composite shaft are analyzed. The numerical results are compared with the results obtained by other models.

2. Spinning Shaft Model

The rotating composite shaft model is shown in Figure 1. The shaft rotates about its axis with constant angular velocity (Ω). d , h , r_T , r_R , and L represent the outer diameter, wall thickness, the radius of tip middle cross section contour line, the radius of root middle cross section contour line, and the length of the shaft, respectively. (X, Y, Z) represents the inertial coordinate system, and the corresponding unit vector is $(\mathbf{I}, \mathbf{J}, \mathbf{K})$; (x, y, z) represents the coordinate systems spinning with shaft and the corresponding unit vector is $(\mathbf{i}, \mathbf{j}, \mathbf{k})$; (ξ, s, x) is a local coordinate system and s represents the axis along the tangent of the cross section middle contour line, while ξ represents the axis along the normal. θ is ply angle.

3. Governing Equations

3.1. Strain Energy of the Rotating Shaft. A displacement function based on variational asymptotic method is derived and utilized to analyze the dynamic characteristics of composite thin-walled closed section structures in [11]; however, transverse shear deformation is neglected in the displacement function. Considering transverse shear deformation, displacement function can be described as the following form [14]:

$$u_1(x, s, t) = U_1(x, t) - y(s)\theta_y(x, t) - z\theta_z(x, t) + g(s, x, t), \quad (1)$$

$$u_2(x, s, t) = U_2(x, t) - z\varphi(x, t),$$

$$u_3(x, s, t) = U_3(x, t) + y\varphi(x, t),$$

$$\theta_y(x, t) = U_2' - 2\gamma_{zx}, \quad (2)$$

$$\theta_z(x, t) = U_3' - 2\gamma_{yx},$$

where u_1, u_2, u_3 are displacements of an arbitrary point on the shaft along x, y, z directions, respectively. $U_1(x, t)$, $U_2(x, t)$, and $U_3(x, t)$ denote the cross mean displacement along x -, y -, and z -axis. $\varphi(x, t)$, $\theta_y(x, t)$, and $\theta_z(x, t)$ denote the twist angle and bending angle about y -, and z -axis, respectively. γ_{zx} and γ_{yx} are the transverse shear strains in planes xz and xy , respectively. $g(s, x, t)$ is a warping displacement function, which is modified as [15]

$$g(s, x, t) = G(s)\varphi'(x, t) + g_1(s)U_1'(x, t) + g_2(s)\theta_y'(x, t) + g_3(s)\theta_z'(x, t), \quad (3)$$

where $g_1(s)$, $g_2(s)$, $g_3(s)$, and $G(s)$ are related to physical behavior of the axial strain, bending curvatures, and the torsion twist rate, respectively. All of the $(\cdot)'$ denote the differentiations with respect to x .

According to (1)~(3), the strains of the composite shaft are obtained as

$$\begin{aligned} \gamma_{xx} &= U_1'(x, t) - y\theta_y'(x, t) - z\theta_z'(x, t), \\ 2\gamma_{xs} &= \frac{dg}{ds} + r_n\varphi' + (U_2'(x, t) - \theta_y(x, t))\frac{dy}{ds} \\ &\quad + (U_3'(x, t) - \theta_z(x, t))\frac{dz}{ds}, \\ 2\gamma_{x\xi} &= (U_2'(x, t) - \theta_y(x, t))\frac{dz}{ds} \\ &\quad - (U_3'(x, t) - \theta_z(x, t))\frac{dy}{ds}, \end{aligned} \quad (4)$$

where r_n is the normal projection of $\mathbf{r}$ which is the position vector of an arbitrary point on the cross section of the deformed shaft in the normal direction:

$$r_n = y\frac{dz}{ds} - z\frac{dy}{ds}. \quad (5)$$

The strain energy of the composite shaft U is

$$U = \frac{1}{2} \int_0^L \iint_A (\sigma_{xx}\epsilon_{xx} + \sigma_{xs}\epsilon_{xs} + \sigma_{x\xi}\epsilon_{x\xi}) dA dx, \quad (6)$$

where σ_{xx} , σ_{xs} , and $\sigma_{x\xi}$ are the engineering stresses. And ϵ_{xx} , ϵ_{xs} , and $\epsilon_{x\xi}$ are associated engineering strains.

Taking the variation of (6), one obtains

$$\begin{aligned} \delta U_s &= \int_0^L \int_A (\sigma_{xx}\delta\epsilon_{xx} + \sigma_{xs}\delta\epsilon_{xs} + \sigma_{x\xi}\delta\epsilon_{x\xi}) dA dx \\ &= \int_0^L [F_x'\delta U_1 + Q_y'\delta U_2 + Q_z'\delta U_3 + M_x'\delta\varphi + (M_z' + Q_y)\delta\theta_y + (M_y' + Q_z)\delta\theta_z] dx. \end{aligned} \quad (7)$$

Stress resultants F_x , Q_y , Q_z and stress couples M_x , M_y , M_z are as follows:

$$\begin{aligned} F_x &= \oint_{\Gamma} N_{xx} ds, \\ M_x &= \oint_{\Gamma} N_{xx} r_n ds, \\ M_y &= -\oint_{\Gamma} N_{xx} z ds, \\ M_z &= -\oint_{\Gamma} N_{xx} y ds, \\ Q_y &= -\oint_{\Gamma} \left(N_{xs} \frac{dy}{ds} + N_{x\xi} \frac{d\eta}{ds} \right) ds, \\ Q_z &= -\oint_{\Gamma} \left(N_{xs} \frac{dz}{ds} - N_{x\xi} \frac{d\eta}{ds} \right) ds, \end{aligned} \quad (8)$$

where N_{xx} , N_{xs} , and $N_{x\xi}$ are shell stress resultants and are of the following form:

$$\begin{bmatrix} N_{xx} \\ N_{xs} \\ N_{x\xi} \end{bmatrix} = \begin{bmatrix} A(s) & \frac{B(s)}{2} & 0 \\ \frac{B(s)}{2} & \frac{C(s)}{4} & 0 \\ 0 & 0 & D(s) \end{bmatrix} \begin{bmatrix} \gamma_{xx} \\ 2\gamma_{xs} \\ \gamma_{x\xi} \end{bmatrix}, \quad (9)$$

where

$$\begin{aligned} A(s) &= \bar{A}_{11} - \frac{\bar{A}_{12}^2}{\bar{A}_{22}}, \\ B(s) &= 2 \left(\bar{A}_{16} - \frac{\bar{A}_{12}\bar{A}_{26}}{\bar{A}_{22}} \right), \\ C(s) &= 4 \left(\bar{A}_{66} - \frac{\bar{A}_{26}^2}{\bar{A}_{22}} \right), \\ D(s) &= \kappa \left(\bar{A}_{44} - \frac{\bar{A}_{45}^2}{\bar{A}_{55}} \right), \\ \bar{A}_{ij} &= \sum_{k=1}^N \bar{Q}_{ij} (z_k - z_{k-1}), \end{aligned} \quad (10)$$

($i, j = 1, 2, 6$; $i, j = 4, 5$).

Parameters $A(s)$ and $B(s)$ denote the reduced axial and coupling stiffness, while parameters $C(s)$ and $D(s)$ are the reduced shear stiffness, $\bar{A}_{ij}$ are the in-plane stiffness components, $\bar{Q}_{ij}$ are transformed stiffness of each layer, N is layer numbers, and $(z_k - z_{k-1})$ is thickness of the k th layer. κ is shear factor of the cross section which changes with the cross section and material properties and is of the following form [16]:

$$\kappa = \frac{6E_{xx} (1 - \bar{m}^4) (1 + \bar{m}^2)}{G_{xy} \nu_{xy} (2\bar{m}^6 + 18\bar{m}^4 - 18\bar{m}^2 - 2) - E_{xx} (7\bar{m}^6 + 27\bar{m}^4 - 27\bar{m}^2 - 7)}, \quad (11)$$

where $\bar{m} = R_{in}/R_{out}$ and R_{in} is the inner radius, while R_{out} is the outer radius. E_{xx} , ν_{xy} , and G_{xy} are elastic modulus, Poisson's ratio, and shear modulus.

Combining (4), (9), and (10), (8) can be altered as

$$\begin{aligned} F_x &= C_{11}U_1' + C_{12}\phi' + C_{13}\theta_z' + C_{14}\theta_y' \\ &\quad + C_{15}(U_2' - \theta_y) + C_{16}(U_3' - \theta_z), \\ M_x &= C_{21}U_1' + C_{22}\phi' + C_{23}\theta_z' + C_{24}\theta_y' \\ &\quad + C_{25}(U_2' - \theta_y) + C_{26}(U_3' - \theta_z), \\ M_y &= C_{31}U_1' + C_{32}\phi' + C_{33}\theta_z' + C_{34}\theta_y' \\ &\quad + C_{35}(U_2' - \theta_y) + C_{36}(U_3' - \theta_z), \\ M_z &= C_{41}U_1' + C_{42}\phi' + C_{43}\theta_z' + C_{44}\theta_y' \\ &\quad + C_{45}(U_2' - \theta_y) + C_{46}(U_3' - \theta_z), \end{aligned}$$

$$\begin{aligned} Q_y &= C_{51}U_1' + C_{52}\phi' + C_{53}\theta_z' + C_{54}\theta_y' \\ &\quad + C_{55}(U_2' - \theta_y) + C_{56}(U_3' - \theta_z), \\ Q_z &= C_{61}U_1' + C_{62}\phi' + C_{63}\theta_z' + C_{64}\theta_y' \\ &\quad + C_{65}(U_2' - \theta_y) + C_{66}(U_3' - \theta_z), \end{aligned} \quad (12)$$

where $C_{ij} = C_{ji}$ ($i, j = 1, \dots, 6$) are the equivalent cross section stiffness coefficients of the composite shaft, which show the cross section geometry and material properties as

$$\begin{aligned} C_{11} &= \oint_{\Gamma} \left(A - \frac{B^2}{C} \right) ds + \left\{ \frac{[\oint_{\Gamma} (B/C) ds]^2}{\oint_{\Gamma} (1/C) ds} \right\}, \\ C_{12} &= \left[\frac{\oint_{\Gamma} (B/C) ds}{\oint_{\Gamma} (1/C) ds} \right] A_e, \\ C_{13} &= -\oint_{\Gamma} \left(A - \frac{B^2}{C} \right) z ds \end{aligned}$$

$$\begin{aligned}
& - \left\{ \frac{\oint_{\Gamma} (B/C) ds \oint_{\Gamma} (B/C) z ds}{\oint_{\Gamma} (1/C) ds} \right\}, \\
C_{14} &= - \oint_{\Gamma} \left(A - \frac{B^2}{C} \right) y ds \\
& - \left\{ \frac{\oint_{\Gamma} (B/C) ds \oint_{\Gamma} (B/C) y ds}{\oint_{\Gamma} (1/C) ds} \right\}, \\
C_{15} &= \frac{1}{2} \oint_{\Gamma} B \frac{dy}{ds} ds, \\
C_{16} &= \frac{1}{2} \oint_{\Gamma} B \frac{dz}{ds} ds, \\
C_{22} &= \left[\frac{1}{\oint_{\Gamma} (1/C) ds} \right] A_e^2, \\
C_{23} &= - \left[\frac{\oint_{\Gamma} (B/C) z ds}{\oint_{\Gamma} (1/C) ds} \right] A_e, \\
C_{24} &= - \left[\frac{\oint_{\Gamma} (B/C) y ds}{\oint_{\Gamma} (1/C) ds} \right] A_e, \\
C_{25} &= \frac{1}{4} \oint_{\Gamma} r_n C \frac{dy}{ds} ds, \\
C_{26} &= \frac{1}{4} \oint_{\Gamma} r_n C \frac{dz}{ds} ds, \\
C_{33} &= \oint_{\Gamma} \left(A - \frac{B^2}{C} \right) z^2 ds + \left\{ \frac{\left[\oint_{\Gamma} (B/C) z ds \right]^2}{\oint_{\Gamma} (1/C) ds} \right\}, \\
C_{34} &= \oint_{\Gamma} \left(A - \frac{B^2}{C} \right) yz ds \\
& + \left\{ \frac{\left[\oint_{\Gamma} (B/C) y ds \oint_{\Gamma} (B/C) z ds \right]}{\oint_{\Gamma} (1/C) ds} \right\}, \\
C_{35} &= - \frac{1}{2} \oint_{\Gamma} Bz \frac{dy}{ds} ds, \\
C_{36} &= - \frac{1}{2} \oint_{\Gamma} Bz \frac{dz}{ds} ds, \\
C_{44} &= \kappa \oint_{\Gamma} \left(A - \frac{B^2}{C} \right) y^2 ds + \left\{ \frac{\left[\oint_{\Gamma} (B/C) y ds \right]^2}{\oint_{\Gamma} (1/C) ds} \right\}, \\
C_{45} &= - \frac{1}{2} \kappa \oint_{\Gamma} By \frac{dy}{ds} ds, \\
C_{46} &= - \frac{1}{2} \kappa \oint_{\Gamma} By \frac{dz}{ds} ds, \\
C_{55} &= \kappa \oint_{\Gamma} \left[\frac{1}{4} C \left(\frac{dy}{ds} \right)^2 + D \left(\frac{dz}{ds} \right)^2 \right] ds,
\end{aligned}$$

$$\begin{aligned}
C_{56} &= \kappa \oint_{\Gamma} \left(\frac{1}{4} C - D \right) \frac{dy}{ds} \frac{dz}{ds} ds, \\
C_{66} &= \kappa \oint_{\Gamma} \left[\frac{1}{4} C \left(\frac{dz}{ds} \right)^2 + D \left(\frac{dy}{ds} \right)^2 \right] ds, \\
A_e &= \frac{1}{2} \oint_{\Gamma} \left(y \frac{dz}{ds} - z \frac{dy}{ds} \right) ds,
\end{aligned} \tag{13}$$

where $\oint_{\Gamma}(\cdot) ds$ denotes the integral around the loop of the midline cross section. A_e is the surrounded area of section's midline. Compared with [12], 16 of the 36 stiffness coefficients are consistent with the ones without shear deformation, while the remaining 20 are new, caused by shear deformation.

3.2. Kinetic Energy of Spinning Shaft. The position vector of an arbitrary point on the spinning shaft is written as

$$\mathbf{r} = (y + u_2) \mathbf{i} + (z + u_3) \mathbf{j} + (x + u_1) \mathbf{k}. \tag{14}$$

From the above equation, the velocity of an arbitrary point in the fixed coordinate system can be given as

$$\mathbf{V} = (\dot{u}_2 - \Omega(z + u_3)) \mathbf{i} + (\dot{u}_3 + \Omega(y + u_2)) \mathbf{j} + \dot{u}_1 \mathbf{k}. \tag{15}$$

The kinetic energy can be written as

$$T = \frac{1}{2} \int_0^L \iint_A \rho (\mathbf{V} \cdot \mathbf{V}) dA dx. \tag{16}$$

Substituting (15) into (16) and omitting the $(\cdot)'$ items due to the negligible effect of warping displacement, the expression of kinematic energy is obtained.

Taking variation of the kinematic energy yields

$$\begin{aligned}
\delta T &= - \int_0^L I_1 \delta U_1 + I_2 \delta U_2 + I_3 \delta U_3 + I_4 \delta \theta_y + I_5 \delta \theta_z \\
&+ I_6 \delta \varphi,
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
I_1 &= m_c \ddot{U}_1 - S_z \ddot{\theta}_y - S_y \ddot{\theta}_z, \\
I_2 &= m_c (\ddot{U}_2 - 2\Omega \dot{U}_3 - \Omega^2 U_2) - S_z (2\Omega \varphi + \Omega^2) \\
&\quad - S_y (\ddot{\varphi} - \Omega^2 \varphi), \\
I_3 &= m_c (\ddot{U}_3 + 2\Omega \dot{U}_2 - \Omega^2 U_3) + S_z (\ddot{\varphi} - \Omega^2 \varphi) \\
&\quad - S_y (2\Omega \varphi + \Omega^2), \\
I_4 &= S_z \ddot{U}_1 - I_z \ddot{\theta}_y - I_{yz} \ddot{\theta}_z,
\end{aligned}$$

$$\begin{aligned}
I_5 &= S_y \ddot{U}_1 - I_{yz} \ddot{\theta}_y - I_y \ddot{\theta}_z, \\
I_6 &= S_z (\ddot{U}_3 + 2\Omega \dot{U}_2 - \Omega^2 U_3) + (I_y + I_z) (\ddot{\phi} - \Omega^2 \phi) \\
&\quad - S_y (\ddot{U}_2 - 2\Omega \dot{U}_3 - \Omega^2 U_2), \\
m_c &= \iint_A \rho dA, \\
S_z &= \iint_A \rho y dA, \\
S_y &= \iint_A \rho z dA, \\
I_z &= \iint_A \rho y^2 dA, \\
I_y &= \iint_A \rho z^2 dA, \\
I_{yz} &= \iint_A \rho yz dA.
\end{aligned} \tag{18}$$

And I_i ($i = 1, \dots, 6$) is equivalent cross section mass coefficients.

3.3. Governing Equations. Hamilton principle is of the following form:

$$\int_{t_0}^{t_1} [\delta U - \delta T] dt = 0. \tag{19}$$

Combining (7), (17), and (19) and considering the variable cross section of the shaft, the governing equations are obtained as

$$\begin{aligned}
-F'_x + I_1 &= 0, \\
-Q'_y + I_2 &= 0, \\
-Q'_z + I_3 &= 0, \\
-M'_x + I_6 &= 0, \\
-M'_z - Q_y + I_4 &= 0, \\
-M'_y - Q_z + I_5 &= 0.
\end{aligned} \tag{20}$$

Substituting internal forces into (20), the free vibration equations are derived, which can be used to analyze the thin-walled shaft with arbitrary cross shape and variable cross section. The special properties of circular cross section shafts with circumferentially uniform stiffness configuration (CUS) [12] lead various components of stiffness coefficient to be zero. So the vibration equations can be simplified as

$$\begin{aligned}
L_1 (U_1, \phi) &= - (C_{11} U'_1)' - (C_{12} \phi')' + m_c \ddot{U}_1 \\
&= 0,
\end{aligned} \tag{21}$$

$$\begin{aligned}
L_2 (U_1, \phi) &= - (C_{12} U'_1)' - (C_{22} \phi')' \\
&\quad + (I_y + I_z) (\ddot{\phi} - \Omega^2 \phi) = 0,
\end{aligned} \tag{22}$$

$$\begin{aligned}
L_3 (U_2, U_3, \theta_y, \theta_z) &= - (C_{35} \theta'_y)' - [C_{55} (U'_2 - \theta_y)]' \\
&\quad + m_c (\ddot{U}_2 - 2\Omega \dot{U}_3 - \Omega^2 U_2) \\
&= 0,
\end{aligned} \tag{23}$$

$$\begin{aligned}
L_4 (U_2, U_3, \theta_y, \theta_z) &= - (C_{46} \theta'_y)' - [C_{66} (U'_3 - \theta_z)]' \\
&\quad + m_c (\ddot{U}_3 + 2\Omega \dot{U}_2 - \Omega^2 U_3) \\
&= 0,
\end{aligned} \tag{24}$$

$$\begin{aligned}
L_5 (U_2, U_3, \theta_y, \theta_z) &= - (C_{44} \theta'_y)' - [C_{46} (U'_3 - \theta_z)]' \\
&\quad - C_{35} \theta'_z - C_{55} (U'_2 - \theta_y) - I_z \ddot{\theta}_y \\
&= 0,
\end{aligned} \tag{25}$$

$$\begin{aligned}
L_6 (U_2, U_3, \theta_y, \theta_z) &= - (C_{33} \theta'_z)' - [C_{35} (U'_2 - \theta_y)]' \\
&\quad - C_{46} \theta'_y - C_{66} (U'_3 - \theta_z) - I_y \ddot{\theta}_z \\
&= 0.
\end{aligned} \tag{26}$$

Equations (23)~(26) are similar to (10a-d) in [6], and the vibration equations of circular cross section shafts with CUS configuration are decoupled as tensile-torsion coupling system ((21) and (22)) and bending-transverse shear coupling system ((23)~(26)). The latter one is emphasis in this paper. Neglecting the effect of transverse shear, the bending-transverse shear coupling system can be simplified due to $\theta_y = U'_2, \theta_z = U'_3$. In this case, the vibration equations become

$$\begin{aligned}
(C_{44} U''_2)'' + (I_z \ddot{U}'_2)' + m_c (\ddot{U}_2 - 2\Omega \dot{U}_3 - \Omega^2 U_2) &= 0, \\
(C_{33} U''_3)'' + (I_y \ddot{U}'_3)' + m_c (\ddot{U}_3 - 2\Omega \dot{U}_2 - \Omega^2 U_3) &= 0.
\end{aligned} \tag{27}$$

The displacement boundary conditions of free vibration cantilever shaft are as

$$U_2(0, t) = U_3(0, t) = \theta_y(0, t) = \theta_z(0, t) = 0. \tag{28}$$

The corresponding force boundary conditions for circular cross section shafts with circumferentially uniform stiffness configuration (CUS) are as

$$\begin{aligned}
Q_y(L) &= 0 \\
\Rightarrow B_3 (U_2, U_3, \theta_y, \theta_z) \\
&= C_{35}(L) \theta'_z(L) + C_{55}(L) [U'_2(L) - \theta_y(L)] \\
&= 0,
\end{aligned} \tag{29}$$

$$\begin{aligned}
Q_z(L) &= 0 \\
&\Rightarrow B_4(U_2, U_3, \theta_y, \theta_z) \\
&= C_{46}(L) \theta'_y(L) + C_{66}(L) [U'_3(L) - \theta_z(L)] \\
&= 0,
\end{aligned} \tag{30}$$

$$\begin{aligned}
M_z(L) &= 0 \\
&\Rightarrow B_5(U_2, U_3, \theta_y, \theta_z) \\
&= C_{44}(L) \theta'_y(L) + C_{46}(L) [U'_3(L) - \theta_z(L)] \\
&= 0,
\end{aligned} \tag{31}$$

$$\begin{aligned}
M_y(L) &= 0 \\
&\Rightarrow B_6(U_2, U_3, \theta_y, \theta_z) \\
&= C_{33}(L) \theta'_z(L) + C_{35}(L) [U'_2(L) - \theta_y(L)] \\
&= 0.
\end{aligned} \tag{32}$$

3.4. Approximate Solution. Because the classical vibration mode function of cantilever shaft satisfies the displacement boundary condition, it does not satisfy the force condition, so General Galerkin method is used to solve vibration and stability of continuous system. Considering the force boundary conditions, and eliminating the shaft's spatial variables by using General Galerkin method based on the assuming mode shape, the free vibration equations are reduced to be ordinary differential equations with respect to time.

Assume that the axial, torsion, and bending deformation is of the following form:

$$\begin{aligned}
U_2 &= \sum_{j=1}^N U_{1j}(t) \psi_j(x), \\
U_3 &= \sum_{j=1}^N U_{3j}(t) \psi_j(x), \\
\theta_y &= \sum_{j=1}^N \theta_{yj}(t) \varphi_j(x), \\
\theta_z &= \sum_{j=1}^N \theta_{zj}(t) \varphi_j(x),
\end{aligned} \tag{33}$$

where $\psi_j(x)$ and $\varphi_j(x)$ denote the mode shape of bending deflection and bending angle, respectively, and are taken to be the mode shapes of a uniform, nonrotating, isotropic, fixed-free Euler-Bernoulli beam; namely,

$$\begin{aligned}
\varphi_j &= \sin \frac{(2j-1)\pi}{2L} x, \\
\cos(\beta_j) \cosh(\beta_j) &= -1,
\end{aligned}$$

$$\lambda_j = -\frac{\cos \beta_j + \cosh \beta_j}{\sin \beta_j + \sinh \beta_j}, \quad j = 1, 2, \dots, N,$$

$$\begin{aligned}
\psi_j &= \cos\left(\frac{\beta_j x}{L}\right) - \cosh\left(\frac{\beta_j x}{L}\right) \\
&\quad + \lambda_j \left(\sin\left(\frac{\beta_j x}{L}\right) - \sinh\left(\frac{\beta_j x}{L}\right) \right).
\end{aligned} \tag{34}$$

Substituting (33) into (23), we get

$$\begin{aligned}
L_3 \left(\sum_{j=1}^N U_{2j}(t) \psi_j(x), \sum_{j=1}^N U_{3j}(t) \psi_j(x), \right. \\
\left. \sum_{j=1}^N \theta_{yj}(t) \varphi_j(x), \sum_{j=1}^N \theta_{zj}(t) \varphi_j(x) \right) = 0.
\end{aligned} \tag{35}$$

Multiply (35) by the mode shape, and integrate it on the whole length of the shaft. The differential equation residual is

$$R_{1m} = \int_0^L L_3 \psi_i dx. \tag{36}$$

Similar to (35), (29) is multiplied by $\psi(L)$, and the boundary residual is

$$\begin{aligned}
R_{1Bm} &= B_3 \left(\sum_{j=1}^N U_{2j}(t) \psi_j(L), \sum_{j=1}^N U_{3j}(t) \psi_j(L), \right. \\
&\quad \left. \sum_{j=1}^N \theta_{yj}(t) \varphi_j(L), \sum_{j=1}^N \theta_{zj}(t) \varphi_j(L) \right) \psi_i(L).
\end{aligned} \tag{37}$$

Based on General Galerkin method, it follows that

$$R_{1m} + R_{1Bm} = 0, \quad m = 1, \dots, N. \tag{38}$$

Similarly, the remaining equations give

$$\begin{aligned}
R_{2m} + R_{2Bm} &= 0, \\
R_{3m} + R_{3Bm} &= 0, \\
R_{4m} + R_{4Bm} &= 0,
\end{aligned} \tag{39}$$

$$m = 1, \dots, N,$$

where

$$R_{2m} = \int_0^L L_4 \psi_i dx,$$

$$R_{3m} = \int_0^L L_5 \varphi_i dx,$$

$$R_{4m} = \int_0^L L_6 \varphi_i dx,$$

$$R_{2Bm} = B_4 \left(\sum_{j=1}^N U_{2j}(t) \psi_j(L), \sum_{j=1}^N U_{3j}(t) \psi_j(L), \right.$$

$$\left. \sum_{j=1}^N \theta_{yj}(t) \varphi_j(L), \sum_{j=1}^N \theta_{zj}(t) \varphi_j(L) \right) \psi_i(L),$$

$$R_{3Bm} = B_5 \left(\sum_{j=1}^N U_{2j}(t) \psi_j(L), \sum_{j=1}^N U_{3j}(t) \psi_j(L), \right.$$

$$\left. \sum_{j=1}^N \theta_{yj}(t) \varphi_j(L), \sum_{j=1}^N \theta_{zj}(t) \varphi_j(L) \right) \varphi_i(L),$$

$$R_{4Bm} = B_6 \left(\sum_{j=1}^N U_{2j}(t) \psi_j(L), \sum_{j=1}^N U_{3j}(t) \psi_j(L), \right. \\ \left. \sum_{j=1}^N \theta_{yj}(t) \varphi_j(L), \sum_{j=1}^N \theta_{zj}(t) \varphi_j(L) \right) \varphi_i(L). \quad (40)$$

The final end product of the procedure is a set of equations and is of the form

$$\mathbf{M} \begin{pmatrix} \ddot{U}_{2j} \\ \ddot{U}_{3j} \\ \ddot{\theta}_{yj} \\ \ddot{\theta}_{zj} \end{pmatrix} + \mathbf{C} \begin{pmatrix} \dot{U}_{2j} \\ \dot{U}_{3j} \\ \dot{\theta}_{yj} \\ \dot{\theta}_{zj} \end{pmatrix} + \mathbf{K} \begin{pmatrix} U_{2j} \\ U_{3j} \\ \theta_{yj} \\ \theta_{zj} \end{pmatrix} = 0, \quad (41)$$

where

$$\mathbf{M} = \begin{bmatrix} m_c \psi_i \psi_j & 0 & 0 & 0 \\ 0 & m_c \psi_i \psi_j & 0 & 0 \\ 0 & 0 & -I_z \varphi_i \varphi_j & 0 \\ 0 & 0 & 0 & -I_z \varphi_i \varphi_j \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 0 & -2\Omega m_c \psi_i \psi_j & 0 & 0 \\ -2\Omega m_c \psi_i \psi_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} -C_{55} \psi_j' \psi_i' - m_c \Omega^2 \psi_j \psi_i & 0 & C_{55} \varphi_j \psi_i' & -C_{35} \varphi_j' \psi_i' \\ 0 & -C_{66} \psi_j' \psi_i' - m_c \Omega^2 \psi_j \psi_i & -C_{46} \varphi_j' \psi_i' & C_{66} \psi_j \psi_i' \\ -C_{55} \psi_j' \varphi_i & -C_{46} \psi_j' \varphi_i' & -C_{44} \varphi_j' \varphi_i' + C_{55} \varphi_j \varphi_i & C_{46} \varphi_j \varphi_i' - C_{35} \varphi_j' \varphi_i \\ -C_{35} \psi_j' \varphi_i' & -C_{66} \psi_j' \varphi_i & C_{35} \varphi_j \varphi_i' - C_{46} \varphi_j' \varphi_i & -C_{33} \varphi_j' \varphi_i' + C_{66} \varphi_j \varphi_i \end{bmatrix}. \quad (42)$$

The free vibration characteristics of the variable cross section shaft can be obtained by solving (41) with MATLAB.

4. Model Validation and Discussion and Free Vibration Analysis

With the application of the vibration differential equations derived in this paper, the variable cross section shaft presented in [10] is analyzed in our work, and the numerical results are compared with those obtained by [10] and finite element method, as presented in Figure 2. It can be seen that the variation trends of the first natural frequencies with the taper ratio (taper is defined in [10]) obtained by the three methods are consistent. Our model is developed based on

variational asymptotic method [11–14] which is exact only for thin-walled and slender shafts, and yet it appears that Kim's model [10] is available for both thin-walled and thick-walled shafts. So Figure 2 shows obvious difference between them using the geometries for thick walled shafts in [10]. As a result, the shaft in Table 1 is analyzed by the present model and compared with Kim's model.

Figures 3-4 and Tables 2-3 illustrate the first natural frequencies of a rotating and a stationary shaft considering or nonconsidering the transverse shear deformation. It is easy to find that the transverse shear deformation has only a little effect when ply angle is in the range of $15^\circ \sim 40^\circ$. The largest effect occurs at the $\theta = 0^\circ$ and $TR = 2$. At the same time, Figures 3 and 4 reveal that shear deformation has same effects on the rotating and stationary shafts.

FIGURE 2: Changes comparison with TR ($\Omega = 0$ rad/s). FIGURE 3: The influence of transverse shear deformation ($\Omega = 0$ rad/s).

Tables 4 and 5 list the effect of mode number on the first natural frequencies and also show the effects of ply method. When the ply method is $[\theta/-\theta]_3$, the present model and Kim's model both show good convergence as shown in Table 4. The convergence of Kim's model is lost when the ply method is $[\theta]_6$, while the present model still shows good convergence.

Figure 6 shows the effect of ply angle on different taper ratio shafts with 0 rotation speed, and it also shows the difference among ANSYS, present model, and Kim's model. The first natural frequency is reduced obviously with the increase of ply angle, while it increased with the increase

FIGURE 4: The influence of transverse shear deformation ($\Omega = 100$ rad/s).

TABLE 1: Properties of the analyzed shaft.

$r_R + r_T = 0.02$ m	$[\theta/-\theta]_3$ Layup
Length (L) = 0.5 m	Ply thickness = 0.15 mm
$\nu_{12} = \nu_{13} = 0.24$	$\nu_{23} = 0.2067$
$E_{11} = 192$ GPa	$E_{22} = E_{33} = 7.24$ GPa
$G_{12} = G_{13} = 4.07$ GPa	$G_{23} = 3$ GPa

TABLE 2: Shear stress effects on the first natural frequency ($\Omega = 0$ rad/s).

Taper ratio	Ply angle (°)	With shear (Hz)	Without shear (Hz)	Change rate (%)
TR = 0	0°	158.76	165	3.8
	10°	154.57	157.56	1.9
	20°	125.85	126.12	0.2
	60°	36.20	34.95	-3.6
TR = 1	0°	209.63	221.98	5.6
	10°	205.72	211.99	3.0
	20°	168.47	169.68	0.71
	60°	48.1	47.02	-2.3
TR = 2	0°	269.77	290.42	7.11
	10°	267.90	277.34	3.4
	20°	222.80	222.0	-0.36
	60°	64.33	61.52	-4.57

of taper ratio. And the three kinds of results are completely consistent with each other.

Figures 7–9 show the effect of rotating speed on different taper ratio and ply angle shafts. And they also reveal the

FIGURE 5: The first natural frequency versus ply angle ($\Omega = 0$ rad/s). FIGURE 6: Changes comparison with rotating speed ($\theta = 0^\circ$).

difference among the three models. When the rotating speed is more than 0, two natural frequencies occur for each mode. The higher value is called the forward precession mode, and the lower one represents a backward precession mode. So 1B and 1F denote the first backward precession frequency and first forward precession, respectively, in all of the figures. The

FIGURE 7: Changes comparison with rotating speed ($\theta = 20^\circ$). FIGURE 8: Changes comparison with rotating speed ($\theta = 60^\circ$).

FIGURE 9: The first natural frequency versus ply angle ($\Omega = 100$ rad/s).

TABLE 3: Shear stress effects on the first natural frequency ($\Omega = 100$ rad/s).

Taper ratio	Ply angle (°)	With shear (Hz)	Without shear (Hz)	Change rate (%)
TR = 0	0°	142.85	149.09	4.2
	10°	138.67	141.67	2.1
	20°	109.95	110.22	0.2
	60°	20.29	19.05	-6.5
TR = 1	0°	193.72	206.08	6
	10°	189.80	196.09	3.2
	20°	152.55	153.78	0.8
	60°	32.20	31.12	-3.4
TR = 2	0°	253.87	274.52	7.5
	10°	255.03	271.74	6.1
	20°	206.88	206.10	-0.37
	60°	48.42	45.62	-6.13

same effects of rotating speed to the first natural frequency can be obtained. The first backward precession frequency (1B) is decreasing with increasing rotating speed for different taper ratio and different ply angle shafts. The first forward precession frequency (1F) is exactly the reverse. In Figures 7–9, the results obtained by the present model are close to the ones obtained by ANSYS software. However, the results obtained by Kim et al. [10] are different from that of ANSYS software obviously; the reason may be that the effect of spinning inertia force has not been considered by Kim et al. [10].

FIGURE 10: The first natural frequency versus taper ratio.

FIGURE 11: Campbell diagrams of the shaft with $\theta = 0^\circ$.

Figure 10 shows the effect of ply angle on 1B with 100 rad/s rotating speed and also displays the difference among the three methods. The effect of ply angle is similar to Figure 5.

Figure 11 shows the effect of taper ratio on natural frequencies of different taper ratio and ply angle rotating shaft or stationary shaft. With the increase of taper ratio, all of the natural frequencies of the shafts are increased. And the lower value ply angle is more obviously affected than high ones.

TABLE 4: Effect of mode number N on the first natural frequencies ($[\theta]_3$, $\Omega = 0$ rad/s).

N		TR = 0			TR = 1			TR = 2		
		0°	20°	60°	0°	20°	60°	0°	20°	60°
2	Kim	159.74	125.07	35.34	213.77	168.12	47.31	279.68	222.62	63.31
	Present	158.76	125.85	36.29	209.63	168.47	48.10	269.77	222.80	64.33
3	Kim	159.48	124.28	34.93	213.63	167.30	46.92	278.54	219.02	61.54
	Present	158.50	125.45	35.14	209.38	168.04	47.09	268.32	219.14	61.97
4	Kim	159.29	124.12	34.86	213.44	167.10	46.87	278.39	218.79	61.36
	Present	158.29	125.29	35.02	209.07	167.99	46.99	267.99	219.02	61.59
5	Kim	159.25	124.06	34.82	213.43	167.07	46.85	278.37	218.55	61.27
	Present	158.24	125.25	34.96	209.03	167.97	46.99	267.86	218.89	61.46

TABLE 5: Effect of mode number N on the first natural frequencies ($[\theta]_6$, $\Omega = 0$ rad/s).

N		TR = 0			TR = 1			TR = 2		
		0°	20°	60°	0°	20°	60°	0°	20°	60°
2	Kim	159.74	816.93	36.66	213.77	127.38	52.46	279.68	220.04	70.05
	Present	158.76	92.04	34.54	209.63	122.05	46.22	269.77	161.33	61.86
3	Kim	159.48	295.27	141.32	213.63	366.79	174.64	278.54	448.68	32.20
	Present	158.50	92.71	34.25	209.38	124.39	45.94	268.32	162.97	60.22
4	Kim	159.29	799.21	49.16	213.44	184.04	70.50	278.39	142.48	94.92
	Present	158.29	92.92	34.16	209.07	124.80	45.90	267.99	163.08	60.06
5	Kim	159.25	219.81	216.16	213.43	371.12	17.24	278.37	453.50	39.785
	Present	158.24	93.35	34.15	209.03	125.54	45.90	267.86	164.05	60.01

A conclusion that lower ply angle and large taper ratio could increase the first natural frequency could be made from Figures 6–11. This conclusion is important for rotating shafts. If rotating speed of the shafts is under the first natural frequency, the shaft can keep dynamic stability. The critical rotating speed could be increased with the increase of first natural frequency, which could improve the transmission efficiency in the fields of car, ship, and plane. If the milling bar is modeled as a rotating cantilever shaft, the machining efficiency could be improved as well.

In a word, the analytical model presented in this work is verified and validated to a decisive extent in Figures 6–10.

5. Stability Analysis

Figures 11–13 show the Campbell diagrams of the shafts with different ply angle and different taper ratio. They illustrate that rotating shaft with 0° ply angle and 2 taper ratio is of maximal rotating critical speed for the shafts. Figures 11–13

FIGURE 12: Campbell diagrams of the shaft with $\theta = 20^\circ$.

FIGURE 13: Campbell diagrams of the shaft with $\theta = 60^\circ$.

show that the critical rotating speeds increase with taper ratio and decrease with ply angle.

All the data and figures illustrate that the stability of rotating shaft can be dramatically improved by utilizing variable cross section and appropriate ply angle.

6. Conclusions

A refined variational asymptotic method and Hamilton's principle are used to establish the vibration differential equation of rotating variable cross section composite shaft which is solved by applying Extended Galerkin method. And the effects of ply angle, taper ratio, and spinning speed on vibration characteristics are analyzed. Critical rotating speeds of the shafts with various ply angle and taper ratio are also calculated. The results illustrate that small ply angle and big taper ratio can improve the natural frequencies and critical rotating speeds effectively. The numerical results are compared with the results obtained by other models.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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