Research Article

A New Approach for Studying Nonlinear Dynamic Response of a Thin Plate with Internal Resonance in a Fractional Viscoelastic Medium

Yury A. Rossikhin, Marina V. Shitikova, and Jean Claude Ngenzi

Research Center for Wave Dynamics of Solids and Structures, Voronezh State University of Architecture and Civil Engineering, 20-Letija Oktjabrja Street 84, Voronezh 394006, Russia

Correspondence should be addressed to Marina V. Shitikova; shitikova@vmail.ru

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In the previous analysis, the dynamic behaviour of a nonlinear plate embedded into a fractional derivative viscoelastic medium has been studied by the method of multiple time scales under the conditions of the internal resonances two-to-one and one-to-one, as well as the internal combinational resonances for the case when the linear parts of nonlinear equations of motion occur to be coupled. A new approach proposed in this paper allows one to uncouple the linear parts of equations of motion of the plate, while the same method, the method of multiple time scales, has been utilized for solving nonlinear equations. The influence of viscosity on the energy exchange mechanism between interacting nonlinear modes has been analyzed. It has been shown that for some internal resonances there exist such particular cases when it is possible to obtain two first integrals, namely, the energy integral and the stream function, which allows one to reduce the problem to the calculation of elliptic integrals. The new approach enables one to solve the problems of vibrations of thin bodies more efficiently.

1. Introduction

It is well known that the nonlinear vibrations of plates are an important area of applied mechanics, since plates are used as structural elements in many fields of industry and technology. Different methods such as analytical [1–4], numerical [5, 6], and experimental methods [5, 7] could be employed to investigate the nonlinear vibrations of plates. Extensive review of recent research developments in the field could be found in Amabili [8, 9] and Sathyamoorthy [10].

The study of free undamped [6, 9], as well as damped [2, 3, 11–13], nonlinear systems is important because the dynamics characteristics of the system—defined by the amplitude-frequency relations and modes of vibrations—are determined [6]. Moreover, nonlinear vibrations could be accompanied by such a phenomenon as the internal resonance, resulting in multimode response with a strong interaction of the modes involved [14] accompanied by the energy exchange phenomenon. The internal resonance could be found within

some combination of natural frequencies of one and the same type of vibrations. As an example, a 1:3 internal resonance was discovered in [6], when the fourth natural frequency of the out-of-plane vibrations was three times larger than the fundamental frequency of the out-of-plane vibrations of the plate. A 1:1 internal resonance was investigated in [11, 15] applying a two-mode Galerkin procedure for the plate deflection in their analysis.

Another type of the internal resonance was investigated in [2, 3], when one frequency of the in-plane vibrations was equal to (a 1:1 internal resonance) or twice as large as (a 1:2 internal resonance) some frequency of the out-of-plane vibrations. The combinational resonances of the additive and difference type were also investigated in [3]. These types of the internal resonance result in the energy exchange between two or three subsystems. Investigations in this field were initiated by Witt and Gorelik [16], who were among the first to show theoretically and experimentally the two-to-one internal resonance with the energy exchange from

one subsystem to another using the simplest two-degree-of-freedom mechanical system as an example, and the interest of researchers in the problems of the internal resonance in mechanical systems does not relax.

In the given paper, nonlinear free vibrations of a thin plate described by three coupled nonlinear differential equations are considered when the plate is being under the conditions of the internal resonance resulting in the interaction of modes corresponding to the mutually orthogonal displacements. The displacement functions are determined in terms of eigenfunctions of linear vibrations. The procedure resulting in decoupling linear parts of equations is proposed with further utilization of the method of multiple scales for solving nonlinear governing equations of motion; in so doing the amplitude functions are expanded into power series in terms of the small parameter and depend on different time scales. It is shown that the phenomenon of internal resonance can be very critical, since in a thin plate internal resonance of the types two-to-one and one-to-one as well as additive and difference combinational resonances are always present.

To describe the nonlinear damped vibrations of the thin plate under consideration, the fractional derivative Kelvin-Voigt model is used, since this model has an advantage over the conventional Kelvin-Voigt model, because its prediction is in a good compliance with experimental data. Thus, for example, the experimental data obtained by Abdel-Ghaffar and Housner [17] and Abdel-Ghaffar and Scanlan [18] during ambient vibration studies of the Vincent-Thomas Suspension Bridge and the Golden Gate Bridge, respectively, show that different vibrational modes feature different amplitude damping factors, and the order of smallness of these coefficients highlights the low damping capacity of suspension combined systems, resulting in prolonged energy transfer from one partial subsystem to another. Besides, as natural frequencies of vibrations increase, the corresponding damping ratios decrease. To lead the theoretical investigations in line with the experiment, fractional derivatives were introduced by Rossikhin and Shitikova [19] for describing the processes of internal friction proceeding in suspension combined systems at nonlinear free vibrations, which allowed the authors to obtain the damping coefficients dependent on the natural frequency of vibrations. Moreover, the good agreement between the theoretical results and the experimental data has been found through the appropriate choice of the fractional parameter (the order of the fractional derivative) and the viscosity coefficient [20].

2. Problem Formulation and the Method of Solution

Let us consider the dynamic behavior of a free supported nonlinear thin rectangular plate (Figure 1), vibrations of which in a viscoelastic medium are described in the Cartesian system of coordinates by the following three differential equations written in the dimensionless form [2, 3, 21]:

$$u_{xx} + \frac{1-\nu}{2}\beta_1^2 u_{yy} + \frac{1+\nu}{2}\beta_1 v_{xy} + w_x \left(w_{xx} + \frac{1-\nu}{2}\beta_1^2 w_{yy} \right)$$

$$+ \frac{1+\nu}{2}\beta_1^2 w_y w_{xy} = \ddot{u} + \alpha_1 D^\nu u, \quad (1)$$

$$\beta_1^2 v_{yy} + \frac{1-\nu}{2}v_{xx} + \frac{1+\nu}{2}\beta_1 u_{xy} + \beta_1 w_y \left(\beta_1^2 w_{yy} + \frac{1-\nu}{2}w_{xx} \right) + \frac{1+\nu}{2}\beta_1 w_x w_{xy} = \ddot{v} + \alpha_2 D^\nu v, \quad (2)$$

$$\begin{aligned} & \frac{\beta_2^2}{12} \left(w_{xxxx} + 2\beta_1^2 w_{xxyy} + \beta_1^4 w_{yyyy} \right) - w_{xx} \left(u_x + \nu\beta_1 v_y \right) \\ & - w_x \left(u_{xx} + \nu\beta_1 v_{xy} \right) \\ & - \frac{1-\nu}{2}\beta_1 \left[w_{xy} \left(\beta_1 u_y + v_x \right) + w_y \left(\beta_1 u_{xy} + v_{xx} \right) \right] \\ & - \beta_1^2 \left[w_{yy} \left(\nu u_x + \beta_1 v_y \right) + w_y \left(\nu u_{xy} + \beta_1 v_{yy} \right) \right] \\ & - \frac{1-\nu}{2}\beta_1 \left[w_{xy} \left(\beta_1 u_y + v_x \right) + w_x \left(\beta_1 u_{yy} + v_{xy} \right) \right] \\ & = -\ddot{w} - \alpha_3 D^\nu w, \end{aligned} \quad (3)$$

subjected to the initial

$$u|_{t=0} = v|_{t=0} = w|_{t=0} = 0, \quad (4)$$

$$\dot{u}|_{t=0} = \varepsilon U^0(x, y), \quad \dot{v}|_{t=0} = \varepsilon V^0(x, y), \quad (5)$$

$$\dot{w}|_{t=0} = \varepsilon W^0(x, y),$$

as well as the boundary conditions

$$\begin{aligned} u|_{x=0} = u|_{x=1} = 0, \quad v|_{x=0} = v|_{x=1} = 0, \\ u_x|_{x=0} = u_x|_{x=1} = 0, \quad w_{xx}|_{x=0} = w_{xx}|_{x=1} = 0, \\ w|_{y=0} = w|_{y=1} = 0, \quad u|_{y=0} = u|_{y=1} = 0, \\ v_y|_{y=0} = v_y|_{y=1} = 0, \quad w_{yy}|_{y=0} = w_{yy}|_{y=1} = 0, \end{aligned} \quad (6)$$

where $u = u(x, y, t)$, $v = v(x, y, t)$, and $w = w(x, y, t)$ are the displacements of points located in the plate's middle surface in the x -, y -, and z -directions, respectively; ν is Poisson's ratio; $\beta_1 = a/b$ and $\beta_2 = h/a$ are the parameters defining the dimensions of the plate; a and b are the plate's dimensions along the x - and y -axes, respectively; h is the thickness; t is the time; an overdot denotes the time-derivative; lower indices label the derivatives with respect to the corresponding coordinates; $U^0(x, y)$, $V^0(x, y)$, and $W^0(x, y)$ are functions describing the distribution of the initial velocities of the points lying in the middle surface of the plate; ε is a small value.

In (1)–(6), the dimensionless values are introduced similarly as it has been done in [2]:

$$\begin{aligned} u &= \frac{u^*}{a}, \quad v = \frac{v^*}{a}, \quad w = \frac{w^*}{a}, \\ x &= \frac{x^*}{a}, \quad y = \frac{y^*}{b}, \quad t = \frac{t^*}{a} \sqrt{\frac{E}{\rho(1-\nu^2)}}, \end{aligned} \quad (7)$$

FIGURE 1: Scheme of a freely supported rectangular plate.

where dimensional values are marked by asterisks, and E is the Young modulus of the plate material.

In (1)–(3), the terms $\mathfrak{a}_1 D^\gamma u$, $\mathfrak{a}_2 D^\gamma v$, and $\mathfrak{a}_3 D^\gamma w$ represent the viscous resistance forces initiated when the plate vibrates in the surrounding viscous medium, where $\mathfrak{a}_i$ ($i = 1, 2, 3$) are damping coefficients and D^γ is the Riemann-Liouville fractional derivative of the γ -order [22]:

$$D^\gamma F = \frac{d}{dt} \int_0^t \frac{F(t-t')}{\Gamma(1-\gamma) t'^\gamma} dt'. \quad (8)$$

But distinct to the traditional modeling the viscous resistance forces via first-order time-derivatives [23], in the present research we adopt the fractional order time-derivatives D^γ , which, as it will be shown below, will allow us to obtain the damping coefficients dependent on the natural frequency of vibrations. It has been demonstrated in [19, 20] that such an approach for modeling the damped nonlinear vibrations of thin bodies provides the good agreement between the theoretical results and the experimental data through the appropriate choice of the fractional parameter (the order of the fractional derivative) and the viscosity coefficient.

It has been noted in [2, 19] that a fractional derivative is the immediate extension of an ordinary derivative. In fact, when $\delta \rightarrow 1$, $D^\gamma x$ tends to $\dot{x}$; that is, at $\gamma \rightarrow 1$ the fractional derivative goes over into the ordinary derivative, and the mathematical model of the viscoelastic plate under consideration transforms into the Kelvin-Voigt model, wherein the elastic element behaves nonlinearly, but the viscous element behaves linearly. When $\gamma \rightarrow 0$, the fractional derivative $D^\gamma x$ tends to $x(t)$. To put it otherwise, the introduction of the new fractional parameter along with the parameters $\mathfrak{a}_i$ allows one to change not only the magnitude of viscosity at the cost of an increase or decrease in the parameters $\mathfrak{a}_i$, but also the character of viscosity at the sacrifice of variations in the fractional parameter.

Reference to (5) shows that free vibrations are excited by weak disturbance from the equilibrium position.

We seek the solution of (1)–(3) in the form

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{1mn}(t) \eta_{1mn}(x, y),$$

$$v(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{2mn}(t) \eta_{2mn}(x, y),$$

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{3mn}(t) \eta_{3mn}(x, y), \quad (9)$$

where $x_{1mn}(t)$, $x_{2mn}(t)$, and $x_{3mn}(t)$ are the generalized displacements corresponding to the displacements in the plane of the plate and to its deflection, respectively, but the natural functions satisfying the boundary conditions (6) have the form

$$\begin{aligned} \eta_{1mn}(x, y) &= \cos \pi m x \sin \pi n y, \\ \eta_{2mn}(x, y) &= \sin \pi m x \cos \pi n y, \\ \eta_{3mn}(x, y) &= \sin \pi m x \sin \pi n y, \end{aligned} \quad (10)$$

and m and n are integers.

Linear undamped natural modes of flexural in-plane and out-of-plane vibrations of the plate are the solution of eigenvalue problem

$$\begin{aligned} -\ddot{x}_{1mn} \eta_{1mn} + x_{1mn} \left(\eta_{1mn,xx} + \frac{1-\gamma}{2} \beta_1^2 \eta_{1mn,yy} \right) \\ + \frac{1+\gamma}{2} \beta_1 \eta_{2mn,xy} x_{2mn} = 0, \end{aligned} \quad (11)$$

$$\begin{aligned} -\ddot{x}_{2mn} \eta_{2mn} + x_{2mn} \left(\beta_1^2 \eta_{2mn,yy} + \frac{1-\gamma}{2} \eta_{2mn,xx} \right) \\ + \frac{1+\gamma}{2} \beta_1 \eta_{1mn,xy} x_{1mn} = 0, \end{aligned} \quad (12)$$

$$\begin{aligned} \ddot{x}_{3mn} \eta_{3mn} + \frac{\beta_2^2}{12} \left(\eta_{3mn,xxxx} + 2\beta_1^2 \eta_{3mn,xyxy} \right. \\ \left. + \beta_1^4 \eta_{3mn,yyyy} \right) x_{3mn} = 0. \end{aligned} \quad (13)$$

The set of equations (11) and (12) has the characteristic equation

$$\omega_{mn}^4 - (S_{11}^{mn} + S_{22}^{mn}) \omega_{mn}^2 + S_{11}^{mn} S_{22}^{mn} - S_{12}^{mn} S_{21}^{mn} = 0, \quad (14)$$

the roots of which are the dimensionless natural frequencies of the flexural in-plane vibrations of the plate

$$\omega_{1mn}^2 = \pi^2 (m^2 + \beta_1^2 n^2), \quad \omega_{2mn}^2 = \frac{1-\gamma}{2} \omega_{1mn}^2, \quad (15)$$

where

$$\begin{aligned} S_{11}^{mn} &= \pi^2 \left(m^2 + \frac{1-\gamma}{2} \beta_1^2 n^2 \right), \\ S_{12}^{mn} &= S_{21}^{mn} = \frac{1+\gamma}{2} \beta_1 \pi^2 m n, \\ S_{22}^{mn} &= \pi^2 \left(\frac{1-\gamma}{2} m^2 + \beta_1^2 n^2 \right). \end{aligned} \quad (16)$$

From (13), the natural frequency of the flexural out-of-plane vibrations of the plate could be obtained:

$$\omega_{3mn}^2 = \frac{\pi^4 \beta_2^2}{12} (m^2 + \beta_1^2 n^2)^2. \quad (17)$$

Substituting (9) into (1)–(3), multiplying (1), (2), and (3) by η_{1lk} , η_{2lk} , and η_{3lk} , respectively, integrating over x and y , and using the orthogonality conditions for linear modes within the domains of $0 \leq x, y \leq 1$, we are led to an infinite number of sets each involving a system of three coupled nonlinear ordinary differential equations of the second order in x_{imn} ($i = 1, 2, 3$):

$$\ddot{x}_{\alpha mn} + \mathfrak{a}_\alpha D^\gamma x_{\alpha mn} + S_{\alpha\beta}^{mn} x_{\beta mn} = -F_{\alpha mn} \quad (\alpha, \beta = 1, 2), \quad (18)$$

$$\ddot{x}_{3mn} + \mathfrak{a}_3 D^\gamma x_{3mn} + \omega_{3mn}^2 x_{3mn} = -F_{3mn}, \quad (19)$$

where the summation is carried out over two repeated indices and the elements of the matrix S_{ij}^{mn} are defined as (16).

The linear parts of nonlinear equations (18) and (19) have been studied in time and frequency domains in detail in [24] using the Laplace integral transform method as a method of solution.

The nonlinear parts F_{imn} of (18) and (19) have the form

$$\begin{aligned} F_{1mn} &= 4 \sum_{m_1} \sum_{n_1} \sum_{m_2} \sum_{n_2} x_{3m_1 n_1} x_{3m_2 n_2} A_{1mn}^{m_1 n_1 m_2 n_2}, \\ F_{2mn} &= 4 \sum_{m_1} \sum_{n_1} \sum_{m_2} \sum_{n_2} x_{3m_1 n_1} x_{3m_2 n_2} A_{2mn}^{m_1 n_1 m_2 n_2}, \\ F_{3mn} &= -4 \sum_{m_1} \sum_{n_1} \sum_{m_2} \sum_{n_2} [x_{3m_1 n_1} x_{1m_2 n_2} C_{mn}^{m_1 n_1 m_2 n_2} \\ &\quad + x_{3m_1 n_1} x_{2m_2 n_2} D_{mn}^{m_1 n_1 m_2 n_2}], \end{aligned} \quad (20)$$

where

$$\begin{aligned} A_{1mn}^{m_1 n_1 m_2 n_2} &= \pi^3 m_1 \left(m_2^2 + \frac{1-\nu}{2} \beta_1^2 n_2^2 \right) a_{1mn}^{m_1 n_1 m_2 n_2} \\ &\quad - \frac{1+\nu}{2} \pi^3 \beta_1^2 n_1 m_2 n_2 a_{2mn}^{m_1 n_1 m_2 n_2}, \\ A_{2mn}^{m_1 n_1 m_2 n_2} &= \pi^3 \beta_1 n_1 \left(\beta_1^2 n_2^2 + \frac{1-\nu}{2} m_2^2 \right) a_{3mn}^{m_1 n_1 m_2 n_2} \\ &\quad - \frac{1+\nu}{2} \pi^3 \beta_1 m_1 m_2 n_2 a_{4mn}^{m_1 n_1 m_2 n_2}, \\ C_{mn}^{m_1 n_1 m_2 n_2} &= \pi^3 m_2 \left(m_1^2 + \nu \beta_1^2 n_1^2 \right) a_{5mn}^{m_1 n_1 m_2 n_2} \\ &\quad + (1-\nu) \pi^3 \beta_1^2 m_1 n_1 n_2 a_{6mn}^{m_1 n_1 m_2 n_2} \\ &\quad - \pi^3 m_1 \left(m_2^2 + \frac{1-\nu}{2} \beta_1^2 n_2^2 \right) a_{7mn}^{m_1 n_1 m_2 n_2} \\ &\quad - \frac{1+\nu}{2} \pi^3 \beta_1^2 n_1 m_2 n_2 a_{8mn}^{m_1 n_1 m_2 n_2}, \\ D_{mn}^{m_1 n_1 m_2 n_2} &= \pi^3 \beta_1 n_2 \left(\nu m_1^2 + \beta_1^2 n_1^2 \right) a_{5mn}^{m_1 n_1 m_2 n_2} \\ &\quad + (1-\nu) \beta_1 \pi^3 m_1 n_1 m_2 a_{6mn}^{m_1 n_1 m_2 n_2} \\ &\quad - \frac{1+\nu}{2} \beta_1 \pi^3 m_1 m_2 n_2 a_{7mn}^{m_1 n_1 m_2 n_2} \\ &\quad - \pi^3 \beta_1 n_1 \left(\frac{1-\nu}{2} m_2^2 + \beta_1^2 n_2^2 \right) a_{8mn}^{m_1 n_1 m_2 n_2}, \end{aligned} \quad (21)$$

and the coefficients $a_{kmn}^{m_1 n_1 m_2 n_2}$ ($k = 1, 2, \dots, 8$) depending on the combinations of sine and cosine functions entering into the eigenfunctions (10) are presented in the appendix.

Since the second-rank tensor $S_{\alpha\beta}^{mn}$ is symmetric, then it has two real eigenvalues $\omega_{\alpha mn}$ defined in (15) which are in correspondence with two mutually orthogonal eigenvectors

$$\begin{aligned} \bar{L}_{mn}^I \left\{ L_{1mn}^I = \frac{\pi m}{\omega_{1mn}}, L_{2mn}^I = \frac{\pi \beta_1 n}{\omega_{1mn}} \right\}, \\ \bar{L}_{mn}^{II} \left\{ L_{1mn}^{II} = \frac{\pi \beta_1 n}{\omega_{1mn}}, L_{2mn}^{II} = -\frac{\pi m}{\omega_{1mn}} \right\}, \end{aligned} \quad (22)$$

that is,

$$L_{\alpha mn}^I L_{\alpha mn}^I = L_{\alpha mn}^{II} L_{\alpha mn}^{II} = 1, \quad L_{\alpha mn}^I L_{\alpha mn}^{II} = 0, \quad (23)$$

where here and below summation is carried out over the repeated index α , while there is no summation over the repeated indices m and n .

Thus, the matrix $S_{\alpha\beta}^{mn}$ and the generalized displacements $x_{\alpha mn}$ entering in (18) and (19) could be expanded in terms of the vectors (22) [25] as

$$S_{\alpha\beta}^{mn} = \omega_{1mn}^2 L_{\alpha mn}^I L_{\beta mn}^I + \omega_{2mn}^2 L_{\alpha mn}^{II} L_{\beta mn}^{II}, \quad (24)$$

$$x_{\alpha mn} = X_{1mn} L_{\alpha mn}^I + X_{2mn} L_{\alpha mn}^{II}. \quad (25)$$

Substituting (24) and (25) in (18) and (19) and then multiplying (18) successively by L_{imn}^I and L_{imn}^{II} with due account for (23), we obtain the following three equations:

$$\begin{aligned} \ddot{X}_{1mn} + \mathfrak{a}_1 D^\gamma X_{1mn} + \omega_{1mn}^2 X_{1mn} &= -F_{\alpha mn} L_{\alpha mn}^I, \\ \ddot{X}_{2mn} + \mathfrak{a}_2 D^\gamma X_{2mn} + \omega_{2mn}^2 X_{2mn} &= -F_{\alpha mn} L_{\alpha mn}^{II}, \\ \ddot{X}_{3mn} + \mathfrak{a}_3 D^\gamma X_{3mn} + \omega_{3mn}^2 X_{3mn} &= -F_{3mn}, \end{aligned} \quad (26)$$

where $X_{3mn} = x_{3mn}$.

It should be emphasized that left-hand side parts of (26) are linear and independent of each other, while (26) are coupled only by nonlinear terms in their right-hand sides.

In order to show the influence of the initial conditions (4) and (5) on the solution to be constructed, let us expand the desired functions X_{imn} ($i = 1, 2, 3$) in a series in terms of the small parameter ε :

$$X_{imn} = \varepsilon X_{imn}^0 + \varepsilon^2 X_{imn}^1 + \dots \quad (i = 1, 2, 3). \quad (27)$$

Substituting (27) in the set of equations (26) and restricting ourselves by the terms of the order of ε , we are led to a linear homogeneous set of differential equations

$$\ddot{X}_{imn}^0 + \omega_{imn}^2 X_{imn}^0 = 0 \quad (i = 1, 2, 3). \quad (28)$$

The solution of (28) has the form

$$\begin{aligned} X_{imn}^0 &= A_{imn}(\varepsilon t) \exp(i\omega_{imn} t) + \bar{A}_{imn}(\varepsilon t) \exp(-i\omega_{imn} t) \\ &\quad (i = 1, 2, 3), \end{aligned} \quad (29)$$

where $A_{imn}(\varepsilon t)$ and $\bar{A}_{imn}(\varepsilon t)$ are complex conjugate functions to be found.

Representing functions $A_{imn}(\varepsilon t)$ in the polar form

$$A_{imn}(\varepsilon t) = a_{imn}(\varepsilon t) e^{i\psi_{imn}(\varepsilon t)} \quad (i = 1, 2, 3), \quad (30)$$

the solution (27) is reduced to

$$X_{imn} = \varepsilon X_{imn}^0 = 2\varepsilon a_{imn}(\varepsilon t) \cos[\omega_{imn}t + \psi_{imn}(\varepsilon t)] \quad (i = 1, 2, 3), \quad (31)$$

where $a_{imn}(\varepsilon t)$ and $\psi_{imn}(\varepsilon t)$ are the amplitudes and phases of nonlinear vibrations, respectively.

Differentiating (31) with respect to time t and ignoring the terms of the order of ε^2 , we obtain

$$\dot{X}_{imn} = -2\varepsilon a_{imn}(\varepsilon t) \omega_{imn} \sin[\omega_{imn}t + \psi_{imn}(\varepsilon t)] \quad (i = 1, 2, 3). \quad (32)$$

Now substituting (25) in (9) with due account for (31), we have

$$\begin{aligned} u(x, y, t) &= 2\varepsilon \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ a_{1mn}(\varepsilon t) \cos[\omega_{1mn}t + \psi_{1mn}(\varepsilon t)] L_{1mn}^I \right. \\ &\quad \left. + a_{2mn}(\varepsilon t) \cos[\omega_{2mn}t + \psi_{2mn}(\varepsilon t)] L_{1mn}^{II} \right\} \\ &\quad \cdot \eta_{1mn}(x, y), \\ v(x, y, t) &= 2\varepsilon \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ a_{1mn}(\varepsilon t) \cos[\omega_{1mn}t + \psi_{1mn}(\varepsilon t)] L_{2mn}^I \right. \\ &\quad \left. + a_{2mn}(\varepsilon t) \cos[\omega_{2mn}t + \psi_{2mn}(\varepsilon t)] L_{2mn}^{II} \right\} \\ &\quad \cdot \eta_{2mn}(x, y), \\ w(x, y, t) &= 2\varepsilon \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{3mn}(\varepsilon t) \\ &\quad \cdot \cos[\omega_{3mn}t + \psi_{3mn}(\varepsilon t)] \eta_{3mn}(x, y). \end{aligned} \quad (33)$$

Using the initial conditions (4) and (5) with due account for relationships (32), from (33) at $t = 0$ we obtain an infinite

set of algebraic equations for determining $a_{imn}(0)$ and $\psi_{imn}(0)$ ($i = 1, 2, 3$):

$$\begin{aligned} a_{1mn}(0) L_{1mn}^I \cos \psi_{1mn}(0) + a_{2mn}(0) L_{1mn}^{II} \cos \psi_{2mn}(0) &= 0, \\ a_{1mn}(0) L_{2mn}^I \cos \psi_{1mn}(0) + a_{2mn}(0) L_{2mn}^{II} \cos \psi_{2mn}(0) &= 0, \\ -a_{1mn}(0) \omega_{1mn} L_{1mn}^I \sin \psi_{1mn}(0) \\ -a_{2mn}(0) \omega_{2mn} L_{1mn}^{II} \sin \psi_{2mn}(0) &= \mathfrak{a}_{1mn}, \\ -a_{1mn}(0) \omega_{1mn} L_{2mn}^I \sin \psi_{1mn}(0) \\ -a_{2mn}(0) \omega_{2mn} L_{2mn}^{II} \sin \psi_{2mn}(0) &= \mathfrak{a}_{2mn}, \\ a_{3mn}(0) \cos \psi_{3mn}(0) &= 0, \\ -a_{3mn}(0) \omega_{3mn} \sin \psi_{3mn}(0) &= \mathfrak{a}_{3mn}, \end{aligned} \quad (34)$$

where

$$\begin{aligned} \mathfrak{a}_{1mn} &= \frac{\int_0^1 \int_0^1 U^0(x, y) \eta_{1mn}(x, y) dx dy}{\int_0^1 \int_0^1 \eta_{1mn}^2(x, y) dx dy}, \\ \mathfrak{a}_{2mn} &= \frac{\int_0^1 \int_0^1 V^0(x, y) \eta_{2mn}(x, y) dx dy}{\int_0^1 \int_0^1 \eta_{2mn}^2(x, y) dx dy}, \\ \mathfrak{a}_{3mn} &= \frac{\int_0^1 \int_0^1 W^0(x, y) \eta_{3mn}(x, y) dx dy}{\int_0^1 \int_0^1 \eta_{3mn}^2(x, y) dx dy}. \end{aligned} \quad (35)$$

All subsequent approximations are determined from an inhomogeneous set of differential equations with known right parts. Since the general solution of such a system is the sum of two solutions, a particular solution of the inhomogeneous system and a general solution of the corresponding homogeneous system, then arbitrary constants could be chosen in such a way that the initial conditions of all subsequent approximations would be zero ones.

It is known [14, 26–28] that during nonstationary excitation of thin bodies not all possible modes of vibration would be excited. Moreover, the modes which are strongly coupled by any of the so-called internal resonance conditions are initiated and dominate in the process of vibration, resulting in the energy transfer from one subsystem to another between the coupled modes; in so doing the types of modes to be excited are dependent on the character of the external excitation. As a result of damping, the modes that are not excited directly by an external excitation or through an internal resonance will die out, and only three modes will dominate the motion of the plate in three directions.

Assume hereafter that the vibration process occurs in such a way that only three natural modes corresponding to the generalized displacements $X_{1s_1s_2}$, $X_{2l_1l_2}$, and $X_{3k_1k_2}$ are excited and dominate over other natural modes. In this

case, the right parts of (26) are significantly simplified, and equations of free vibrations (26) take the form

$$\begin{aligned}\ddot{X}_{1s_1s_2} + \mathfrak{a}_1 D^\gamma X_{1s_1s_2} + \omega_{1s_1s_2}^2 X_{1s_1s_2} &= -\zeta_{1s_1s_2}^{k_1k_2} X_{3k_1k_2}^2, \\ \ddot{X}_{2l_1l_2} + \mathfrak{a}_2 D^\gamma X_{2l_1l_2} + \omega_{2l_1l_2}^2 X_{2l_1l_2} &= -\zeta_{2l_1l_2}^{k_1k_2} X_{3k_1k_2}^2, \\ \ddot{X}_{3k_1k_2} + \mathfrak{a}_3 D^\gamma X_{3k_1k_2} + \omega_{3k_1k_2}^2 X_{3k_1k_2} &= -\zeta_{13\ k_1k_2}^{s_1s_2} X_{1s_1s_2} X_{3k_1k_2} \\ &\quad - \zeta_{23\ k_1k_2}^{l_1l_2} X_{2l_1l_2} X_{3k_1k_2},\end{aligned}\quad (36)$$

where

$$\zeta_{1s_1s_2}^{k_1k_2} = 4 \left(A_{1s_1s_2}^{k_1k_2k_1k_2} L_{1s_1s_2}^I + A_{2s_1s_2}^{k_1k_2k_1k_2} L_{2s_1s_2}^I \right), \quad (37)$$

$$\zeta_{2l_1l_2}^{k_1k_2} = 4 \left(A_{1l_1l_2}^{k_1k_2k_1k_2} L_{1l_1l_2}^{II} + A_{2l_1l_2}^{k_1k_2k_1k_2} L_{2l_1l_2}^{II} \right), \quad (38)$$

$$\zeta_{13k_1k_2}^{s_1s_2} = 4 \left(C_{k_1k_2}^{k_1k_2s_1s_2} L_{1s_1s_2}^I + D_{k_1k_2}^{k_1k_2s_1s_2} L_{2s_1s_2}^I \right), \quad (39)$$

$$\zeta_{23k_1k_2}^{l_1l_2} = 4 \left(C_{k_1k_2}^{k_1k_2l_1l_2} L_{1l_1l_2}^{II} + D_{k_1k_2}^{k_1k_2l_1l_2} L_{2l_1l_2}^{II} \right), \quad (40)$$

$$\begin{aligned}A_{1s_1s_2}^{k_1k_2k_1k_2} &= \pi^3 k_1 \left(k_1^2 + \frac{1-\gamma}{2} \beta_1^2 k_2^2 \right) a_{1s_1s_2}^{k_1k_2k_1k_2} \\ &\quad - \frac{1+\gamma}{2} \pi^3 \beta_1^2 k_1 k_2 a_{2s_1s_2}^{k_1k_2k_1k_2},\end{aligned}\quad (41)$$

$$\begin{aligned}A_{2l_1l_2}^{k_1k_2k_1k_2} &= \pi^3 \beta_1 k_2 \left(\beta_1^2 k_2^2 + \frac{1-\gamma}{2} k_1^2 \right) a_{3l_1l_2}^{k_1k_2k_1k_2} \\ &\quad - \frac{1+\gamma}{2} \pi^3 \beta_1 k_1^2 k_2 a_{4l_1l_2}^{k_1k_2k_1k_2},\end{aligned}\quad (42)$$

$$\begin{aligned}C_{k_1k_2}^{k_1k_2s_1s_2} &= \pi^3 s_1 \left(k_1^2 + \gamma \beta_1^2 k_2^2 \right) a_{5k_1k_2}^{k_1k_2s_1s_2} \\ &\quad + (1-\gamma) \pi^3 \beta_1^2 k_1 k_2 s_2 a_{6k_1k_2}^{k_1k_2s_1s_2} \\ &\quad - \pi^3 k_1 \left(s_1^2 + \frac{1-\gamma}{2} \beta_1^2 s_2^2 \right) a_{7k_1k_2}^{k_1k_2s_1s_2} \\ &\quad - \frac{1+\gamma}{2} \beta_1^2 \pi^3 s_1 s_2 k_2 a_{8k_1k_2}^{k_1k_2s_1s_2},\end{aligned}\quad (43)$$

$$\begin{aligned}D_{k_1k_2}^{k_1k_2l_1l_2} &= \pi^3 \beta_1 l_2 \left(\gamma k_1^2 + \beta_1^2 k_2^2 \right) a_{5k_1k_2}^{k_1k_2l_1l_2} \\ &\quad + (1-\gamma) \beta_1 \pi^3 l_1 k_1 k_2 a_{6k_1k_2}^{k_1k_2l_1l_2} \\ &\quad - \frac{1+\gamma}{2} \beta_1 \pi^3 k_1 l_1 l_2 a_{7k_1k_2}^{k_1k_2l_1l_2} \\ &\quad - \pi^3 \beta_1 k_2 \left(\frac{1-\gamma}{2} l_1^2 + \beta_1^2 l_2^2 \right) a_{8k_1k_2}^{k_1k_2l_1l_2},\end{aligned}\quad (44)$$

and the coefficients $a_{kmn}^{m_1n_1m_2n_2}$ ($k = 1, 2, \dots, 8$) involving in (41)–(44) are presented in the appendix.

From relationships (37)–(44) we could calculate all coefficients entering in (36).

Omitting hereafter the subindices s_1s_2 , k_1k_2 , and l_1l_2 for ease of presentation, (36) could be rewritten as

$$\begin{aligned}\ddot{X}_1 + \mathfrak{a}_1 D^\gamma X_1 + \omega_1^2 X_1 + \zeta_1 X_3^2 &= 0, \\ \ddot{X}_2 + \mathfrak{a}_2 D^\gamma X_2 + \omega_2^2 X_2 + \zeta_2 X_3^2 &= 0, \\ \ddot{X}_3 + \mathfrak{a}_3 D^\gamma X_3 + \omega_3^2 X_3 + X_3 (\zeta_{13} X_1 + \zeta_{23} X_2) &= 0.\end{aligned}\quad (45)$$

3. Method of Solution

An approximate solution of (45) for small but finite amplitudes weakly varying with time can be represented by a third-order uniform expansion in terms of different time scales in the following form [29]:

$$\begin{aligned}X_i &= \varepsilon X_{i1}(T_0, T_1, T_2, \dots) + \varepsilon^2 X_{i2}(T_0, T_1, T_2, \dots) \\ &\quad + \varepsilon^3 X_{i3}(T_0, T_1, T_2, \dots) + \dots,\end{aligned}\quad (46)$$

where $i = 1, 2, 3$, ε is a small dimensionless parameter of the same order of magnitude as the amplitudes, and $T_n = \varepsilon^n t$ are new independent variables; among them, $T_0 = t$ is a fast scale characterizing motions with the natural frequencies and $T_1 = \varepsilon t$ and $T_2 = \varepsilon^2 t$ are slow scales characterizing the modulation of the amplitudes and phases of the modes with nonlinearity.

Recall that the first and the second time-derivatives are defined, respectively, as follows:

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \dots, \quad (47)$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) + \dots. \quad (48)$$

Further as a fractional derivative we will utilize the Grünvald-Letnikov definition which is defined on a semiaxis [30]

$$\left(\frac{d}{dt} \right)^\gamma \varphi(t) = \lim_{h \rightarrow 0} \frac{\Delta_h^\gamma \varphi(t)}{h^\gamma} \quad 0 < \gamma < 1, \quad (49)$$

where

$$\Delta_h^\gamma \varphi(t) = \sum_{k=0}^{\infty} (-1)^k \binom{\gamma}{k} \varphi(t - kh) \quad (50)$$

is the γ th-order finite difference of the function $\varphi(t)$ with the step h and with the point t as a center and $\binom{\gamma}{k}$ are the binomial coefficients which are determined by the formula

$$\binom{\gamma}{k} = (-1)^{k-1} \frac{\gamma \Gamma(k - \gamma)}{\Gamma(1 - \gamma) \Gamma(k + 1)}. \quad (51)$$

Substituting (47) in (49), we obtain following Rossikhin and Shitikova [19] that

$$\begin{aligned}\left(\frac{d}{dt} \right)^\gamma &= (D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \dots)^\gamma \\ &= D_0^\gamma + \varepsilon \gamma D_0^{\gamma-1} D_1 \\ &\quad + \frac{1}{2} \varepsilon^2 \gamma [(\gamma - 1) D_0^{\gamma-2} D_1^2 + 2D_0^{\gamma-1} D_2] + \dots,\end{aligned}\quad (52)$$

where $D_n = \partial/\partial T_n$ and $D_0^\gamma, D_0^{\gamma-1}, D_0^{\gamma-2}, \dots$ are the Grünvald-Letnikov fractional derivatives, which for “reasonably good” functions, such as exponential functions $e^{\pm i\omega t}$, coincide with the Riemann-Liouville fractional derivatives in time t defined on a semiaxis [30].

Considering that viscosity of the medium surrounding the plate under consideration is small, that is,

$$\mathfrak{a}_i = \varepsilon^k \mu_i \tau_i^\gamma, \quad (53)$$

where τ_i is the relaxation time of the i th generalized displacement, μ_i is a finite value, and the choice of k depends on the order of smallness of the viscosity coefficients $\mathfrak{a}_i$, substituting (46)–(52) in (45), after equating the coefficients at like powers of ε to zero, we are led to a set of recurrence equations to various orders:

to order ε

$$\begin{aligned} D_0^2 X_{11} + \omega_1^2 X_{11} &= 0, \\ D_0^2 X_{21} + \omega_2^2 X_{21} &= 0, \\ D_0^2 X_{31} + \omega_3^2 X_{31} &= 0; \end{aligned} \quad (54)$$

to order ε^2

$$\begin{aligned} D_0^2 X_{12} + \omega_1^2 X_{12} &= -2D_0 D_1 X_{11} - \zeta_1 X_{31}^2 \\ &\quad - \mu_1 (2-k) \tau_1^\gamma D_0^\gamma X_{11}; \\ D_0^2 X_{22} + \omega_2^2 X_{22} &= -2D_0 D_1 X_{21} - \zeta_2 X_{31}^2 \\ &\quad - \mu_2 (2-k) \tau_2^\gamma D_0^\gamma X_{21}; \\ D_0^2 X_{32} + \omega_3^2 X_{32} &= -2D_0 D_1 X_{31} - \zeta_{13} X_{11} X_{31} \\ &\quad - \zeta_{23} X_{21} X_{31} - \mu_3 (2-k) \tau_3^\gamma D_0^\gamma X_{31}; \end{aligned} \quad (55)$$

to order ε^3

$$\begin{aligned} D_0^2 X_{13} + \omega_1^2 X_{13} &= -2D_0 D_1 X_{12} - (D_1^2 + 2D_0 D_2) X_{11} \\ &\quad - 2\zeta_1 X_{31} X_{32} - \mu_1 (2-k) \tau_1^\gamma \\ &\quad \cdot (D_0^\gamma X_{12} + D_0^{\gamma-1} D_1 X_{11}) \\ &\quad - \mu_1 (k-1) \tau_1^\gamma D_0^\gamma X_{11}, \end{aligned} \quad (56)$$

$$\begin{aligned} D_0^2 X_{23} + \omega_2^2 X_{23} &= -2D_0 D_1 X_{22} - (D_1^2 + 2D_0 D_2) X_{21} \\ &\quad - 2\zeta_2 X_{31} X_{32} - \mu_2 (2-k) \tau_2^\gamma \\ &\quad \cdot (D_0^\gamma X_{22} + D_0^{\gamma-1} D_1 X_{21}) \\ &\quad - \mu_2 (k-1) \tau_2^\gamma D_0^\gamma X_{21}, \end{aligned} \quad (57)$$

$$\begin{aligned} D_0^2 X_{33} + \omega_3^2 X_{33} &= -2D_0 D_1 X_{32} - (D_1^2 + 2D_0 D_2) X_{31} \\ &\quad - \zeta_{13} (X_{11} X_{32} + X_{12} X_{31}) \end{aligned}$$

$$\begin{aligned} &- \zeta_{23} (X_{21} X_{32} + X_{22} X_{31}) \\ &- \mu_3 (2-k) \tau_3^\gamma (D_0^\gamma X_{32} + D_0^{\gamma-1} D_1 X_{31}) \\ &- \mu_3 (k-1) \tau_3^\gamma D_0^\gamma X_{31}. \end{aligned} \quad (58)$$

In order to construct the uniformly valid solution, it is necessary on each step to use the solution from the preceding step and to eliminate secular terms during integration [29].

We will seek the solution of (54) in the form

$$\begin{aligned} X_{j1} &= A_j(T_1, T_2) \exp(i\omega_j T_0) + \bar{A}_j(T_1, T_2) \exp(-i\omega_j T_0) \\ &\quad (j = 1, 2, 3), \end{aligned} \quad (59)$$

where $A_j(T_1, T_2)$ ($j = 1, 2, 3$) are unknown complex functions and $\bar{A}_j(T_1, T_2)$ are the complex conjugates of $A_j(T_1, T_2)$.

To solve the sets of (55) and (56)–(58), it is necessary to specify the action of the fractional derivative D_0^γ (8) on the functions X_{j1} , that is, to calculate $D_0^\gamma e^{i\omega_j t}$. It has been shown in [31] that

$$D_0^\gamma e^{i\omega_j t} = (i\omega_j)^\gamma e^{i\omega_j t} + \frac{\sin \pi \gamma}{\pi} \int_0^\infty \frac{u^\gamma}{u + i\omega_j} e^{-ut} du. \quad (60)$$

The first term in (60) is equivalent to the action of the fractional derivative with the low limit tending to $-\infty$:

$$D_0^\gamma x(t) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_{-\infty}^t \frac{x(s) ds}{(t-s)^\gamma}, \quad (61)$$

the application of which for the exponential function is reduced to

$$D_0^\gamma e^{i\omega_j t} = (i\omega_j)^\gamma e^{i\omega_j t}, \quad (62)$$

while the second term of (60), as it has been proved in [32], does not influence the solution constructed via the method of multiple time scales restricted to the zeroth- and first-order approximations.

In other words, even the utilization of exact formula (60) in the problem under consideration produces completely equivalent results given by the approximate formula (62) if the solution is constructed via the method of multiple time scales within the considered orders of approximation. Thus, in further analysis we will utilize formula (62).

For further analysis it is also a need to specify the order of weak damping.

3.1. Viscosity of the Order of ε . Let us first consider the case of viscosity of the order of ε . Then at $k = 1$ (55) are reduced to

$$\begin{aligned} D_0^2 X_{12} + \omega_1^2 X_{12} &= -2D_0 D_1 X_{11} - \zeta_1 X_{31}^2 - \mu_1 \tau_1^\gamma D_0^\gamma X_{11}, \\ D_0^2 X_{22} + \omega_2^2 X_{22} &= -2D_0 D_1 X_{21} - \zeta_2 X_{31}^2 - \mu_2 \tau_2^\gamma D_0^\gamma X_{21}, \\ D_0^2 X_{32} + \omega_3^2 X_{32} &= -2D_0 D_1 X_{31} \\ &\quad - X_{31} (\zeta_{13} X_{11} + \zeta_{23} X_{21}) \\ &\quad - \mu_3 \tau_3^\gamma D_0^\gamma X_{31}. \end{aligned} \quad (63)$$

Substituting (59) in the right-hand side of (63) with due account for (62) yields

$$\begin{aligned}
D_0^2 X_{12} + \omega_1^2 X_{12} &= -2i\omega_1 D_1 A_1 (T_1) \exp(i\omega_1 T_0) \\
&\quad - \zeta_1 [A_3^2 \exp(2i\omega_3 T_0) + A_3 \bar{A}_3] \\
&\quad - \mu_1 \tau_1^\gamma A_1 (i\omega_1)^\gamma \exp(i\omega_1 T_0) + \text{cc}, \\
D_0^2 X_{22} + \omega_2^2 X_{22} &= -2i\omega_2 D_1 A_2 (T_1) \exp(i\omega_2 T_0) \\
&\quad - \zeta_2 [A_3^2 \exp(2i\omega_3 T_0) + A_3 \bar{A}_3] \\
&\quad - \mu_2 \tau_2^\gamma A_2 (i\omega_2)^\gamma \exp(i\omega_2 T_0) + \text{cc}, \\
D_0^2 X_{32} + \omega_3^2 X_{32} &= -2i\omega_3 D_1 A_3 (T_1) \exp(i\omega_3 T_0) \\
&\quad - \mu_3 \tau_3^\gamma A_3 (i\omega_3)^\gamma \exp(i\omega_3 T_0) \\
&\quad - \zeta_{13} \{A_1 A_3 \exp[i(\omega_1 + \omega_3) T_0] \\
&\quad \quad + A_1 \bar{A}_3 \exp[i(\omega_1 - \omega_3) T_0]\} \\
&\quad - \zeta_{23} \{A_2 A_3 \exp[i(\omega_2 + \omega_3) T_0] \\
&\quad \quad + A_2 \bar{A}_3 \exp[i(\omega_2 - \omega_3) T_0]\} + \text{cc},
\end{aligned} \tag{64}$$

where cc is the complex conjugate part to the preceding terms. Reference to (64) shows that the following types of the internal resonance could occur on this step:

- (1) the two-to-one internal resonance, when one natural frequency is twice the other natural frequency,

$$\omega_1 = 2\omega_3 \quad (\omega_2 \neq \omega_1, \omega_2 \neq 2\omega_3), \tag{65}$$

$$\omega_2 = 2\omega_3 \quad (\omega_1 \neq \omega_2, \omega_1 \neq 2\omega_3); \tag{66}$$

- (2) the one-to-one-to-two internal resonance, that is,

$$\omega_1 = \omega_2 = 2\omega_3, \quad \text{or} \quad 1:1:2. \tag{67}$$

3.2. Viscosity of the Order of ε^2 . Now let us consider vibrations of a nonlinear plate putting $k = 2$ in its equations of motion (55):

$$\begin{aligned}
D_0^2 X_{12} + \omega_1^2 X_{12} &= -2D_0 D_1 X_{11} - \zeta_1 X_{31}^2, \\
D_0^2 X_{22} + \omega_2^2 X_{22} &= -2D_0 D_1 X_{21} - \zeta_2 X_{31}^2, \\
D_0^2 X_{32} + \omega_3^2 X_{32} &= -2D_0 D_1 X_{31} - X_{31} (\zeta_{13} X_{11} + \zeta_{23} X_{21}).
\end{aligned} \tag{68}$$

Substituting (59) in the right-hand side of (68) yields

$$\begin{aligned}
D_0^2 X_{12} + \omega_1^2 X_{12} &= -2i\omega_1 D_1 A_1 (T_1) \exp(i\omega_1 T_0) \\
&\quad - \zeta_1 [A_3^2 \exp(2i\omega_3 T_0) + A_3 \bar{A}_3] + \text{cc},
\end{aligned} \tag{69}$$

$$\begin{aligned}
D_0^2 X_{22} + \omega_2^2 X_{22} &= -2i\omega_2 D_1 A_2 (T_1) \exp(i\omega_2 T_0) \\
&\quad - \zeta_2 [A_3^2 \exp(2i\omega_3 T_0) + A_3 \bar{A}_3] + \text{cc},
\end{aligned} \tag{70}$$

$$\begin{aligned}
D_0^2 X_{32} + \omega_3^2 X_{32} &= -2i\omega_3 D_1 A_3 (T_1) \exp(i\omega_3 T_0) \\
&\quad - \zeta_{13} \{A_1 A_3 \exp[i(\omega_1 + \omega_3) T_0] \\
&\quad \quad + A_1 \bar{A}_3 \exp[i(\omega_1 - \omega_3) T_0]\} \\
&\quad - \zeta_{23} \{A_2 A_3 \exp[i(\omega_2 + \omega_3) T_0] \\
&\quad \quad + A_2 \bar{A}_3 \exp[i(\omega_2 - \omega_3) T_0]\} + \text{cc}.
\end{aligned} \tag{71}$$

Assume that the natural frequencies ω_j ($j = 1, 2, 3$) possess such magnitudes that any of the combinations from (65) to (67) could not occur. Then the functions $\exp(\pm i\omega_j T_0)$ entering into the right-hand side of (69) produce secular terms.

To eliminate circular terms in (69), it is necessary to vanish to zero the coefficients standing at $\exp(\pm i\omega_j T_0)$, that is,

$$D_1 A_j (T_1, T_2) = 0, \tag{72}$$

whence it follows that A_j ($j = 1, 2, 3$) are T_1 -independent.

Considering (72), solution of (69)–(71) at $j = 1, 2, 3$ has the form

$$\begin{aligned}
X_{12} &= F_1 (T_2) \exp(i\omega_1 T_0) + k_1 A_3^2 \exp(2i\omega_3 T_0) \\
&\quad + k_2 A_3 \bar{A}_3 + \text{cc}, \\
X_{22} &= F_2 (T_2) \exp(i\omega_2 T_0) + k_3 A_3^2 \exp(2i\omega_3 T_0) \\
&\quad + k_4 A_3 \bar{A}_3 + \text{cc}, \\
X_{32} &= F_3 (T_2) \exp(i\omega_3 T_0) + k_5 A_1 A_3 \exp[i(\omega_1 + \omega_3) T_0] \\
&\quad + k_6 A_2 A_3 \exp[i(\omega_2 + \omega_3) T_0] \\
&\quad + k_7 A_3 \bar{A}_1 \exp[i(\omega_3 - \omega_1) T_0] \\
&\quad + k_8 A_3 \bar{A}_2 \exp[i(\omega_3 - \omega_2) T_0] + \text{cc},
\end{aligned} \tag{73}$$

where $F_j(T_2)$ ($j = 1, 2, 3$) are new functions to be determined and coefficients k_i ($i = 1, 2, \dots, 8$) have the form

$$\begin{aligned} k_1 &= \frac{\zeta_1}{4\omega_3^2 - 1}, & k_2 &= -\frac{\zeta_1}{\omega_1^2}, \\ k_3 &= \frac{\zeta_2}{4\omega_3^2 - 1}, & k_4 &= -\frac{\zeta_2}{\omega_2^2}, \\ k_5 &= \frac{\zeta_{13}}{\omega_1(\omega_1 + 2\omega_3)}, & k_6 &= \frac{\zeta_{23}}{\omega_2(\omega_2 + 2\omega_3)}, \\ k_7 &= \frac{\zeta_{13}}{\omega_1(\omega_1 - 2\omega_3)}, & k_8 &= \frac{\zeta_{23}}{\omega_2(\omega_2 - 2\omega_3)}. \end{aligned} \quad (74)$$

Substituting (59) and (73) with due account for (72) in (56)–(58) at $k = 2$, we have

$$\begin{aligned} & D_0^2 X_{13} + \omega_1^2 X_{13} \\ &= - \left[2i\omega_1 D_2 A_1 + \mu_1 \tau_1^\gamma (i\omega_1)^\gamma A_1 + 2\zeta_1 k_5 A_1 A_3 \bar{A}_3 \right] \\ & \quad \cdot \exp(i\omega_1 T_0) \\ & \quad - 2\zeta_1 \left\{ F_3 A_3 \exp(2i\omega_3 T_0) + \bar{F}_3 A_3 \right. \\ & \quad \quad + k_5 A_1 A_3^2 \exp[i(\omega_1 + 2\omega_3) T_0] \\ & \quad \quad + k_6 A_2 A_3^2 \exp[i(\omega_2 + 2\omega_3) T_0] \\ & \quad \quad + k_6 A_2 A_3 \bar{A}_3 \exp(i\omega_2 T_0) \\ & \quad \quad + k_7 \bar{A}_1 A_3^2 \exp[i(2\omega_3 - \omega_1) T_0] \\ & \quad \quad \left. + k_8 A_3^2 \bar{A}_2 \exp[i(2\omega_3 - \omega_2) T_0] \right\} + \text{cc}, \\ & D_0^2 X_{23} + \omega_2^2 X_{23} \\ &= - \left[2i\omega_2 D_2 A_2 + \mu_2 \tau_2^\gamma (i\omega_2)^\gamma A_2 + 2\zeta_2 k_6 A_2 A_3 \bar{A}_3 \right] \\ & \quad \cdot \exp(i\omega_2 T_0) \\ & \quad - 2\zeta_2 \left\{ F_3 A_3 \exp(2i\omega_3 T_0) + \bar{F}_3 A_3 \right. \\ & \quad \quad + k_5 A_1 A_3^2 \exp[i(\omega_1 + 2\omega_3) T_0] \\ & \quad \quad + k_6 A_2 A_3^2 \exp[i(\omega_2 + 2\omega_3) T_0] \\ & \quad \quad + k_5 A_1 A_3 \bar{A}_3 \exp(i\omega_1 T_0) \\ & \quad \quad + k_7 \bar{A}_1 A_3^2 \exp[i(2\omega_3 - \omega_1) T_0] \\ & \quad \quad \left. + k_8 A_3^2 \bar{A}_2 \exp[i(2\omega_3 - \omega_2) T_0] \right\} + \text{cc}, \\ & D_0^2 X_{33} + \omega_3^2 X_{33} \\ &= - \left[2i\omega_3 D_2 A_3 + \mu_3 \tau_3^\gamma (i\omega_3)^\gamma A_3 \right. \\ & \quad + (\zeta_{13} k_2 + \zeta_{23} k_4) A_3^2 \bar{A}_3 \\ & \quad + \zeta_{13} k_7 A_1 \bar{A}_1 A_3 + \zeta_{23} k_8 A_2 \bar{A}_2 A_3 \left. \right] \\ & \quad \cdot \exp(i\omega_3 T_0) - (\zeta_{13} k_1 + \zeta_{23} k_3) A_3^3 \exp(3i\omega_3 T_0) \end{aligned}$$

$$\begin{aligned} & - \zeta_{13} \left\{ (F_3 A_1 + F_1 A_3) \exp[i(\omega_3 + \omega_1) T_0] \right. \\ & \quad \left. + (A_3 \bar{F}_1 + \bar{A}_1 F_3) \exp[i(\omega_3 - \omega_1) T_0] \right\} \\ & - \zeta_{23} \left\{ (F_3 A_2 + F_2 A_3) \exp[i(\omega_3 + \omega_2) T_0] \right. \\ & \quad \left. + (A_3 \bar{F}_2 + \bar{A}_2 F_3) \exp[i(\omega_3 - \omega_2) T_0] \right\} \\ & - \zeta_{13} \left\{ k_5 A_1^2 A_3 \exp[i(2\omega_1 + \omega_3) T_0] \right. \\ & \quad \left. + k_7 A_1^2 \bar{A}_3 \exp[i(2\omega_1 - \omega_3) T_0] \right\} \\ & - \zeta_{23} \left\{ k_6 A_2^2 A_3 \exp[i(2\omega_2 + \omega_3) T_0] \right. \\ & \quad \left. + k_8 A_2^2 \bar{A}_3 \exp[i(2\omega_2 - \omega_3) T_0] \right\} \\ & - (\zeta_{13} k_6 + \zeta_{23} k_5) A_1 A_2 A_3 \exp[i(\omega_1 + \omega_2 + \omega_3) T_0] \\ & - (\zeta_{13} k_6 + \zeta_{23} k_7) \bar{A}_1 A_2 A_3 \exp[i(\omega_2 + \omega_3 - \omega_1) T_0] \\ & - (\zeta_{13} k_8 + \zeta_{23} k_5) A_1 \bar{A}_2 A_3 \exp[i(\omega_1 - \omega_2 + \omega_3) T_0] \\ & - (\zeta_{13} k_8 + \zeta_{23} k_7) \bar{A}_1 \bar{A}_2 A_3 \\ & \quad \cdot \exp[i(\omega_3 - \omega_1 - \omega_2) T_0] + \text{cc}. \end{aligned} \quad (75)$$

Reference to (75) shows that the following types of the internal resonance could occur on this step:

- (1) the one-to-one internal resonance, when one natural frequency is nearly equal to the other natural frequency,

$$\omega_1 = \omega_2 \quad (\omega_3 \neq \omega_1, \omega_3 \neq \omega_2), \quad (76)$$

$$\omega_1 = \omega_3 \quad (\omega_2 \neq \omega_1, \omega_2 \neq \omega_3), \quad (77)$$

$$\omega_2 = \omega_3 \quad (\omega_1 \neq \omega_2, \omega_1 \neq \omega_3); \quad (78)$$

- (2) the one-to-one-to-one internal resonance

$$\omega_1 = \omega_2 = \omega_3; \quad (79)$$

- (3) the combinational resonance of the additive-difference type, that is,

$$\omega_1 = \omega_2 + 2\omega_3, \quad \text{or} \quad \omega_2 = \omega_1 - 2\omega_3, \quad \text{or} \quad (80)$$

$$\begin{aligned} & 2\omega_3 = \omega_1 - \omega_2, \\ & \omega_1 = 2\omega_3 - \omega_2, \quad \text{or} \quad \omega_2 = 2\omega_3 - \omega_1, \quad \text{or} \end{aligned} \quad (81)$$

$$\begin{aligned} & 2\omega_3 = \omega_1 + \omega_2, \\ & \omega_1 = \omega_2 - 2\omega_3, \quad \text{or} \quad \omega_2 = 2\omega_3 + \omega_1, \quad \text{or} \end{aligned} \quad (82)$$

4. Governing Nonlinear Differential Equations Describing Amplitude-Phase Modulation for Different Types of the Internal Resonance of the Order of ε

To deduce the nonlinear differential equations describing the modulation of amplitudes and phases of the nonlinear plate under consideration, we should consider separately each type of the internal resonance which could occur with due account for weak damping of the order ε .

4.1. Internal Resonance 2:1. Let us consider case (65), when $\omega_1 = 2\omega_3$, while $\omega_2 \neq \omega_1$ and $\omega_2 \neq 2\omega_3$. Case (66) could be treated similarly.

Then eliminating secular terms in (64), we obtain the following solvability equations:

$$2i\omega_1 D_1 A_1(T_1) + \mu_1 (i\omega_1 \tau_1)^\gamma A_1 + \zeta_1 A_1^2 = 0, \quad (83)$$

$$2i\omega_2 D_1 A_2(T_1) + \mu_2 (i\omega_2 \tau_2)^\gamma A_2 = 0, \quad (84)$$

$$2i\omega_3 D_1 A_3(T_1) + \mu_3 (i\omega_3 \tau_3)^\gamma A_3 + \zeta_{13} A_1 \bar{A}_3 = 0. \quad (85)$$

Reference to the set of equations (83)–(85) shows that its second equation is independent of other two, while the first and third ones represent a set of two nonlinear equations. The similar situation was noted in [2] for a fractionally damped nonlinear plate in the case of the two-to-one internal resonance, when equations describing plate's in-plane motion are coupled.

Let us multiply (83)–(85), respectively, by $\bar{A}_1$, $\bar{A}_2$, and $\bar{A}_3$ and find their complex conjugates. Adding every pair of the mutually adjoint equations with each other and subtracting one from another and considering that

$$A_i = a_i e^{i\varphi_i} \quad (i = 1, 2, 3), \quad (86)$$

as a result we have

$$\begin{aligned} (a_1^2)' + s_1 a_1^2 + \omega_1^{-1} \zeta_1 a_1 a_3^2 \sin \delta &= 0, \\ (a_3^2)' + s_3 a_3^2 - \omega_3^{-1} \zeta_{13} a_1 a_3^2 \sin \delta &= 0, \\ \dot{\varphi}_1 - \frac{1}{2} \sigma_1 - \frac{1}{2} \zeta_1 \omega_1^{-1} a_3^2 a_1^{-1} \cos \delta &= 0, \\ \dot{\varphi}_3 - \frac{1}{2} \sigma_3 - \frac{1}{2} \zeta_{13} \omega_3^{-1} a_1 \cos \delta &= 0, \\ (a_2^2)' + s_2 a_2^2 &= 0, \\ \dot{\varphi}_2 - \frac{1}{2} \sigma_2 &= 0, \end{aligned} \quad (87)$$

where a dot denotes differentiation with respect to T_1 , $\delta = 2\varphi_3 - \varphi_1$, and

$$\begin{aligned} s_i &= \mu_i \tau_i^\gamma \omega_i^{\gamma-1} \sin \psi, & \sigma_i &= \mu_i \tau_i^\gamma \omega_i^{\gamma-1} \cos \psi, \\ \psi &= \frac{1}{2} \pi \gamma \quad (i = 1, 2, 3). \end{aligned} \quad (88)$$

The set of nonlinear equations (87)–(88) was solved in [2]. It has been found that during free vibrations of the plate being under the conditions of the two-to-one internal resonance three regimes can be observed: stationary (absence of damping at $\gamma = 0$ and $\mu = 0$), quasi-stationary (damping is defined by an ordinary derivative at $\gamma = 1$), and transient (damping is defined by a fractional derivative at $0 < \gamma < 1$).

4.2. Internal Resonance 1:1:2. Let us consider case (67), when $\omega_1 = \omega_2 = 2\omega_3$. Then eliminating secular terms in (64), we obtain the following solvability equations:

$$\begin{aligned} 2i\omega_1 D_1 A_1(T_1) + \mu_1 (i\omega_1 \tau_1)^\gamma A_1 + \zeta_1 A_1^2 &= 0, \\ 2i\omega_2 D_1 A_2(T_1) + \mu_2 (i\omega_2 \tau_2)^\gamma A_2 + \zeta_2 A_2^2 &= 0, \\ 2i\omega_3 D_1 A_3(T_1) + \mu_3 (i\omega_3 \tau_3)^\gamma A_3 + \zeta_{13} A_1 \bar{A}_3 &+ \zeta_{23} A_2 \bar{A}_3 = 0. \end{aligned} \quad (89)$$

Reference to the set of equations (89) shows that, in contrast to (83)–(85) corresponding to case (65) when $\omega_1 = 2\omega_3$, while $\omega_2 \neq \omega_1$ and $\omega_2 \neq 2\omega_3$, now we obtain the set of three coupled nonlinear equations.

Representing functions A_i entering in (89) in the polar form (86) and applying the same procedure as it has been carried out above for the 2:1 internal resonance, we have

$$(a_1^2)' + s_1 a_1^2 + \omega_1^{-1} \zeta_1 a_1 a_3^2 \sin(2\varphi_3 - \varphi_1) = 0, \quad (90)$$

$$(a_2^2)' + s_2 a_2^2 + \omega_2^{-1} \zeta_2 a_2 a_3^2 \sin(2\varphi_3 - \varphi_2) = 0, \quad (91)$$

$$\begin{aligned} (a_3^2)' + s_3 a_3^2 - \omega_3^{-1} \zeta_{13} a_1 a_3^2 \sin(2\varphi_3 - \varphi_1) \\ - \omega_3^{-1} \zeta_{23} a_2 a_3^2 \sin(2\varphi_3 - \varphi_2) &= 0, \end{aligned} \quad (92)$$

$$\dot{\varphi}_1 - \frac{1}{2} \sigma_1 - \frac{1}{2} \zeta_1 \omega_1^{-1} a_3^2 a_1^{-1} \cos(2\varphi_3 - \varphi_1) = 0, \quad (93)$$

$$\dot{\varphi}_2 - \frac{1}{2} \sigma_2 - \frac{1}{2} \zeta_2 \omega_2^{-1} a_3^2 a_2^{-1} \cos(2\varphi_3 - \varphi_2) = 0, \quad (94)$$

$$\begin{aligned} \dot{\varphi}_3 - \frac{1}{2} \sigma_3 - \frac{1}{2} \zeta_{13} \omega_3^{-1} a_1 \cos(2\varphi_3 - \varphi_1) \\ - \frac{1}{2} \zeta_{23} \omega_3^{-1} a_2 \cos(2\varphi_3 - \varphi_2) &= 0. \end{aligned} \quad (95)$$

The nonlinear set of equations (90)–(95) with the corresponding initial conditions completely describes the vibrational process of the mechanical system being investigated under the condition of the internal resonance 1:1:2 and could be solved numerically. This case has been discussed in [33].

Introducing new functions $\xi_1(T_1)$, $\xi_2(T_1)$, and $\xi_3(T_1)$ such that

$$a_1^2 = \frac{\zeta_1 \omega_3}{\zeta_{13} \omega_1} \xi_1 e^{-s_1 T_1}, \quad a_2^2 = \frac{\zeta_2 \omega_3}{\zeta_{23} \omega_2} \xi_2 e^{-s_2 T_1}, \quad (96)$$

$$a_3^2 = \xi_3 e^{-s_3 T_1}$$

and adding (90)–(95) with due account for (96), we obtain

$$\dot{\xi}_1 e^{-s_1 T_1} + \dot{\xi}_2 e^{-s_2 T_1} + \dot{\xi}_3 e^{-s_3 T_1} = 0. \quad (97)$$

Equation (97) describes the law of energy variation for this case of the internal resonance.

5. Governing Nonlinear Differential Equations Describing Amplitude-Phase Modulation for Different Types of the Internal Resonance of the Order of ε^2

To deduce the nonlinear differential equations describing the modulation of amplitudes and phases of the nonlinear plate under consideration, we should consider separately each type of the internal resonance which could occur with due account for weak damping of the order ε^2 .

5.1. Internal Resonance 1:1. The one-to-one internal resonance could be of two types: when the natural frequencies of two modes of in-plane vibrations are close to each other, that is, when $\omega_1 = \omega_2 = \omega$, or when the natural frequency of the out-of-plane mode is close to the natural frequency of one of the in-plane modes, that is, $\omega_3 = \omega_1 = \omega$ or $\omega_3 = \omega_2 = \omega$.

5.1.1. Internal Resonance $\omega_1 = \omega_2 = \omega$. Now let us first consider case (76) of the one-to-one internal resonance, when $\omega_1 = \omega_2 = \omega$, but $\omega_3 \neq \omega$.

Then eliminating secular terms in (75), we obtain the following solvability equations:

$$2i\omega_1 D_2 A_1 + \mu_1 (i\omega_1 \tau_1)^y A_1 + 2\zeta_1 k_5 A_1 A_3 \bar{A}_3 + 2\zeta_1 k_6 A_2 A_3 \bar{A}_3 = 0, \quad (98)$$

$$2i\omega_2 D_2 A_2 + \mu_2 (i\omega_2 \tau_2)^y A_2 + 2\zeta_2 k_6 A_2 A_3 \bar{A}_3 + 2\zeta_2 k_5 A_1 A_3 \bar{A}_3 = 0, \quad (99)$$

$$2i\omega_3 D_2 A_3 + \mu_3 (i\omega_3 \tau_3)^y A_3 + (\zeta_{13} k_2 + \zeta_{23} k_4) A_3^2 \bar{A}_3 + \zeta_{13} k_7 A_1 \bar{A}_1 A_3 + \zeta_{23} k_8 A_2 \bar{A}_2 A_3 + (\zeta_{13} k_8 + \zeta_{23} k_5) A_1 \bar{A}_2 A_3 + (\zeta_{13} k_6 + \zeta_{23} k_7) \bar{A}_1 A_2 A_3 = 0. \quad (100)$$

Let us multiply (98)–(100), respectively, by $\bar{A}_1$, $\bar{A}_2$, and $\bar{A}_3$ and find their complex conjugates. Adding every pair of the mutually adjoint equations with each other and subtracting one from another and considering that the functions $A_i(T_2)$ could be represented in the polar form (86), where a_i and φ_i ($i = 1, 2, 3$) are T_2 -dependent, as a result we have

$$(a_1^2)^{\cdot} + s_1 a_1^2 = -2\omega^{-1} \zeta_1 k_6 a_1 a_2 a_3^2 \sin \delta, \quad (101)$$

$$(a_2^2)^{\cdot} + s_2 a_2^2 = 2\omega^{-1} \zeta_2 k_5 a_1 a_2 a_3^2 \sin \delta, \quad (102)$$

$$(a_3^2)^{\cdot} + s_3 a_3^2 = -\omega_3^{-1} [\zeta_{13} (k_6 - k_8) + \zeta_{23} (k_7 - k_5)] \cdot a_1 a_2 a_3^2 \sin \delta, \quad (103)$$

$$\dot{\varphi}_1 = \frac{1}{2} \sigma_1 + \omega^{-1} \zeta_1 k_5 a_3^2 + \omega^{-1} \zeta_1 k_6 a_1^{-1} a_2 a_3^2 \cos \delta, \quad (104)$$

$$\dot{\varphi}_2 = \frac{1}{2} \sigma_2 + \omega^{-1} \zeta_2 k_6 a_3^2 + \omega^{-1} \zeta_2 k_5 a_1 a_2^{-1} a_3^2 \cos \delta, \quad (105)$$

$$\begin{aligned} \dot{\varphi}_3 = & \frac{1}{2} \sigma_3 + \frac{1}{2} \omega_3^{-1} \zeta_{13} k_7 a_1^2 + \frac{1}{2} \omega_3^{-1} \zeta_{23} k_8 a_2^2 \\ & + \frac{1}{2} \omega_3^{-1} (\zeta_{13} k_2 + \zeta_{23} k_4) a_3^2 \\ & + \frac{1}{2} \omega_3^{-1} [\zeta_{13} (k_6 - k_8) + \zeta_{23} (k_7 - k_5)] \\ & \cdot a_1 a_2 \cos \delta, \end{aligned} \quad (106)$$

where the phase difference has the form $\delta = \varphi_2 - \varphi_1$ and an overdot denotes the differentiation with respect to T_2 .

It could be shown that the expression in square brackets in (103) and (106) is vanished to zero when $\omega_1 = \omega_2 = \omega$. Then (103) turns out to be uncoupled with respect to all other equations from (101) to (106):

$$(a_3^2)^{\cdot} + s_3 a_3^2 = 0, \quad (107)$$

the solution of which has the form

$$a_3^2 = c_3 e^{-s_3 T_2}, \quad (108)$$

where c_3 is a constant of integration to be determined from the initial conditions, while (106) is reduced to

$$\begin{aligned} \dot{\varphi}_3 = & \frac{1}{2} \sigma_3 + \frac{1}{2} \omega_3^{-1} \\ & \cdot [\zeta_{13} k_7 a_1^2 + \zeta_{23} k_8 a_2^2 + (\zeta_{13} k_2 + \zeta_{23} k_4) a_3^2], \end{aligned} \quad (109)$$

whence it follows that the phase φ_3 is dependent only on the squared values of all amplitudes of the three interacting modes of vibrations.

Introducing new functions $\xi_1(T_2)$ and $\xi_2(T_2)$ such that

$$a_1^2 = \frac{\zeta_1 k_6}{\omega} \xi_1 \exp(-s_1 T_2), \quad a_2^2 = \frac{\zeta_2 k_5}{\omega} \xi_2 \exp(-s_2 T_2) \quad (110)$$

and adding (101) and (102) with due account for (110) yield

$$\dot{\xi}_2 + \xi_1 e^{(s_2 - s_1) T_2} = 0, \quad (111)$$

while subtracting (104) from (105) we obtain

$$\begin{aligned} \dot{\delta} = & \dot{\varphi}_2 - \dot{\varphi}_1 = \Sigma + \omega^{-1} (\zeta_2 k_6 - \zeta_1 k_5) a_3^2 \\ & + \omega^{-1} (\zeta_2 k_5 a_1^2 - \zeta_1 k_6 a_2^2) \frac{a_3^2}{a_1 a_2} \cos \delta, \end{aligned} \quad (112)$$

where $\Sigma = (1/2)(\sigma_2 - \sigma_1)$.

From (101) and (102) we could find that

$$\frac{a_2 a_3^2}{a_1} = -\frac{1}{2} \frac{\omega}{\zeta_1 k_6} \frac{1}{\sin \delta} \frac{\dot{\xi}_1}{\xi_1}, \quad \frac{a_1 a_3^2}{a_2} = \frac{1}{2} \frac{\omega}{\zeta_2 k_5} \frac{1}{\sin \delta} \frac{\dot{\xi}_2}{\xi_2}. \quad (113)$$

Considering (110) and (113), (112) could be reduced to

$$\dot{\delta} - \Sigma = Ke^{-s_3 T_2} + \frac{1}{2} \left(\frac{\dot{\xi}_1}{\xi_1} + \frac{\dot{\xi}_2}{\xi_2} \right) \cot \delta, \quad (114)$$

where $K = \omega^{-1}(\zeta_2 k_6 - \zeta_1 k_5) c_3$.

Equations (101) and (112) could be rewritten in another form if we substitute (110) and (108) in all its terms:

$$\dot{\xi}_1 = -2b\sqrt{\xi_1 \xi_2} e^{-(s_3+1/2s_2-1/2s_1)T_2} \sin \delta, \quad (115)$$

$$\begin{aligned} \dot{\delta} - \Sigma &= Ke^{-s_3 T_2} + b(\xi_1 e^{-s_1 T_2} - \xi_2 e^{-s_2 T_2}) \\ &\cdot \frac{\cos \delta}{\sqrt{\xi_1 \xi_2}} e^{-(s_3+1/2s_2+1/2s_1)T_2}, \end{aligned} \quad (116)$$

where $b = \omega^{-1} \sqrt{\zeta_1 \zeta_2 k_5 k_6}$.

The nonlinear set of equations (111), (115), and (116), or (114), with the initial conditions

$$\begin{aligned} \xi_1|_{T_2=0} &= \xi_{10}, & \xi_2|_{T_2=0} &= \xi_{20}, \\ \xi_3|_{T_2=0} &= \xi_{30}, & \delta|_{T_2=0} &= \delta_0, \end{aligned} \quad (117)$$

completely describes the vibrational process of the mechanical system being investigated under the condition of the one-to-one internal resonance $\omega_1 = \omega_2$ and could be solved numerically.

It should be noted that the equation defining the stream function could be obtained from (114), if we neglect the first term in (114) which decays rather rapidly as compared with its last term.

Particular Case. In the particular case at $\Sigma = 0$ and $s_1 = s_2 = s_3 = s$, (111) has the form

$$\dot{\xi}_2 + \dot{\xi}_1 = 0, \quad (118)$$

whence it follows that

$$\xi_2 + \xi_1 = E_0, \quad (119)$$

$$\xi_1 = E_0 \xi, \quad \xi_2 = E_0 (1 - \xi), \quad (120)$$

where E_0 is the initial energy.

Considering (120), (115) and (116), wherein the term rapidly decaying with time has been neglected, are reduced to the following:

$$\dot{\xi} = -2b\sqrt{\xi(1-\xi)} e^{-sT_2} \sin \delta, \quad (121)$$

$$\begin{aligned} \dot{\delta} &= -b \left[-\xi^{1/2} (1-\xi)^{-1/2} + \xi^{-1/2} (1-\xi)^{1/2} \right] \\ &\cdot \cos \delta e^{-sT_2} = -b \frac{(1-2\xi)}{\sqrt{\xi(1-\xi)}} \cos \delta e^{-sT_2}, \end{aligned} \quad (122)$$

while (114), wherein its first term rapidly decaying with time has been neglected, takes the form

$$\dot{\delta} \tan \delta \approx \frac{1}{2} \frac{\dot{\xi}_2}{\xi_2} + \frac{1}{2} \frac{\dot{\xi}_1}{\xi_1}, \quad (123)$$

and it could be integrated, resulting in its first integral

$$G(\xi, \delta) = \xi^{1/2} (1-\xi)^{1/2} \cos \delta = G_0(\xi_0, \delta_0). \quad (124)$$

The second first integral (124) defines the stream function $G(\xi, \delta)$ such that

$$v_\xi = \dot{\xi} = 2b \frac{\partial G}{\partial \delta} e^{-sT_2}, \quad v_\delta = \dot{\delta} = -2b \frac{\partial G}{\partial \xi} e^{-sT_2}, \quad (125)$$

which describes steady-state vibrations of an elastic plate decaying with time.

Eliminating the variable δ from (121) and (124) and integrating over T_2 , we have

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi(1-\xi)} - G_0^2} = \frac{2b}{s} (e^{-sT_2} - 1). \quad (126)$$

The integral in the left-hand side of (126) can be transformed to an incomplete elliptic integral of the first kind and can be easily calculated using special tables [34].

The solution of (126) allows one to find the value $\xi(T_2)$ and, thus, to solve the problem under consideration.

If we represent $\dot{\delta}$ considering (121) as

$$\dot{\delta} = \frac{d\delta}{dT_2} = \frac{d\delta}{d\xi} \dot{\xi} = -2b \frac{d\delta}{d\xi} \xi^{1/2} (1-\xi)^{1/2} e^{-sT_2} \sin \delta \quad (127)$$

and retain all terms in (116) with due account for (120), then (116) could be rewritten as follows:

$$\frac{d \cos \delta}{d\xi} + \frac{1}{2} \frac{1-2\xi}{\xi(1-\xi)} \cos \delta = \frac{K}{2b\sqrt{\xi(1-\xi)}}. \quad (128)$$

Integrating (128) yields

$$G_1(\xi, \delta) = \xi^{1/2} (1-\xi)^{1/2} \cos \delta - \frac{1}{2} K b^{-1} \xi = G_1^0(\xi_0, \delta_0). \quad (129)$$

Note that relationships (125) and (126) are valid in this case as well.

5.1.2. Internal Resonance $\omega_1 = \omega_3 = \omega$. Now let us consider case (77), when $\omega_1 = \omega_3 = \omega$, but $\omega_2 \neq \omega$. Note that case (78), when $\omega_2 = \omega_3 = \omega$, but $\omega_1 \neq \omega$, could be treated similarly.

Then eliminating secular terms in (75), we obtain the following solvability equations:

$$\begin{aligned} 2i\omega_1 D_2 A_1 + \mu_1 (i\omega_1 \tau_1)^y A_1 + 2\zeta_1 k_5 A_1 A_3 \bar{A}_3 \\ + 2\zeta_1 k_7 \bar{A}_1 A_3^2 &= 0, \\ 2i\omega_2 D_2 A_2 + \mu_2 (i\omega_2 \tau_2)^y A_2 + 2\zeta_2 k_6 A_2 A_3 \bar{A}_3 &= 0, \\ 2i\omega_3 D_2 A_3 + \mu_3 (i\omega_3 \tau_3)^y A_3 + (\zeta_{13} k_2 + \zeta_{23} k_4) A_3^2 \bar{A}_3 \\ + \zeta_{13} k_7 A_1 \bar{A}_1 A_3 + \zeta_{23} k_8 A_2 \bar{A}_2 A_3 + \zeta_{13} k_7 A_1^2 \bar{A}_3 \\ &= 0. \end{aligned} \quad (130)$$

Representing functions A_i entering in (130) in the polar form and applying the same procedure as it has been carried out above for the other case of the 1:1 internal resonance, as a result we have

$$(a_1^2)' + s_1 a_1^2 = -2\omega^{-1} \zeta_1 k_7 a_1^2 a_3^2 \sin \delta, \quad (131)$$

$$(a_2^2)' + s_2 a_2^2 = 0, \quad (132)$$

$$(a_3^2)' + s_3 a_3^2 = \omega^{-1} \zeta_{13} k_7 a_1^2 a_3^2 \sin \delta, \quad (133)$$

$$\dot{\phi}_1 = \frac{1}{2} \sigma_1 + \omega^{-1} \zeta_1 k_5 a_3^2 + \omega^{-1} \zeta_1 k_7 a_3^2 \cos \delta, \quad (134)$$

$$\dot{\phi}_2 = \frac{1}{2} \sigma_2 + \omega_2^{-1} \zeta_2 k_6 a_3^2, \quad (135)$$

$$\begin{aligned} \dot{\phi}_3 = & \frac{1}{2} \sigma_3 + \frac{1}{2} \omega^{-1} \zeta_{13} k_7 a_1^2 + \frac{1}{2} \omega^{-1} \zeta_{23} k_8 a_2^2 \\ & + \frac{1}{2} \omega^{-1} (\zeta_{13} k_2 + \zeta_{23} k_4) a_3^2 \\ & + \frac{1}{2} \omega^{-1} \zeta_{13} k_7 a_1^2 \cos \delta, \end{aligned} \quad (136)$$

where the phase difference has the form $\delta = 2(\phi_3 - \phi_1)$.

Since (132) is independent of others from (131) to (136), its solution has the form

$$a_2^2 = c_2 e^{-s_2 T_2}, \quad (137)$$

where c_2 is a constant of integration to be determined from the initial conditions.

Introducing new functions $\xi_1(T_2)$ and $\xi_3(T_2)$ such that

$$a_1^2 = \frac{\zeta_1 k_7}{\omega} \xi_1 \exp(-s_1 T_2), \quad a_3^2 = \frac{\zeta_{13} k_7}{\omega} \xi_3 \exp(-s_3 T_2) \quad (138)$$

and adding (131) and (133) with due account for (138) yield

$$\dot{\xi}_1 e^{-s_1 T_2} + 2\dot{\xi}_3 e^{-s_3 T_2} = 0, \quad (139)$$

while subtracting (134) from (136) we obtain

$$\begin{aligned} \dot{\delta} = 2(\dot{\phi}_3 - \dot{\phi}_1) = & \Sigma + \omega^{-1} (\zeta_{13} k_2 + \zeta_{23} k_4 - 2\zeta_1 k_5) a_3^2 \\ & + \omega^{-1} \zeta_{13} k_7 a_1^2 + \omega^{-1} \zeta_{23} k_8 a_2^2 \\ & + \omega^{-1} k_7 (\zeta_{13} a_1^2 - 2\zeta_1 a_3^2) \cos \delta, \end{aligned} \quad (140)$$

where $\Sigma = \sigma_3 - \sigma_1$.

From (131) and (133) we could find that

$$-2 \frac{\zeta_1 k_7}{\omega} a_3^2 = \frac{\dot{\xi}_1}{\xi_1} \frac{1}{\sin \delta}, \quad \frac{\zeta_{13} k_7}{\omega} a_1^2 = \frac{\dot{\xi}_3}{\xi_3} \frac{1}{\sin \delta}. \quad (141)$$

Considering (138) and (141), (131) and (140) could be reduced to

$$\begin{aligned} \dot{\xi}_1 = & -2b \xi_1 \xi_3 e^{-s_3 T_2} \sin \delta, \\ \dot{\delta} - \Sigma = & b \xi_1 e^{-s_1 T_2} + K_2 e^{-s_2 T_2} + K_3 \xi_3 e^{-s_3 T_2} \\ & + \left(\frac{\dot{\xi}_1}{\xi_1} + \frac{\dot{\xi}_3}{\xi_3} \right) \cot \delta \\ = & b \xi_1 e^{-s_1 T_2} + K_2 e^{-s_2 T_2} \\ & + K_3 \xi_3 e^{-s_3 T_2} + b (\xi_1 e^{-s_1 T_2} - 2\xi_3 e^{-s_3 T_2}) \cos \delta, \end{aligned} \quad (142)$$

where

$$\begin{aligned} b = & \omega^{-2} k_7^2 \zeta_1 \zeta_{13}, \quad K_2 = \omega^{-1} \zeta_{23} k_8 c_2, \\ K_3 = & \omega^{-2} \zeta_{13} k_7 (\zeta_{13} k_2 + \zeta_{23} k_4 - 2\zeta_1 k_5). \end{aligned} \quad (143)$$

The nonlinear set of equations (139) and (142) with the initial conditions (117) completely describes the vibrational process of the mechanical system being investigated under the condition of the one-to-one internal resonance $\omega_1 = \omega_3$ and could be solved numerically.

Particular Case. In the particular case at $\Sigma = 0$ and $s_1 = s_2 = s_3 = s$, (139) has the form

$$\dot{\xi}_1 + 2\dot{\xi}_3 = 0, \quad (144)$$

whence it follows that

$$\xi_1 + 2\xi_3 = E_0, \quad (145)$$

$$\xi_1 = E_0 \xi, \quad \xi_3 = \frac{1}{2} E_0 (1 - \xi). \quad (146)$$

Considering (146), (142) are reduced to

$$\begin{aligned} \dot{\xi} = & -b E_0 \xi (1 - \xi) e^{-s T_2} \sin \delta, \\ \dot{\delta} = & \left(b E_0 \xi + K_2 + \frac{1}{2} K_3 E_0 (1 - \xi) \right) e^{-s T_2} + \left(\frac{\dot{\xi}}{\xi} + \frac{\dot{\xi}_3}{\xi_3} \right) \cot \delta \\ = & \left[b E_0 \xi + K_2 + \frac{1}{2} K_3 (1 - \xi) \right] e^{-s T_2} \\ & - b E_0 (1 - 2\xi) e^{-s T_2} \cos \delta, \end{aligned} \quad (147)$$

while (148), wherein its first term rapidly decaying with time has been neglected, takes the form

$$\dot{\delta} \tan \delta \approx \frac{\dot{\xi}_1}{\xi_1} + \frac{\dot{\xi}_3}{\xi_3}, \quad (149)$$

and it could be integrated, resulting in its first integral

$$G(\xi, \delta) = \xi (1 - \xi) \cos \delta = G_0(\xi_0, \delta_0). \quad (150)$$

This first integral (150) defines the stream function $G(\xi, \delta)$ such that

$$v_\xi = \dot{\xi} = bE_0 \frac{\partial G}{\partial \delta} e^{-sT_2}, \quad v_\delta = \dot{\delta} = -bE_0 \frac{\partial G}{\partial \xi} e^{-sT_2}, \quad (151)$$

which describes steady-state vibrations of an elastic plate decaying with time.

Eliminating the variable δ from (147) and (150) and integrating over T_2 , we have

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi^2 (1-\xi)^2 - G_0^2}} = \frac{bE_0}{s} (e^{-sT_2} - 1). \quad (152)$$

The solution of (152) allows one to find the value $\xi(T_2)$ and, thus, to solve the problem under consideration.

If we represent $\dot{\delta}$ considering (147) as

$$\begin{aligned} \dot{\delta} &= \frac{d\delta}{dT_2} = \frac{d\delta}{d\xi} \dot{\xi} = -bE_0 \frac{d\delta}{d\xi} \xi (1-\xi) e^{-sT_2} \sin \delta \\ &= bE_0 \frac{d \cos \delta}{d\xi} \xi (1-\xi) e^{-sT_2} \end{aligned} \quad (153)$$

and retain all terms in (148) with due account for (146), then (148) could be rewritten as follows:

$$\frac{d \cos \delta}{d\xi} + \frac{1-2\xi}{\xi(1-\xi)} \cos \delta = \frac{K_3}{b\xi} + \frac{1}{1-\xi} + \frac{K_2}{bE_0 \xi (1-\xi)}. \quad (154)$$

Integrating (154) yields

$$\begin{aligned} G_1(\xi, \delta) &= \xi(1-\xi) \cos \delta + \frac{1}{2} K_3 b^{-1} (1-\xi)^2 \\ &\quad - \frac{1}{2} \xi^2 - K_2 (bE_0)^{-1} \xi = G_1^0(\xi_0, \delta_0). \end{aligned} \quad (155)$$

Note that relationships (151) and (152) are valid in this case as well.

Thus, during free vibrations of the plate being under the conditions of the one-to-one internal resonance three regimes can be observed: stationary (absence of damping at $\gamma = 0$), quasi-stationary (damping is defined by an ordinary derivative at $\gamma = 1$), and transient (damping is defined by a fractional derivative at $0 < \gamma < 1$).

5.1.3. Internal Resonance 1:1:1. Now let us consider case (79), when $\omega_1 = \omega_2 = \omega_3 = \omega$.

Then eliminating secular terms in (75), we obtain the following solvability equations:

$$\begin{aligned} 2i\omega_1 D_2 A_1 + \mu_1 (i\omega_1 \tau_1)^y A_1 + 2\zeta_1 k_5 A_1 A_3 \bar{A}_3 \\ + 2\zeta_1 k_6 A_2 A_3 \bar{A}_3 + 2\zeta_1 k_7 \bar{A}_1 A_3^2 + 2\zeta_1 k_8 \bar{A}_2 A_3^2 = 0, \\ 2i\omega_2 D_2 A_2 + \mu_2 (i\omega_2 \tau_2)^y A_2 + 2\zeta_2 k_6 A_2 A_3 \bar{A}_3 \\ + 2\zeta_2 k_5 A_1 A_3 \bar{A}_3 + 2\zeta_2 k_7 \bar{A}_1 A_3^2 + 2\zeta_2 k_8 \bar{A}_2 A_3^2 = 0, \\ 2i\omega_3 D_2 A_3 + \mu_3 (i\omega_3 \tau_3)^y A_3 + (\zeta_{13} k_2 + \zeta_{23} k_4) A_3^2 \bar{A}_3 \\ + \zeta_{13} k_7 A_1 \bar{A}_1 A_3 + \zeta_{23} k_8 A_2 \bar{A}_2 A_3 + \zeta_{13} k_7 A_1^2 \bar{A}_3 \\ + \zeta_{23} k_8 A_2^2 \bar{A}_3 + (\zeta_{13} k_6 + \zeta_{23} k_7) \bar{A}_1 A_2 A_3 \\ + (\zeta_{13} k_8 + \zeta_{23} k_5) A_1 \bar{A}_2 A_3 = 0. \end{aligned} \quad (156)$$

Representing functions A_i entering in (151) in the polar form and applying the same procedure as it has been carried out above for the cases of the 1:1 internal resonance, as a result we have

$$\begin{aligned} (a_1^2)' + s_1 a_1^2 &= -2\omega^{-1} \zeta_1 \\ &\quad \cdot [k_7 a_1^2 \sin 2(\varphi_3 - \varphi_1) \\ &\quad + k_6 a_1 a_2 \sin(\varphi_2 - \varphi_1) \\ &\quad + k_8 a_1 a_2 \sin(2\varphi_3 - \varphi_2 - \varphi_1)] a_3^2, \\ (a_2^2)' + s_2 a_2^2 &= -2\omega^{-1} \zeta_2 \\ &\quad \cdot [k_8 a_2^2 \sin 2(\varphi_3 - \varphi_2) \\ &\quad - k_5 a_1 a_2 \sin(\varphi_2 - \varphi_1) \\ &\quad + k_7 a_1 a_2 \sin(2\varphi_3 - \varphi_2 - \varphi_1)] a_3^2, \\ (a_3^2)' + s_3 a_3^2 &= \omega^{-1} [\zeta_{13} k_7 a_1^2 \sin 2(\varphi_3 - \varphi_1) \\ &\quad + \zeta_{23} k_8 a_2^2 \sin 2(\varphi_3 - \varphi_2)] a_3^2, \\ \dot{\varphi}_1 &= \frac{1}{2} \sigma_1 + \omega^{-1} \zeta_1 \\ &\quad \cdot [k_5 + k_7 \cos 2(\varphi_3 - \varphi_1) \\ &\quad + k_6 a_1^{-1} a_2 \cos(\varphi_2 - \varphi_1) \\ &\quad + k_8 a_1^{-1} a_2 \cos(2\varphi_3 - \varphi_2 - \varphi_1)] a_3^2, \\ \dot{\varphi}_2 &= \frac{1}{2} \sigma_2 + \omega^{-1} \zeta_2 \\ &\quad \cdot [k_6 + k_8 \cos 2(\varphi_3 - \varphi_2) \\ &\quad + k_5 a_1 a_2^{-1} \cos(\varphi_2 - \varphi_1) \\ &\quad + k_7 a_1 a_2^{-1} \cos(2\varphi_3 - \varphi_2 - \varphi_1)] a_3^2, \end{aligned}$$

$$\begin{aligned}
\dot{\varphi}_3 = & \frac{1}{2}\sigma_3 + \frac{1}{2}\omega^{-1} \\
& \cdot [\zeta_{13}k_7a_1^2 + \zeta_{23}k_8a_2^2 + (\zeta_{13}k_2 + \zeta_{23}k_4)a_3^2] \\
& + \frac{1}{2}\omega^{-1} [\zeta_{13}k_7a_1^2 \cos 2(\varphi_3 - \varphi_1) \\
& + \zeta_{23}k_8a_2^2 \cos 2(\varphi_3 - \varphi_2)] \\
& - \frac{2}{3}\omega^{-3}\zeta_{13}\zeta_{23}a_1a_2 \cos(\varphi_2 - \varphi_1).
\end{aligned} \tag{157}$$

The nonlinear set of equations (157) with the initial conditions (117) completely describes the vibrational process of the mechanical system being investigated under the condition of the internal resonance 1:1:1 and could be solved numerically.

5.2. Combinational Resonances of the Additive-Difference Type. Reference to (75) shows that three cases of the combinational internal resonance of the additive-difference type (80)–(82) could occur for the plate under consideration.

5.2.1. Combinational Resonance: $2\omega_3 = \omega_1 + \omega_2$. Let us first consider case (81), when $2\omega_3 = \omega_1 + \omega_2$.

Then eliminating secular terms in (75), we obtain the following solvability equations:

$$\begin{aligned}
2i\omega_1 D_2 A_1 + \mu_1 (i\omega_1 \tau_1)^y A_1 + 2\zeta_1 k_5 A_1 A_3 \bar{A}_3 \\
+ 2\zeta_1 k_8 \bar{A}_2 A_3^2 = 0,
\end{aligned} \tag{158}$$

$$\begin{aligned}
2i\omega_2 D_2 A_2 + \mu_2 (i\omega_2 \tau_2)^y A_2 + 2\zeta_2 k_6 A_2 A_3 \bar{A}_3 \\
+ 2\zeta_2 k_7 \bar{A}_1 A_3^2 = 0,
\end{aligned} \tag{159}$$

$$\begin{aligned}
2i\omega_3 D_2 A_3 + \mu_3 (i\omega_3 \tau_3)^y A_3 + (\zeta_{13}k_2 + \zeta_{23}k_4) A_3^2 \bar{A}_3 \\
+ \zeta_{13}k_7 A_1 \bar{A}_1 A_3 + \zeta_{23}k_8 A_2 \bar{A}_2 A_3 \\
+ (\zeta_{13}k_8 + \zeta_{23}k_7) A_1 A_2 \bar{A}_3 = 0.
\end{aligned} \tag{160}$$

Let us multiply (158)–(160), respectively, by $\bar{A}_1$, $\bar{A}_2$, and $\bar{A}_3$ and find their complex conjugates. Adding every pair of the mutually adjoint equations with each other and subtracting one from another and considering that the functions $A_i(T_2)$ could be represented in the polar form (86), where a_i and φ_i ($i = 1, 2, 3$) are T_2 -dependent, as a result we have

$$(a_1^2)' + s_1 a_1^2 = -2\omega_1^{-1} \zeta_1 k_8 a_1 a_2 a_3^2 \sin \delta, \tag{161}$$

$$(a_2^2)' + s_2 a_2^2 = -2\omega_2^{-1} \zeta_2 k_7 a_1 a_2 a_3^2 \sin \delta, \tag{162}$$

$$(a_3^2)' + s_3 a_3^2 = \omega_3^{-1} (\zeta_{13}k_8 + \zeta_{23}k_7) a_1 a_2 a_3^2 \sin \delta, \tag{163}$$

$$\dot{\varphi}_1 = \frac{1}{2}\sigma_1 + \omega_1^{-1} \zeta_1 k_5 a_3^2 + \omega_1^{-1} \zeta_1 k_8 a_1^{-1} a_2 a_3^2 \cos \delta, \tag{164}$$

$$\dot{\varphi}_2 = \frac{1}{2}\sigma_2 + \omega_2^{-1} \zeta_2 k_6 a_3^2 + \omega_2^{-1} \zeta_2 k_7 a_1 a_2^{-1} a_3^2 \cos \delta, \tag{165}$$

$$\begin{aligned}
\dot{\varphi}_3 = & \frac{1}{2}\sigma_3 + \frac{1}{2}\omega_3^{-1} \zeta_{13} k_7 a_1^2 + \frac{1}{2}\omega_3^{-1} \zeta_{23} k_8 a_2^2 \\
& + \frac{1}{2}\omega_3^{-1} (\zeta_{13}k_2 + \zeta_{23}k_4) a_3^2 \\
& + \frac{1}{2}\omega_3^{-1} (\zeta_{13}k_8 + \zeta_{23}k_7) a_1 a_2 \cos \delta,
\end{aligned} \tag{166}$$

where the phase difference has the form $\delta = 2\varphi_3 - \varphi_2 - \varphi_1$. Introducing new functions $\xi_1(T_2)$, $\xi_2(T_2)$, and $\xi_3(T_2)$ such that

$$\begin{aligned}
a_1^2 = \frac{\zeta_1 k_8}{\omega_1} \xi_1 \exp(-s_1 T_2), \quad a_2^2 = \frac{\zeta_2 k_7}{\omega_2} \xi_2 \exp(-s_2 T_2), \\
a_3^2 = \frac{\zeta_{13} k_8 + \zeta_{23} k_7}{\omega_3} \xi_3 \exp(-s_3 T_2)
\end{aligned} \tag{167}$$

and adding (161)–(163) with due account for (167) yield

$$\dot{\xi}_1 e^{-s_1 T_2} + \dot{\xi}_2 e^{-s_2 T_2} + 4\dot{\xi}_3 e^{-s_3 T_2} = 0, \tag{168}$$

while subtracting (164) and (165) from the doubled (166) we obtain

$$\begin{aligned}
\dot{\delta} = & 2\dot{\varphi}_3 - \dot{\varphi}_1 - \dot{\varphi}_2 \\
= & \Sigma + \left[\omega_3^{-1} (\zeta_{13}k_2 + \zeta_{23}k_4) \right. \\
& - \omega_1^{-1} \zeta_1 k_5 - \omega_2^{-1} \zeta_2 k_6 \left. \right] a_3^2 \\
& + \omega_3^{-1} \zeta_{13} k_7 a_1^2 + \omega_3^{-1} \zeta_{23} k_8 a_2^2 \\
& + \left[\omega_3^{-1} (\zeta_{13}k_8 + \zeta_{23}k_7) a_1 a_2 - \omega_1^{-1} \zeta_1 k_8 \frac{a_2 a_3^2}{a_1} \right. \\
& \left. - \omega_2^{-1} \zeta_2 k_7 \frac{a_1 a_3^2}{a_2} \right] \cos \delta,
\end{aligned} \tag{169}$$

where $2\Sigma = 2\sigma_3 - \sigma_1 - \sigma_2$.

From (161)–(163) we could find that

$$\begin{aligned}
-2 \frac{\zeta_1 k_8}{\omega_1} \frac{a_2 a_3^2}{a_1} = \frac{\dot{\xi}_1}{\xi_1} \frac{1}{\sin \delta}, \quad -2 \frac{\zeta_2 k_7}{\omega_2} \frac{a_1 a_3^2}{a_2} = \frac{\dot{\xi}_2}{\xi_2} \frac{1}{\sin \delta}, \\
\frac{\zeta_{13} k_8 + \zeta_{23} k_7}{\omega_3} a_1 a_2 = \frac{\dot{\xi}_3}{\xi_3} \frac{1}{\sin \delta}.
\end{aligned} \tag{170}$$

Then (161), (163), and (169) could be rewritten in the following form:

$$\dot{\xi}_1 = -2b\sqrt{\xi_1\xi_2\xi_3}e^{-(s_3+1/2s_2-1/2s_1)T_2}\sin\delta, \quad (171)$$

$$\dot{\xi}_3 = b\sqrt{\xi_1\xi_2\xi_3}e^{-1/2(s_2+s_1)T_2}\sin\delta, \quad (172)$$

$$\begin{aligned} \delta - \Sigma &= K_1\xi_1e^{-s_1T_2} + K_2\xi_2e^{-s_2T_2} + K_3\xi_3e^{-s_3T_2} \\ &+ \left(\frac{1}{2}\frac{\dot{\xi}_1}{\xi_1} + \frac{1}{2}\frac{\dot{\xi}_2}{\xi_2} + \frac{\dot{\xi}_3}{\xi_3}\right)\cot\delta \\ &= K_1\xi_1e^{-s_1T_2} + K_2\xi_2e^{-s_2T_2} + K_3\xi_3e^{-s_3T_2} \\ &+ b\left(\sqrt{\xi_1\xi_2}e^{-1/2(s_1+s_2)T_2} \right. \\ &\quad \left. - \frac{\xi_3\sqrt{\xi_2}}{\sqrt{\xi_1}}e^{-(s_3+1/2s_2-1/2s_1)T_2} \right. \\ &\quad \left. - \frac{\xi_3\sqrt{\xi_1}}{\sqrt{\xi_2}}e^{-(s_3+1/2s_1-1/2s_2)T_2}\right)\cos\delta, \end{aligned} \quad (173)$$

where

$$\begin{aligned} b &= \frac{\zeta_{13}k_8 + \zeta_{23}k_7}{\omega_3}\sqrt{\frac{\zeta_1\zeta_2k_7k_8}{\omega_1\omega_2}}, \quad K_1 = \frac{\zeta_1\zeta_{13}k_7k_8}{\omega_1\omega_3}, \\ K_2 &= \frac{\zeta_2\zeta_{23}k_7k_8}{\omega_2\omega_3}, \\ K_3 &= \frac{\zeta_{13}k_8 + \zeta_{23}k_7}{2\omega_3}\left(\frac{\zeta_{13}k_2 + \zeta_{23}k_4}{\omega_3} - \frac{\zeta_1k_5}{\omega_1} - \frac{\zeta_2k_6}{\omega_2}\right). \end{aligned} \quad (174)$$

The nonlinear set of equations (168), (171), (172), and (173) with the initial conditions (117) completely describes the vibrational process of the mechanical system being investigated under the condition of the combinational internal resonance $2\omega_3 = \omega_1 + \omega_2$ and could be solved numerically. This case has been discussed in [35].

Particular Case. In the particular case at $\Sigma = 0$ and $s_1 = s_2 = s_3 = s$, (168) has the form

$$\dot{\xi}_1 + \dot{\xi}_2 + 4\dot{\xi}_3 = 0, \quad (175)$$

whence it follows that

$$\xi_1 + \xi_2 + 4\xi_3 = E_0, \quad (176)$$

$$\begin{aligned} \xi_1 &= 2E_0(c_1 - \xi), \quad \xi_2 = 2E_0(c_2 - \xi), \\ \xi_3 &= E_0(c_3 + \xi), \end{aligned} \quad (177)$$

where c_1 , c_2 , and c_3 are constants of integration such that

$$2c_1 + 2c_2 + 4c_3 = 1. \quad (178)$$

Considering (177), (171) and (173) are reduced to

$$\dot{\xi} = 2bE_0\sqrt{(c_1 - \xi)(c_2 - \xi)(c_3 + \xi)}e^{-sT_2}\sin\delta, \quad (179)$$

$$\begin{aligned} \delta &= 2E_0[K_1(c_1 - \xi) + K_2(c_2 - \xi) + K_3(c_3 + \xi)]e^{-sT_2} \\ &+ \left(\frac{1}{2}\frac{\dot{\xi}_1}{\xi_1} + \frac{1}{2}\frac{\dot{\xi}_2}{\xi_2} + \frac{\dot{\xi}_3}{\xi_3}\right)\cot\delta \\ &= 2E_0[K_1(c_1 - \xi) + K_2(c_2 - \xi) + K_3(c_3 + \xi)]e^{-sT_2} \\ &- bE_0\left[\sqrt{(c_1 - \xi)(c_2 - \xi)} - \frac{(c_3 + \xi)\sqrt{c_2 - \xi}}{\sqrt{c_1 - \xi}} \right. \\ &\quad \left. - \frac{(c_3 + \xi)\sqrt{c_1 - \xi}}{\sqrt{c_2 - \xi}}\right]e^{-sT_2}\cos\delta, \end{aligned} \quad (180)$$

while (180), wherein its first term rapidly decaying with time has been neglected, takes the form

$$\dot{\delta}\tan\delta \approx \frac{1}{2}\frac{\dot{\xi}_1}{\xi_1} + \frac{1}{2}\frac{\dot{\xi}_2}{\xi_2} + \frac{\dot{\xi}_3}{\xi_3}, \quad (181)$$

and it could be integrated, resulting in its first integral

$$G(\xi, \delta) = (c_3 + \xi)\sqrt{(c_1 - \xi)(c_2 - \xi)}\cos\delta = G_0(\xi_0, \delta_0). \quad (182)$$

This first integral (182) defines the stream function $G(\xi, \delta)$ such that

$$\nu_\xi = \dot{\xi} = -2bE_0\frac{\partial G}{\partial \delta}e^{-sT_2}, \quad \nu_\delta = \dot{\delta} = 2bE_0\frac{\partial G}{\partial \xi}e^{-sT_2}, \quad (183)$$

which describes steady-state vibrations of an elastic plate decaying with time.

Eliminating the variable δ from (179) and (182) and integrating over T_2 , we have

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{(c_3 + \xi)^2(c_1 - \xi)(c_2 - \xi) - G_0^2}} = \frac{2bE_0}{s}(1 - e^{-sT_2}). \quad (184)$$

The solution of (184) allows one to find the value $\xi(T_2)$ and, thus, to solve the problem under consideration.

If we represent $\dot{\delta}$ considering (179) as

$$\begin{aligned} \dot{\delta} &= \frac{d\delta}{dT_2} = \frac{d\delta}{d\xi}\dot{\xi} \\ &= 2bE_0\frac{d\delta}{d\xi}\sqrt{(c_1 - \xi)(c_2 - \xi)(c_3 + \xi)}e^{-sT_2}\sin\delta \\ &= -2bE_0\frac{d\cos\delta}{d\xi}\sqrt{(c_1 - \xi)(c_2 - \xi)(c_3 + \xi)}e^{-sT_2} \end{aligned} \quad (185)$$

and retain all terms in (180) with due account for (177), then (180) could be rewritten as follows:

$$\begin{aligned}
 & -\frac{d \cos \delta}{d\xi} + \left(\frac{1}{2} \frac{1}{c_1 - \xi} + \frac{1}{2} \frac{1}{c_2 - \xi} - \frac{1}{c_3 + \xi} \right) \cos \delta \\
 & = \frac{K_1 \sqrt{c_1 - \xi}}{b(c_3 + \xi) \sqrt{c_2 - \xi}} + \frac{K_2 \sqrt{c_2 - \xi}}{b(c_3 + \xi) \sqrt{c_1 - \xi}} \\
 & + \frac{K_3}{b \sqrt{(c_1 - \xi)(c_2 - \xi)}}.
 \end{aligned} \quad (186)$$

Integrating (186) yields

$$\begin{aligned}
 G_1(\xi, \delta) &= (c_3 + \xi) \sqrt{(c_1 - \xi)(c_2 - \xi)} \cos \delta \\
 & - \frac{1}{2} K_1 b^{-1} (c_1 - \xi)^2 - \frac{1}{2} K_2 b^{-1} (c_2 - \xi)^2 \\
 & + \frac{1}{2} K_3 b^{-1} (c_3 + \xi)^2 = G_1^0(\xi_0, \delta_0).
 \end{aligned} \quad (187)$$

Note that relationships (183) and (184) are valid in this case as well.

5.2.2. Combinational Resonance: $2\omega_3 = \omega_1 - \omega_2$. Now let us consider case (80), when $2\omega_3 = \omega_1 - \omega_2$.

Then eliminating secular terms in (75), we obtain the following solvability equations:

$$\begin{aligned}
 & 2i\omega_1 D_2 A_1 + \mu_1 (i\omega_1 \tau_1)^y A_1 + 2\zeta_1 k_5 A_1 A_3 \bar{A}_3 \\
 & + 2\zeta_1 k_6 A_2 A_3^2 = 0, \\
 & 2i\omega_2 D_2 A_2 + \mu_2 (i\omega_2 \tau_2)^y A_2 + 2\zeta_2 k_6 A_2 A_3 \bar{A}_3 \\
 & + 2\zeta_2 k_7 A_1 \bar{A}_3^2 = 0, \\
 & 2i\omega_3 D_2 A_3 + \mu_3 (i\omega_3 \tau_3)^y A_3 + (\zeta_{13} k_2 + \zeta_{23} k_4) A_3^2 \bar{A}_3 \\
 & + \zeta_{13} k_7 A_1 \bar{A}_1 A_3 + \zeta_{23} k_8 A_2 \bar{A}_2 A_3 \\
 & + (\zeta_{13} k_6 + \zeta_{23} k_7) A_1 \bar{A}_2 \bar{A}_3 = 0.
 \end{aligned} \quad (188)$$

Applying to (188) the same procedure as it has been done above for (158)–(160), as a result we have

$$(a_1^2)' + s_1 a_1^2 = -2\omega_1^{-1} \zeta_1 k_6 a_1 a_2 a_3^2 \sin \delta, \quad (189)$$

$$(a_2^2)' + s_2 a_2^2 = 2\omega_2^{-1} \zeta_2 k_7 a_1 a_2 a_3^2 \sin \delta, \quad (190)$$

$$(a_3^2)' + s_3 a_3^2 = \omega_3^{-1} (\zeta_{13} k_6 + \zeta_{23} k_7) a_1 a_2 a_3^2 \sin \delta, \quad (191)$$

$$\dot{\varphi}_1 = \frac{1}{2} \sigma_1 + \omega_1^{-1} \zeta_1 k_5 a_3^2 + \omega_1^{-1} \zeta_1 k_6 a_1^{-1} a_2 a_3^2 \cos \delta, \quad (192)$$

$$\dot{\varphi}_2 = \frac{1}{2} \sigma_2 + \omega_2^{-1} \zeta_2 k_6 a_3^2 + \omega_2^{-1} \zeta_2 k_7 a_1 a_2^{-1} a_3^2 \cos \delta, \quad (193)$$

$$\begin{aligned}
 \dot{\varphi}_3 &= \frac{1}{2} \sigma_3 + \frac{1}{2} \omega_3^{-1} \zeta_{13} k_7 a_1^2 + \frac{1}{2} \omega_3^{-1} \zeta_{23} k_8 a_2^2 \\
 & + \frac{1}{2} \omega_3^{-1} (\zeta_{13} k_2 + \zeta_{23} k_4) a_3^2 \\
 & + \frac{1}{2} \omega_3^{-1} (\zeta_{13} k_6 + \zeta_{23} k_7) a_1 a_2 \cos \delta,
 \end{aligned} \quad (194)$$

where the phase difference has the form $\delta = 2\varphi_3 + \varphi_2 - \varphi_1$.

Introducing new functions $\xi_1(T_2)$, $\xi_2(T_2)$, and $\xi_3(T_2)$ such that

$$\begin{aligned}
 a_1^2 &= \frac{\zeta_1 k_6}{\omega_1} \xi_1 \exp(-s_1 T_2), \quad a_2^2 = \frac{\zeta_2 k_7}{\omega_2} \xi_2 \exp(-s_2 T_2), \\
 a_3^2 &= \frac{\zeta_{13} k_6 + \zeta_{23} k_7}{\omega_3} \xi_3 \exp(-s_3 T_2)
 \end{aligned} \quad (195)$$

and adding (189)–(191) with due account for (195) yield

$$2\dot{\xi}_1 e^{-s_1 T_2} + \dot{\xi}_2 e^{-s_2 T_2} + 2\dot{\xi}_3 e^{-s_3 T_2} = 0, \quad (196)$$

while subtracting (192) from the sum of (193) and the doubled (194) we obtain

$$\begin{aligned}
 \dot{\delta} &= 2\dot{\varphi}_3 - \dot{\varphi}_1 + \dot{\varphi}_2 \\
 &= \Sigma + \left[\omega_3^{-1} (\zeta_{13} k_2 + \zeta_{23} k_4) \right. \\
 & \quad \left. - \omega_1^{-1} \zeta_1 k_5 + \omega_2^{-1} \zeta_2 k_6 \right] a_3^2 \\
 & + \omega_3^{-1} \zeta_{13} k_7 a_1^2 + \omega_3^{-1} \zeta_{23} k_8 a_2^2 \\
 & + \left[\omega_3^{-1} (\zeta_{13} k_6 + \zeta_{23} k_7) a_1 a_2 - \omega_1^{-1} \zeta_1 k_6 \frac{a_2 a_3^2}{a_1} \right. \\
 & \quad \left. + \omega_2^{-1} \zeta_2 k_7 \frac{a_1 a_3^2}{a_2} \right] \cos \delta,
 \end{aligned} \quad (197)$$

where $2\Sigma = 2\sigma_3 - \sigma_1 + \sigma_2$.

From (189)–(191) we could find that

$$\begin{aligned}
 -2 \frac{\zeta_1 k_6}{\omega_1} \frac{a_2 a_3^2}{a_1} &= \frac{\dot{\xi}_1}{\xi_1} \frac{1}{\sin \delta}, \quad 2 \frac{\zeta_2 k_7}{\omega_2} \frac{a_1 a_3^2}{a_2} = \frac{\dot{\xi}_2}{\xi_2} \frac{1}{\sin \delta}, \\
 \frac{\zeta_{13} k_6 + \zeta_{23} k_7}{\omega_3} a_1 a_2 &= \frac{\dot{\xi}_3}{\xi_3} \frac{1}{\sin \delta}.
 \end{aligned} \quad (198)$$

Then (189), (191), and (197) could be rewritten in the following form:

$$\dot{\xi}_1 = -2b\sqrt{\xi_1\xi_2}\xi_3e^{-(s_3+1/2s_2-1/2s_1)T_2}\sin\delta, \quad (199)$$

$$\dot{\xi}_3 = b\sqrt{\xi_1\xi_2\xi_3}e^{-1/2(s_2+s_1)T_2}\sin\delta, \quad (200)$$

$$\begin{aligned} \dot{\delta} - \Sigma &= K_1\xi_1e^{-s_1T_2} + K_2\xi_2e^{-s_2T_2} + K_3\xi_3e^{-s_3T_2} \\ &+ \left(\frac{1}{2}\frac{\dot{\xi}_1}{\xi_1} + \frac{1}{2}\frac{\dot{\xi}_2}{\xi_2} + \frac{\dot{\xi}_3}{\xi_3}\right)\cot\delta \\ &= K_1\xi_1e^{-s_1T_2} + K_2\xi_2e^{-s_2T_2} + K_3\xi_3e^{-s_3T_2} \\ &+ b\left(\sqrt{\xi_1\xi_2}e^{-1/2(s_1+s_2)T_2} \right. \\ &\quad \left. - \frac{\xi_3\sqrt{\xi_2}}{\sqrt{\xi_1}}e^{-(s_3+1/2s_2-1/2s_1)T_2} \right. \\ &\quad \left. + \frac{\xi_3\sqrt{\xi_1}}{\sqrt{\xi_2}}e^{-(s_3+1/2s_1-1/2s_2)T_2}\right)\cos\delta, \end{aligned} \quad (201)$$

where

$$\begin{aligned} b &= \frac{\zeta_{13}k_6 + \zeta_{23}k_7}{\omega_3}\sqrt{\frac{\zeta_1\zeta_2k_6k_7}{\omega_1\omega_2}}, \quad K_1 = \frac{\zeta_1\zeta_{13}k_6k_7}{\omega_1\omega_3}, \\ K_2 &= \frac{\zeta_2\zeta_{23}k_7k_8}{\omega_2\omega_3}, \quad (202) \\ K_3 &= \frac{\zeta_{13}k_6 + \zeta_{23}k_7}{2\omega_3}\left(\frac{\zeta_{13}k_2 + \zeta_{23}k_4}{\omega_3} - \frac{\zeta_1k_5}{\omega_1} + \frac{\zeta_2k_6}{\omega_2}\right). \end{aligned}$$

The nonlinear set of equations (196), (199), (200), and (201) with the initial conditions (117) completely describes the vibrational process of the mechanical system being investigated under the condition of the combinational internal resonance $2\omega_3 = \omega_1 - \omega_2$ and could be solved numerically.

Particular Case. In the particular case at $\Sigma = 0$ and $s_1 = s_2 = s_3 = s$, (196) has the form

$$2\dot{\xi}_1 + \dot{\xi}_2 + 2\dot{\xi}_3 = 0, \quad (203)$$

whence it follows that

$$2\xi_1 + \xi_2 + 2\xi_3 = E_0, \quad (204)$$

$$\begin{aligned} \xi_1 &= 2E_0(c_1 - \xi), \quad \xi_2 = 2E_0(c_2 + \xi), \\ \xi_3 &= E_0(c_3 + \xi), \end{aligned} \quad (205)$$

where c_1, c_2 , and c_3 are constants of integration such that

$$4c_1 + 2c_2 + 2c_3 = 1. \quad (206)$$

Considering (205), (199) and (201) are reduced to

$$\dot{\xi} = 2bE_0\sqrt{(c_1 - \xi)(c_2 + \xi)(c_3 + \xi)}e^{-sT_2}\sin\delta, \quad (207)$$

$$\begin{aligned} \dot{\delta} &= 2E_0[K_1(c_1 - \xi) + K_2(c_2 + \xi) + K_3(c_3 + \xi)]e^{-sT_2} \\ &+ \left(\frac{1}{2}\frac{\dot{\xi}_1}{\xi_1} + \frac{1}{2}\frac{\dot{\xi}_2}{\xi_2} + \frac{\dot{\xi}_3}{\xi_3}\right)\cot\delta \\ &= 2E_0[K_1(c_1 - \xi) + K_2(c_2 + \xi) + K_3(c_3 + \xi)]e^{-sT_2} \\ &- bE_0\left[\sqrt{(c_1 - \xi)(c_2 + \xi)} - \frac{(c_3 + \xi)\sqrt{c_2 + \xi}}{\sqrt{c_1 - \xi}} \right. \\ &\quad \left. + \frac{(c_3 + \xi)\sqrt{c_1 - \xi}}{\sqrt{c_2 + \xi}}\right]e^{-sT_2}\cos\delta, \end{aligned} \quad (208)$$

while (208), wherein its first term rapidly decaying with time has been neglected, takes the form

$$\dot{\delta}\tan\delta \approx \frac{1}{2}\frac{\dot{\xi}_1}{\xi_1} + \frac{1}{2}\frac{\dot{\xi}_2}{\xi_2} + \frac{\dot{\xi}_3}{\xi_3}, \quad (209)$$

and it could be integrated, resulting in its first integral

$$G(\xi, \delta) = (c_3 + \xi)\sqrt{(c_1 - \xi)(c_2 + \xi)}\cos\delta = G_0(\xi_0, \delta_0). \quad (210)$$

This first integral (210) defines the stream function $G(\xi, \delta)$ such that

$$v_\xi = \dot{\xi} = -2bE_0\frac{\partial G}{\partial \delta}e^{-sT_2}, \quad v_\delta = \dot{\delta} = 2bE_0\frac{\partial G}{\partial \xi}e^{-sT_2}, \quad (211)$$

which describes steady-state vibrations of an elastic plate decaying with time.

Eliminating the variable δ from (207) and (210) and integrating over T_2 , we have

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{(c_3 + \xi)^2(c_1 - \xi)(c_2 + \xi) - G_0^2}} = \frac{2bE_0}{s}(1 - e^{-sT_2}). \quad (212)$$

The solution of (212) allows one to find the value $\xi(T_2)$ and, thus, to solve the problem under consideration.

If we represent $\dot{\delta}$ considering (207) as

$$\begin{aligned} \dot{\delta} &= \frac{d\delta}{dT_2} = \frac{d\delta}{d\xi}\dot{\xi} \\ &= 2bE_0\frac{d\delta}{d\xi}\sqrt{(c_1 - \xi)(c_2 + \xi)(c_3 + \xi)}e^{-sT_2}\sin\delta \\ &= -2bE_0\frac{d\cos\delta}{d\xi}\sqrt{(c_1 - \xi)(c_2 + \xi)(c_3 + \xi)}e^{-sT_2}, \end{aligned} \quad (213)$$

and retain all terms in (208) with due account for (205), then (208) could be rewritten as follows:

$$\begin{aligned}
 & -\frac{d \cos \delta}{d \xi} + \left(\frac{1}{2} \frac{1}{c_1 - \xi} - \frac{1}{2} \frac{1}{c_2 + \xi} - \frac{1}{c_3 + \xi} \right) \cos \delta \\
 & = \frac{K_1 \sqrt{c_1 - \xi}}{b(c_3 + \xi) \sqrt{c_2 + \xi}} + \frac{K_2 \sqrt{c_2 + \xi}}{b(c_3 + \xi) \sqrt{c_1 - \xi}} \\
 & + \frac{K_3}{b \sqrt{(c_1 - \xi)(c_2 + \xi)}}.
 \end{aligned} \quad (214)$$

Integrating (214) yields

$$\begin{aligned}
 G_1(\xi, \delta) &= (c_3 + \xi) \sqrt{(c_1 - \xi)(c_2 + \xi)} \cos \delta \\
 & - \frac{1}{2} K_1 b^{-1} (c_1 - \xi)^2 + \frac{1}{2} K_2 b^{-1} (c_2 + \xi)^2 \\
 & + \frac{1}{2} K_3 b^{-1} (c_3 + \xi)^2 \\
 & = G_1^0(\xi_0, \delta_0).
 \end{aligned} \quad (215)$$

Note that relationships (211) and (212) are valid in this case as well.

5.2.3. Combinational Resonance: $2\omega_3 = \omega_2 - \omega_1$. Now let us consider the third possible case of the combinational resonance (82), that is, when $2\omega_3 = \omega_2 - \omega_1$.

Then eliminating secular terms in (75), we obtain the following solvability equations:

$$\begin{aligned}
 & 2i\omega_1 D_2 A_1 + \mu_1 (i\omega_1 \tau_1)^y A_1 + 2\zeta_1 k_5 A_1 A_3 \bar{A}_3 \\
 & + 2\zeta_1 k_8 A_2 \bar{A}_3^2 = 0, \\
 & 2i\omega_2 D_2 A_2 + \mu_2 (i\omega_2 \tau_2)^y A_2 + 2\zeta_2 k_6 A_2 A_3 \bar{A}_3 \\
 & + 2\zeta_2 k_5 A_1 A_3^2 = 0, \\
 & 2i\omega_3 D_2 A_3 + \mu_3 (i\omega_3 \tau_3)^y A_3 + (\zeta_{13} k_2 + \zeta_{23} k_4) A_3^2 \bar{A}_3 \\
 & + \zeta_{13} k_7 A_1 \bar{A}_1 A_3 + \zeta_{23} k_8 A_2 \bar{A}_2 A_3 \\
 & + (\zeta_{13} k_8 + \zeta_{23} k_5) \bar{A}_1 A_2 \bar{A}_3 = 0,
 \end{aligned} \quad (216)$$

reducing to six differential equations in amplitudes and phases of the three interacting modes:

$$(a_1^2)' + s_1 a_1^2 = 2\omega_1^{-1} \zeta_1 k_8 a_1 a_2 a_3^2 \sin \delta, \quad (217)$$

$$(a_2^2)' + s_2 a_2^2 = -2\omega_2^{-1} \zeta_2 k_5 a_1 a_2 a_3^2 \sin \delta, \quad (218)$$

$$(a_3^2)' + s_3 a_3^2 = \omega_3^{-1} (\zeta_{13} k_8 + \zeta_{23} k_5) a_1 a_2 a_3^2 \sin \delta, \quad (219)$$

$$\dot{\phi}_1 = \frac{1}{2} \sigma_1 + \omega_1^{-1} \zeta_1 k_5 a_3^2 + \omega_1^{-1} \zeta_1 k_8 a_1^{-1} a_2 a_3^2 \cos \delta, \quad (220)$$

$$\dot{\phi}_2 = \frac{1}{2} \sigma_2 + \omega_2^{-1} \zeta_2 k_6 a_3^2 + \omega_2^{-1} \zeta_2 k_5 a_1 a_2^{-1} a_3^2 \cos \delta, \quad (221)$$

$$\begin{aligned}
 \dot{\phi}_3 &= \frac{1}{2} \sigma_3 + \frac{1}{2} \omega_3^{-1} \zeta_{13} k_7 a_1^2 + \frac{1}{2} \omega_3^{-1} \zeta_{23} k_8 a_2^2 \\
 & + \frac{1}{2} \omega_3^{-1} (\zeta_{13} k_2 + \zeta_{23} k_4) a_3^2 \\
 & + \frac{1}{2} \omega_3^{-1} (\zeta_{13} k_8 + \zeta_{23} k_5) a_1 a_2 \cos \delta,
 \end{aligned} \quad (222)$$

where the phase difference has the form $\delta = 2\varphi_3 + \varphi_1 - \varphi_2$.

Introducing new functions $\xi_1(T_2)$, $\xi_2(T_2)$, and $\xi_3(T_2)$ such that

$$\begin{aligned}
 a_1^2 &= \frac{\zeta_1 k_8}{\omega_1} \xi_1 \exp(-s_1 T_2), \quad a_2^2 = \frac{\zeta_2 k_5}{\omega_2} \xi_2 \exp(-s_2 T_2), \\
 a_3^2 &= \frac{\zeta_{13} k_8 + \zeta_{23} k_5}{\omega_3} \xi_3 \exp(-s_3 T_2),
 \end{aligned} \quad (223)$$

and adding (217)–(219) with due account for (223) yield

$$\dot{\xi}_1 e^{-s_1 T_2} + 2\dot{\xi}_2 e^{-s_2 T_2} + 2\dot{\xi}_3 e^{-s_3 T_2} = 0, \quad (224)$$

while subtracting (221) from the sum of (220) and the doubled (222) we obtain

$$\begin{aligned}
 \dot{\delta} &= 2\dot{\phi}_3 + \dot{\phi}_1 - \dot{\phi}_2 \\
 &= \Sigma + \left[\omega_3^{-1} (\zeta_{13} k_2 + \zeta_{23} k_4) \right. \\
 & \quad \left. + \omega_1^{-1} \zeta_1 k_5 - \omega_2^{-1} \zeta_2 k_6 \right] a_3^2 \\
 & + \omega_3^{-1} \zeta_{13} k_7 a_1^2 + \omega_3^{-1} \zeta_{23} k_8 a_2^2 \\
 & + \left[\omega_3^{-1} (\zeta_{13} k_8 + \zeta_{23} k_5) a_1 a_2 + \omega_1^{-1} \zeta_1 k_8 \frac{a_2 a_3^2}{a_1} \right. \\
 & \quad \left. - \omega_2^{-1} \zeta_2 k_5 \frac{a_1 a_3^2}{a_2} \right] \cos \delta,
 \end{aligned} \quad (225)$$

where $2\Sigma = 2\sigma_3 + \sigma_1 - \sigma_2$.

From (217)–(219) we could find that

$$\begin{aligned}
 2 \frac{\zeta_1 k_8}{\omega_1} \frac{a_2 a_3^2}{a_1} &= \frac{\dot{\xi}_1}{\xi_1} \frac{1}{\sin \delta}, \quad -2 \frac{\zeta_2 k_5}{\omega_2} \frac{a_1 a_3^2}{a_2} = \frac{\dot{\xi}_2}{\xi_2} \frac{1}{\sin \delta}, \\
 \frac{\zeta_{13} k_8 + \zeta_{23} k_5}{\omega_3} a_1 a_2 &= \frac{\dot{\xi}_3}{\xi_3} \frac{1}{\sin \delta}.
 \end{aligned} \quad (226)$$

Then (217), (219), and (225) could be rewritten in the following form:

$$\dot{\xi}_1 = -2b\sqrt{\xi_1\xi_2\xi_3}e^{-(s_3+1/2s_2-1/2s_1)T_2}\sin\delta, \quad (227)$$

$$\dot{\xi}_3 = b\sqrt{\xi_1\xi_2\xi_3}e^{-1/2(s_2+s_1)T_2}\sin\delta, \quad (228)$$

$$\begin{aligned} \dot{\delta} - \Sigma &= K_1\xi_1e^{-s_1T_2} + K_2\xi_2e^{-s_2T_2} + K_3\xi_3e^{-s_3T_2} \\ &+ \left(\frac{1}{2}\frac{\dot{\xi}_1}{\xi_1} + \frac{1}{2}\frac{\dot{\xi}_2}{\xi_2} + \frac{\dot{\xi}_3}{\xi_3}\right)\cot\delta \\ &= K_1\xi_1e^{-s_1T_2} + K_2\xi_2e^{-s_2T_2} + K_3\xi_3e^{-s_3T_2} \\ &+ b\left(\sqrt{\xi_1\xi_2}e^{-1/2(s_1+s_2)T_2} \right. \\ &\quad \left. + \frac{\xi_3\sqrt{\xi_2}}{\sqrt{\xi_1}}e^{-(s_3+1/2s_2-1/2s_1)T_2} \right. \\ &\quad \left. - \frac{\xi_3\sqrt{\xi_1}}{\sqrt{\xi_2}}e^{-(s_3+1/2s_1-1/2s_2)T_2}\right)\cos\delta, \end{aligned} \quad (229)$$

where

$$\begin{aligned} b &= \frac{\zeta_{13}k_8 + \zeta_{23}k_5}{\omega_3}\sqrt{\frac{\zeta_1\zeta_2k_5k_8}{\omega_1\omega_2}}, \quad K_1 = \frac{\zeta_1\zeta_{13}k_7k_8}{\omega_1\omega_3}, \\ K_2 &= \frac{\zeta_2\zeta_{23}k_5k_8}{\omega_2\omega_3}, \quad (230) \\ K_3 &= \frac{\zeta_{13}k_8 + \zeta_{23}k_5}{2\omega_3}\left(\frac{\zeta_{13}k_2 + \zeta_{23}k_4}{\omega_3} + \frac{\zeta_1k_5}{\omega_1} - \frac{\zeta_2k_6}{\omega_2}\right). \end{aligned}$$

The nonlinear set of equations (224), (227), (228), and (229) with the initial conditions (117) completely describes the vibrational process of the mechanical system being investigated under the condition of the combinational internal resonance $2\omega_3 = \omega_2 - \omega_1$ and could be solved numerically.

Particular Case. In the particular case at $\Sigma = 0$ and $s_1 = s_2 = s_3 = s$, (224) has the form

$$\dot{\xi}_1 + 2\dot{\xi}_2 + 2\dot{\xi}_3 = 0, \quad (231)$$

whence it follows that

$$\xi_1 + 2\xi_2 + 2\xi_3 = E_0, \quad (232)$$

$$\begin{aligned} \xi_1 &= 2E_0(c_1 + \xi), \quad \xi_2 = 2E_0(c_2 - \xi), \\ \xi_3 &= E_0(c_3 + \xi), \end{aligned} \quad (233)$$

where c_1 , c_2 , and c_3 are constants of integration such that

$$2c_1 + 4c_2 + 2c_3 = 1. \quad (234)$$

Considering (233), (227) and (229) are reduced to

$$\dot{\xi} = 2bE_0\sqrt{(c_1 + \xi)(c_2 - \xi)(c_3 + \xi)}e^{-sT_2}\sin\delta, \quad (235)$$

$$\begin{aligned} \dot{\delta} &= 2E_0[K_1(c_1 + \xi) + K_2(c_2 - \xi) + K_3(c_3 + \xi)]e^{-sT_2} \\ &+ \left(\frac{1}{2}\frac{\dot{\xi}_1}{\xi_1} + \frac{1}{2}\frac{\dot{\xi}_2}{\xi_2} + \frac{\dot{\xi}_3}{\xi_3}\right)\cot\delta \\ &= 2E_0[K_1(c_1 + \xi) + K_2(c_2 - \xi) + K_3(c_3 + \xi)]e^{-sT_2} \\ &- bE_0\left[\sqrt{(c_1 + \xi)(c_2 - \xi)} + \frac{(c_3 + \xi)\sqrt{c_2 - \xi}}{\sqrt{c_1 + \xi}} \right. \\ &\quad \left. - \frac{(c_3 + \xi)\sqrt{c_1 + \xi}}{\sqrt{c_2 - \xi}}\right]e^{-sT_2}\cos\delta, \end{aligned} \quad (236)$$

while (236), wherein its first term rapidly decaying with time has been neglected, takes the form

$$\dot{\delta}\tan\delta \approx \frac{1}{2}\frac{\dot{\xi}_1}{\xi_1} + \frac{1}{2}\frac{\dot{\xi}_2}{\xi_2} + \frac{\dot{\xi}_3}{\xi_3}, \quad (237)$$

and it could be integrated, resulting in its first integral

$$G(\xi, \delta) = (c_3 + \xi)\sqrt{(c_1 + \xi)(c_2 - \xi)}\cos\delta = G_0(\xi_0, \delta_0). \quad (238)$$

This first integral (238) defines the stream function $G(\xi, \delta)$ such that

$$v_\xi = \dot{\xi} = -2bE_0\frac{\partial G}{\partial \delta}e^{-sT_2}, \quad v_\delta = \dot{\delta} = 2bE_0\frac{\partial G}{\partial \xi}e^{-sT_2}, \quad (239)$$

which describes steady-state vibrations of an elastic plate decaying with time.

Eliminating the variable δ from (235) and (238) and integrating over T_2 , we have

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{(c_3 + \xi)^2(c_1 + \xi)(c_2 - \xi) - G_0^2}} = \frac{2bE_0}{s}(1 - e^{-sT_2}). \quad (240)$$

The solution of (240) allows one to find the value $\xi(T_2)$ and, thus, to solve the problem under consideration.

If we represent $\dot{\delta}$ considering (235) as

$$\begin{aligned} \dot{\delta} &= \frac{d\delta}{dT_2} = \frac{d\delta}{d\xi}\dot{\xi} \\ &= 2bE_0\frac{d\delta}{d\xi}\sqrt{(c_1 + \xi)(c_2 - \xi)(c_3 + \xi)}e^{-sT_2}\sin\delta \\ &= -2bE_0\frac{d\cos\delta}{d\xi}\sqrt{(c_1 + \xi)(c_2 - \xi)(c_3 + \xi)}e^{-sT_2} \end{aligned} \quad (241)$$

and retain all terms in (236) with due account for (233), then (236) could be rewritten as follows:

$$\begin{aligned}
 & -\frac{d \cos \delta}{d \xi} + \left(-\frac{1}{2} \frac{1}{c_1 + \xi} + \frac{1}{2} \frac{1}{c_2 - \xi} - \frac{1}{c_3 + \xi} \right) \cos \delta \\
 & = \frac{K_1 \sqrt{c_1 + \xi}}{b(c_3 + \xi) \sqrt{c_2 - \xi}} + \frac{K_2 \sqrt{c_2 - \xi}}{b(c_3 + \xi) \sqrt{c_1 + \xi}} \\
 & + \frac{K_3}{b \sqrt{(c_1 + \xi)(c_2 - \xi)}}.
 \end{aligned} \quad (242)$$

Integrating (242) yields

$$\begin{aligned}
 G_1(\xi, \delta) &= (c_3 + \xi) \sqrt{(c_1 + \xi)(c_2 - \xi)} \cos \delta \\
 &+ \frac{1}{2} K_1 b^{-1} (c_1 + \xi)^2 - \frac{1}{2} K_2 b^{-1} (c_2 - \xi)^2 \\
 &+ \frac{1}{2} K_3 b^{-1} (c_3 + \xi)^2 = G_1^0(\xi_0, \delta_0).
 \end{aligned} \quad (243)$$

Note that relationships (239) and (240) are valid in this case as well.

6. Numerical Investigations

As examples, let us carry out the qualitative analysis of the cases of one-to-one internal resonance (76), when two different modes of in-plane vibrations are coupled, and (77), when one mode of in-plane vibrations is coupled with a certain mode of out-of-plane vibrations.

6.1. One-to-One Internal Resonance $\omega_1 = \omega_2$ (76). For this case, the stream function $G(\xi, \delta)$ is defined by relationship (129), and the phase portrait to be constructed according to (129) depends essentially on the magnitudes of the coefficient K .

6.1.1. Case (76) at $K = 0$. For this case, the stream function $G(\xi, \delta)$ is constructed according to (124) and the streamlines of the phase fluid in the phase plane ξ - δ are presented in Figure 2. Magnitudes of G are indicated by digits near the curves which correspond to the streamlines; the flow directions of the phase fluid elements are shown by arrows on the streamlines.

Reference to Figure 2 shows that the phase fluid flows within the circulation zones, which tend to be located around the perimeter of the rectangles bounded by the lines $\xi = 0$, $\xi = 1$, and $\delta = \pm(\pi/2) \pm 2\pi n$ ($n = 0, 1, 2, \dots$). As this takes place, the flow in each such rectangle becomes isolated. On all four rectangle sides $G = 0$ and inside it the value G preserves its sign. The function G attains its extreme magnitudes ± 0.5 at the points with the coordinates $\xi = 1/2$, $\delta = \pm\pi n$ ($n = 0, 1, 2, \dots$).

The transition of fluid elements from the points $\xi = 0$, $\delta = \pi/2 \pm 2\pi n$ to the points $\xi = 0$, $\delta = -\pi/2 \pm 2\pi n$ ($n = 0, 1, 2, \dots$) and from the points $\xi = 1$, $\delta = -\pi/2 \pm 2\pi n$ to the points $\xi = 1$, $\delta = \pi/2 \pm 2\pi n$ ($n = 0, 1, 2, \dots$) proceeds

FIGURE 2: Phase portrait for the case of 1:1 internal resonance $\omega_1 = \omega_2$ at $K = 0$.

instantly, because according to the distribution of the phase velocity along the section $\delta = 0$ the magnitude of $\mathbf{v}$ tends to infinity as $\xi \rightarrow 0$ and $\xi \rightarrow 1$, respectively. The distribution of the velocity along the vertical lines $\delta = \pm\pi n$ has the aperiodic character, while in the vicinity of the line $\xi = 1/2$ it possesses the periodic character.

The transition of fluid elements from the points $\xi = 1$, $\delta = -\pi/2 \pm 2\pi n$ to the points $\xi = 1$, $\delta = \pi/2 \pm 2\pi n$ ($n = 0, 1, 2, \dots$) proceeds instantly, because according to the distribution of the phase velocity along the section $\delta = 0$ the magnitude of $\mathbf{v}$ tends to infinity as $\xi \rightarrow 1$.

Along the lines $\delta = \pm(\pi/2) \pm 2\pi n$ ($n = 0, 1, 2, \dots$) pure amplitude modulated aperiodic motions are realized, since with an increase in time t the value ξ increases from ξ_0 to 1 (along the line $\delta = -\pi/2$) or decreases from ξ_0 to 0 (along the line $\delta = \pi/2$), and solution (126) could be written in the form of a soliton-like solution:

$$-\arcsin(1 - 2\xi)|_{\xi_0}^{\xi} = \frac{2b}{s} (e^{-sT_2} - 1), \quad (244)$$

$$\delta(T_2) = \delta_0 = \frac{\pi}{2} \pm \pi n, \quad n = 0, 1, 2, \dots \quad (245)$$

Streamlines give a pictorial estimate of the connection of G with all types of the energy-exchange mechanism. Thus, in the case of undamped vibrations, that is, when the damping coefficient is equal to zero and $s = 0$, the points with the coordinates $\xi_0 = 1/2$, $\delta_0 = \pm\pi n$ ($n = 0, 1, 2, \dots$) correspond to the stationary regime, since $\dot{\xi} = 0$ and $\dot{\delta} = 0$ according to (125). The stationary points $\xi_0 = 1/2$, $\delta_0 = \pm\pi n$ are centers; as with a small deviation from a center, a phase element begins to move around the stationary point along a closed trajectory. Closed streamlines correspond to the periodic change of both amplitudes and phases.

The T_2 -dependence of the values ξ and $1 - \xi$ which are proportional to the square of the amplitudes a_1 and a_2 , respectively, at $\delta_0 = \pi/4$ and $\xi_0 = 0.5$ is presented in Figure 2, wherein the solid and dashed lines correspond, respectively, to the cases of damped and undamped vibrations. Reference to Figure 3 shows the damping of the energy exchange between two subsystems. Moreover, the points of touching of solid curves, that is, the points of tangency of the envelopes of vibrations ($\xi_i = 1 - \xi_i$, T_2, i), allow one to determine the value

FIGURE 3: The T_2^* -dependence of the dimensionless amplitudes of vibrations for the case of 1:1 internal resonance.

FIGURE 4: Phase portrait for the case of 1:1 internal resonance $\omega_1 = \omega_2$ at $(1/2)Kb^{-1} = 0.1$.

characterizing the damping of energy during vibrations via the formula

$$\frac{\ln(a_{i+1}/a_i)}{T_{2(i-1)} - T_{2i}}, \quad (246)$$

where a_i are defined by (110) with account for (120).

6.1.2. Case (76) at $(1/2)Kb^{-1} = 0.1$. In this case, the stream function $G(\xi, \delta)$ (129) is defined as

$$G(\xi, \delta) = \sqrt{\xi(1-\xi)} \cos \delta - 0.1\xi = G(\xi_0, \delta_0), \quad (247)$$

and Figure 4 shows the streamlines of the phase fluid in the phase plane.

As in the previous case, the phase fluid flows in an infinitely long channel, the boundaries of which are the straight lines $\xi = 0$ and $\xi = 1$, corresponding to the phase modulated motions. In one part the streamlines are nonclosed, which corresponds to the periodic change of amplitudes and the aperiodic change of phases; in another part they are closed, which corresponds to the periodic change of both amplitudes and phases. The aperiodic regime lines are the boundaries of the closed and unclosed streamline areas. From the phase portrait in Figure 4 it is seen that the circulation zones are located in a staggered arrangement by the right and left channel sides (this configuration resembles that of von Kármán staggered vortex tracks).

Each zone by the side $\xi = 1$ is surrounded by a line with the value $G = -0.1$. However, only those parts of the line $G =$

-0.1 which bound these zones from below and come closer to the side $\xi = 1$ at the points $\xi = 1, \delta = \pi/2 \pm \pi n$ belong to the domain of the fluid flow. The transition of fluid elements from the points $\xi = 1, \delta = (3\pi/2) \pm \pi n$ to the points $\xi = 0, \delta = (\pi/2) \pm \pi n$ proceeds instantly. The soliton-like solution along the separatrix is written as

$$\frac{1}{\sqrt{1.01}} \arcsin [1 - 2.02(1 - \xi)] \Big|_{1-\xi_0}^{1-\xi} = \frac{2b}{s} (e^{-sT_2} - 1),$$

$$\cos \delta = -0.1 \sqrt{\frac{1-\xi}{\xi}}, \quad (248)$$

wherein ξ varies from $\xi_{\min} = 0.0099$ to $\xi_{\max} = 1$.

The circulation zones by the side $\xi = 0$ are surrounded by the line with the value $G = 0$. However, only those parts of the line $G = 0$ which bound these zones from above and come closer to the side $\xi = 0$ at the points $\xi = 0, \delta = \pi/2 \pm \pi n$ belong to the domain of the fluid flow. The transition of fluid elements from the points $\xi = 0, \delta = (\pi/2) \pm \pi n$ to the points $\xi = 0, \delta = (-\pi/2) \pm \pi n$ proceeds instantly. The soliton-like solution along the separatrix line $G = 0$ has the form

$$-\frac{1}{\sqrt{1.01}} \arcsin (1 - 2.02\xi) \Big|_{\xi_0}^{\xi} = \frac{2b}{s} (e^{-sT_2} - 1),$$

$$\cos \delta = 0.1 \sqrt{\frac{\xi}{1-\xi}}, \quad (249)$$

wherein ξ varies from $\xi_{\min} = 0$ to $\xi_{\max} = 0.99$.

Inside the two circulation zones there are points with the extreme values of the stream function: maximal $G_{\max} = 0.4525$ and minimal $G_{\min} = -0.5525$, respectively. These points are the centers corresponding to the stable stationary regimes $\xi = \xi_0 = 0.45, \delta = \delta_0 = \pm 2\pi n$ and $\xi = \xi_0 = 0.55, \delta = \delta_0 = \pi \pm 2\pi n$, respectively.

Between the lines corresponding to $G = 0$ and $G = -0.1$, unclosed streamlines are located which are in accordance with the periodic change of the amplitudes and the aperiodic change of the phase difference.

6.1.3. Case (76) at $(1/2)Kb^{-1} = \pm 1$. In this case, the stream function $G(\xi, \delta)$ (129) is defined as

$$G(\xi, \delta) = \sqrt{\xi(1-\xi)} \cos \delta \mp \xi = G(\xi_0, \delta_0), \quad (250)$$

and Figures 5 and 6 show the streamlines of the phase fluid in the phase plane, respectively.

From the comparison of Figures 5 and 6 it is evident that with the increase of the parameter K the zone of the unclosed streamlines becomes wider while the circulation zones locating in a staggered arrangement by the right and left channel sides $\xi = 0$ and $\xi = 1$ turn out to be more narrow.

FIGURE 5: Phase portrait for the case of 1:1 internal resonance $\omega_1 = \omega_2$ at $(1/2)Kb^{-1} = 1$.

The soliton-like solution along the separatrix line $G = 0$ near the bank $\xi = 0$ is found to be as follows:

$$\begin{aligned} -\frac{1}{\sqrt{2}} \arcsin(1 - 4\xi) \Big|_{\xi_0}^{\xi} &= \frac{2b}{s} (e^{-sT_2} - 1), \\ \cos \delta &= \sqrt{\frac{\xi}{1 - \xi}}, \end{aligned} \quad (251)$$

wherein ξ varies from $\xi_{\min} = 0$ to $\xi_{\max} = 0.5$, while along the separatrix $G = -1$ attached to the side $\xi = 1$ it has the form

$$\begin{aligned} \frac{1}{\sqrt{2}} \arcsin[1 - 4(1 - \xi)] \Big|_{1 - \xi_0}^{1 - \xi} &= \frac{2b}{s} (e^{-sT_2} - 1), \\ \cos \delta &= -\sqrt{\frac{1 - \xi}{\xi}}, \end{aligned} \quad (252)$$

wherein ξ varies from $\xi_{\min} = 0.5$ to $\xi_{\max} = 1$.

Inside the circulation zones there are the stationary points, that is, the centers corresponding to the stable stationary regimes, where the stream function takes the extreme values: $G_{\max} = 0.207$ at the points with the coordinates $\xi = \xi_0 = 0.1465$, $\delta = \delta_0 = \pm 2\pi n$ and $G_{\min} = -1.207$ at the points with the coordinates $\xi = \xi_0 = 0.8535$, $\delta = \delta_0 = \pi \pm 2\pi n$.

At the negative value of parameter Kb^{-1} , the phase fluid flow portrait shown in Figure 5 is shifted along the δ -axis on the value π , the flow is opposite in direction (see Figure 6), and the maxima and minima change places. In so doing the solutions along the curve separatrices with $G = 0$ and $G = 1$ are written according to (251) and (252), respectively.

6.2. One-to-One Internal Resonance $\omega_1 = \omega_3$ (77). For this case, the stream function $G(\xi, \delta)$ is defined by relationship (155), and the phase portrait to be constructed according to (155) depends essentially on the magnitudes of the coefficients K_2 and K_3 .

6.2.1. Case (77) at $K_2 = K_3 = 0$. For this case, the stream function $G(\xi, \delta)$ is constructed according to (150) and the streamlines of the phase fluid in the phase plane ξ - δ are presented in Figure 7. Magnitudes of G are indicated by digits near the curves which correspond to the streamlines; the flow

FIGURE 6: Phase portrait for the case of 1:1 internal resonance $\omega_1 = \omega_2$ at $(1/2)Kb^{-1} = -1$.

FIGURE 7: Phase portrait for the case of 1:1 internal resonance $\omega_1 = \omega_3$ at $K_2 = K_3 = 0$.

directions of the phase fluid elements are shown by arrows on the streamlines.

Reference to Figure 7 shows that the phase fluid flows within the circulation zones, which tend to be located around the perimeter of the rectangles bounded by the lines $\xi = 0$, $\xi = 1$, and $\delta = \pm(\pi/2) \pm 2\pi n$ ($n = 0, 1, 2, \dots$). As this takes place, the flow in each such rectangle becomes isolated. On all four rectangle sides $G = 0$ and inside it the value G preserves its sign. The function G attains its extreme magnitudes ± 0.25 at the points with the coordinates $\xi = 1/2$, $\delta = \pm\pi n$ ($n = 0, 1, 2, \dots$).

Along the boundaries $\xi = 0$ and $\xi = 1$ the phase modulated regimes are realized because the particles of the phase fluid flow with finite velocities (as distinct to the similar case shown in Figure 2 for $\omega_1 = \omega_2$, where velocities of fluid particles tend to infinity as $\xi \rightarrow 0$ and $\xi \rightarrow 1$), while along the lines $\delta = \pm(\pi/2) \pm 2\pi n$ ($n = 0, 1, 2, \dots$) there exist pure amplitude modulated aperiodic motions, since with an increase in time t the value ξ increases from ξ_0 to 1 (along the line $\delta = -\pi/2$) or decreases from ξ_0 to 0 (along the line $\delta = \pi/2$), and the solution (152) could be written in the form of a soliton-like solution:

$$\begin{aligned} \ln \left| \frac{1 - \xi}{\xi} \right|_{\xi_0}^{\xi} &= \frac{bE_0}{s} (1 - e^{-sT_2}), \\ \delta(T_2) &= \delta_0 = \frac{\pi}{2} \pm \pi n, \quad n = 0, 1, 2, \dots \end{aligned} \quad (253)$$

FIGURE 8: Phase portrait for the case of 1:1 internal resonance $\omega_1 = \omega_3$ at $K_2(bE_0)^{-1} = -1$ and $K_3b^{-1} = 1/2$.

6.2.2. *Case (77) at $K_2(bE_0)^{-1} = -1$ and $K_3b^{-1} = 1/2$.* In this particular case, the stream function $G(\xi, \delta)$ (150) is defined as

$$G(\xi, \delta) = \xi(1 - \xi) \cos \delta + \frac{1}{4} + \frac{1}{2}\xi - \frac{1}{4}\xi^2 = G_0(\xi_0, \delta_0), \quad (254)$$

and Figure 8 shows the streamlines of the phase fluid in the phase plane.

As in the previous case, the phase fluid flows in an infinitely long channel, the boundaries of which are the straight lines $\xi = 0$ and $\xi = 1$, corresponding to the phase modulated motions. In one part the streamlines are nonclosed, which corresponds to the periodic change of amplitudes and the aperiodic change of phases; in another part they are closed, which corresponds to the periodic change of both amplitudes and phases. The aperiodic regime lines are the boundaries of the closed and unclosed streamline areas. From the phase portrait in Figure 8 it is seen that the circulation zones are located in a staggered arrangement by the right and left channel sides (this configuration resembles that of von Kármán staggered vortex tracks).

Each zone by the side $\xi = 1$ is surrounded by a line with the value $G = 0.5$. This line consists of two parts connected with each other at the branch points with the coordinates $\xi = 1, \delta = \pi/2 \pm \pi n$ ($n = 0, 1, 2, \dots$). One branch of this line corresponds to the phase modulated regime $\xi = 1$, and the other corresponds to the aperiodic regime, wherein ξ varies from $\xi_{\min} = 0.2$ to $\xi_{\max} = 1$. At the branch point itself, the phase fluid flow velocity is equal to zero. The soliton-like solution along the separatrix with $G = 0.5$ is written as

$$\begin{aligned} \ln \left| \frac{8}{1 - \xi} \left(\sqrt{\left(\xi - \frac{1}{5} \right) \left(\xi + \frac{1}{3} \right)} - 4(1 - \xi) + 4 \right) \right|_{1 - \xi_0}^{1 - \xi} \\ = \frac{bE_0}{s} (e^{-sT_2} - 1), \quad (255) \\ \cos \delta = \frac{1 - \xi}{4\xi}, \quad \frac{1}{5} \leq \xi \leq 1. \end{aligned}$$

The circulation zones by the side $\xi = 0$ are surrounded by the line with the value $G = 0.25$. This line consists of two parts connected with each other at the branch points with the

FIGURE 9: Phase portrait for the case of 1:1 internal resonance $\omega_1 = \omega_3$ at $K_2(bE_0)^{-1} = -1$ and $K_3b^{-1} = -1$.

coordinates $\xi = 0, \delta = 2\pi/3 \pm \pi n$ ($n = 0, 1, 2, \dots$). One branch of this line corresponds to the phase modulated regime $\xi = 0$, and the other corresponds to the aperiodic regime, wherein ξ varies from $\xi_{\min} = 0$ to $\xi_{\max} = 2/3$. At the branch point itself, the phase fluid flow velocity is equal to zero. The soliton-like solution along the separatrix line $G = 0.25$ has the form

$$\begin{aligned} \frac{2}{\sqrt{3}} \ln \left| \frac{4}{\xi} \left(\sqrt{\left(\frac{2}{3} - \xi \right) \left(\frac{6}{5} - \xi \right)} - 7\xi + 6 \right) \right|_{\xi_0}^{\xi} \\ = \frac{bE_0}{s} (1 - e^{-sT_2}), \quad (256) \\ \cos \delta = -\frac{2 - \xi}{4(1 - \xi)}, \quad 0 \leq \xi \leq \frac{2}{3}. \end{aligned}$$

Inside the two circulation zones there are points with the extreme values of the stream function: maximal $G_{\max} = 0.7$ and minimal $G_{\min} = 1/6$, respectively. These points are the centers corresponding to the stable stationary regimes $\xi = \xi_0 = 0.6, \delta = \delta_0 = \pm 2\pi n$ and $\xi = \xi_0 = 1/3, \delta = \delta_0 = \pi \pm 2\pi n$, respectively.

Between the lines corresponding to $G = 0.25$ and $G = 0.5$, unclosed streamlines are located which are in accordance with the periodic change of the amplitudes and the aperiodic change of the phase difference.

6.2.3. *Case (77) at $K_2(bE_0)^{-1} = -1$ and $K_3b^{-1} = -1$.* In this case, the stream function $G(\xi, \delta)$ (155) is defined as

$$G(\xi, \delta) = \xi(1 - \xi) \cos \delta - \frac{1}{2} + 2\xi - \xi^2 = G_0(\xi_0, \delta_0), \quad (257)$$

and Figure 9 shows the streamlines of the phase fluid in the phase plane.

From Figure 9 it is seen that, unlike the previous case presented in Figure 8, the circulation zones by the side $\xi = 0$ and the aperiodic regime disappear. If $\xi \rightarrow 0$, then the streamlines level off and tend to the line $\xi = 0$ with $G = G_{\min} = -0.5$, where the phase modulated regime is realized.

Each circulation zone by the side $\xi = 1$ is surrounded by a line with the value $G = 0.5$. This line consists of two parts connected with each other at the branch points with the coordinates $\xi = 1, \delta = \pi/2 \pm \pi n$ ($n = 0, 1, 2, \dots$). One branch

FIGURE 10: Phase portrait for the case of 1:1 internal resonance $\omega_1 = \omega_3$ at $K_2(bE_0)^{-1} = 1$ and $K_3b^{-1} = 1$.

of this line corresponds to the phase modulated regime $\xi = 1$, and the other corresponds to the aperiodic regime, wherein ξ varies from $\xi_{\min} = 0.5$ to $\xi_{\max} = 1$. At the branch point itself, the phase fluid flow velocity is equal to zero. The soliton-like solution along the separatrix with $G = 0.5$ is written as

$$\ln \left| \frac{\sqrt{2\xi - 1} - 1}{\sqrt{2\xi - 1} + 1} \right|_{1-\xi_0}^{1-\xi} = \frac{bE_0}{s} (1 - e^{-sT_2}), \quad (258)$$

$$\cos \delta = \frac{1 - \xi}{\xi}, \quad \frac{1}{2} \leq \xi \leq 1.$$

Inside the circulation zones there are points with the maximal value $G_{\max} = 0.625$. These points are the centers corresponding to the stable stationary regimes $\xi = \xi_0 = 0.75$, $\delta = \delta_0 = \pm 2\pi n$. The lines corresponding to $-0.5 < G < 0.5$ are unclosed streamlines which are in accordance with the periodic change of the amplitudes and the aperiodic change of the phase difference.

6.2.4. Case (77) at $K_2(bE_0)^{-1} = 1$ and $K_3b^{-1} = 1$. In this case, the stream function $G(\xi, \delta)$ (155) is defined as

$$G(\xi, \delta) = \xi(1 - \xi) \cos \delta + \frac{1}{2} - 2\xi = G_0(\xi_0, \delta_0), \quad (259)$$

and Figure 10 shows the streamlines of the phase fluid in the phase plane.

Figure 10 illustrates the phase portrait with only unclosed phase fluid streamlines along which the fluid flows in the direction of a decrease in δ . With $\xi \rightarrow 0$ and $\xi \rightarrow 1$, the streamlines level off and tend, respectively, to the lines $\xi = 0$ with $G = G_{\max} = 0.5$ and $\xi = 1$ with $G = G_{\min} = -0.5$, where the boundary phase modulated regimes are realized.

6.2.5. Case (77) at $K_2 = 0$ and $K_3b^{-1} = 1$. In this particular case, the stream function $G(\xi, \delta)$ (155) is defined as

$$G(\xi, \delta) = \xi(1 - \xi) \cos \delta - \frac{1}{2} + \xi - \xi^2 = G_0(\xi_0, \delta_0), \quad (260)$$

and Figure 11 shows the streamlines of the phase fluid in the phase plane.

FIGURE 11: Phase portrait for the case of 1:1 internal resonance $\omega_1 = \omega_3$ at $K_2 = 0$ and $K_3b^{-1} = 1$.

Reference to Figure 11 shows that the phase fluid flows within the circulation zones, which tend to be located around the perimeter of the rectangles bounded by the lines $\xi = 0$, $\xi = 1$, and $\delta = \pm(\pi) \pm 2\pi n$ ($n = 0, 1, 2, \dots$). As this takes place, the flow in each such rectangle becomes isolated. On all four rectangle sides $G = -0.5$, and inside it the value $-0.5 < G < 0$ attaining its maximal magnitude $G_{\max} = 0$ at the points with the coordinates $\xi = 1/2$, $\delta = \pm 2\pi n$ ($n = 0, 1, 2, \dots$).

Along the boundaries $\xi = 0$ and $\xi = 1$ the phase modulated regimes are realized because the particles of the phase fluid flow with finite velocities:

$$v_\delta = \mp bE_0 (1 + \cos \delta) e^{-sT_2}, \quad (261)$$

in so doing

$$\tan \frac{\delta}{2} \Big|_{\delta_0}^{\delta} = \mp \frac{bE_0}{s} (1 - e^{-sT_2}), \quad (262)$$

where the signs “-” and “+” refer to the boundaries $\xi = 0$ and $\xi = 1$, respectively.

Along the vertical lines $\delta = \delta_0 = \text{const} = \pi \pm 2\pi n$ ($n = 1, 2, \dots$) there exist stationary regimes for all $0 \leq \xi_0 \leq 1$ attenuating with time.

7. Conclusion

Free damped vibrations of a nonlinear plate in a fractional derivative viscoelastic medium have been investigated. Nonlinear vibrations are described by coupled three equations with respect to the three orthogonal displacements.

The proposed analytical approach for investigating the damped vibrations of the nonlinear plate subjected to the conditions of the internal resonance has been possible owing to the new procedure suggested in this paper, resulting in decoupling linear parts of equations with further utilization of the method of multiple scales for solving nonlinear governing equations of motion.

It has been shown that the phenomenon of internal resonance could be very critical, since in the thin plate under consideration the internal resonance is always present. Moreover, its type depends on the order of smallness of the viscosity involved into consideration. Thus, at the ε -order, damped vibrations occur within the two-to-one and

one-to-one-to-two internal resonance. Other types of the internal resonance, such as one-to-one, one-to-one-to-one, and combinational resonances of the additive and difference types could be found at ε^2 -order; that is, the type of the resonance depends on the order of smallness of the fractional derivative entering in the equations of motion of the plate.

For each type of the resonance, the nonlinear sets of resolving equations in terms of amplitudes and phase differences have been obtained. It has been shown that for the one-to-one and combinational internal resonances there exist such particular cases when it is possible to obtain two first integrals, namely, the energy integral and the stream function, which allows one to reduce the problem to the calculation of elliptic integrals.

The influence of viscosity on the energy exchange mechanism has been analyzed. It has been shown that each mode is characterized by its damping coefficient connected with the natural frequency by the exponential relationship with a negative fractional exponent. Thus, during free vibrations of the plate with internal resonances three regimes can be observed: stationary (absence of damping at $\gamma = 0$), quasi-stationary (damping is defined by an ordinary derivative at $\gamma = 1$), and transient (damping is defined by a fractional derivative at $0 < \gamma < 1$).

The phenomenological analysis carried out for two types of the one-to-one internal resonance with the help of the phase portraits constructed for different magnitudes of the plate parameters reveals the great variety of vibrational motions: stationary vibrations, two-sided energy exchange between two subsystems under consideration, and one-sided energy interchange resulting in the complete one-sided energy transfer. Under the presence of small viscosity all regimes attenuate with time.

The analysis of the phase portraits for various oscillatory regimes cited above demonstrates that they contain closed as well as nonclosed streamlines, which are separated by the curves separatrixes. Along the separatrixes one succeeds in finding analytic solutions which determine the irreversible energy exchange from one subsystem into another and which are inherently soliton-like solutions in the theory of vibrations.

Comparison of the results obtained in this paper via the new suggested procedure with those for a nonlinear plate, the motion of which is described either by one equation in terms of plate's deflection [6] or also by three coupled nonlinear equations in terms of three displacements but without decoupling the linear parts of governing equations [2, 3], reveals the fact that the new approach allows one to discover and analyze much more diversified variety of internal resonances, including the one-to-one-to-one and one-to-one-to-two internal resonances and additional one-to-one, two-to-one, and combinational resonances (ten possible cases in all), than all other existing previously in literature approaches.

It should be noted that the proposed analytical approach could be used for other cases if the matrix of the natural modes coupled by the internal resonance is Hermitian matrix; that is, it should be symmetric and all its elements should be real.

Appendix

$$\begin{aligned}
 a_{1mn}^{m_1 n_1 m_2 n_2} &= \int_0^1 \int_0^1 \cos \pi m_1 x \sin \pi n_1 y \sin \pi m_2 x \sin \pi n_2 y \\
 &\quad \cdot \cos \pi m x \sin \pi n y \, dx \, dy, \\
 a_{2mn}^{m_1 n_1 m_2 n_2} &= \int_0^1 \int_0^1 \sin \pi m_1 x \cos \pi n_1 y \cos \pi m_2 x \cos \pi n_2 y \\
 &\quad \cdot \cos \pi m x \sin \pi n y \, dx \, dy, \\
 a_{3mn}^{m_1 n_1 m_2 n_2} &= \int_0^1 \int_0^1 \sin \pi m_1 x \cos \pi n_1 y \sin \pi m_2 x \sin \pi n_2 y \\
 &\quad \cdot \sin \pi m x \cos \pi n y \, dx \, dy, \\
 a_{4mn}^{m_1 n_1 m_2 n_2} &= \int_0^1 \int_0^1 \cos \pi m_1 x \sin \pi n_1 y \cos \pi m_2 x \cos \pi n_2 y \\
 &\quad \cdot \sin \pi m x \cos \pi n y \, dx \, dy, \\
 a_{5mn}^{m_1 n_1 m_2 n_2} &= \int_0^1 \int_0^1 \sin \pi m_1 x \sin \pi n_1 y \sin \pi m_2 x \sin \pi n_2 y \\
 &\quad \cdot \sin \pi m x \sin \pi n y \, dx \, dy, \\
 a_{6mn}^{m_1 n_1 m_2 n_2} &= \int_0^1 \int_0^1 \cos \pi m_1 x \cos \pi n_1 y \cos \pi m_2 x \cos \pi n_2 y \\
 &\quad \cdot \sin \pi m x \sin \pi n y \, dx \, dy, \\
 a_{7mn}^{m_1 n_1 m_2 n_2} &= \int_0^1 \int_0^1 \cos \pi m_1 x \sin \pi n_1 y \cos \pi m_2 x \sin \pi n_2 y \\
 &\quad \cdot \sin \pi m x \sin \pi n y \, dx \, dy, \\
 a_{8mn}^{m_1 n_1 m_2 n_2} &= \int_0^1 \int_0^1 \sin \pi m_1 x \cos \pi n_1 y \sin \pi m_2 x \cos \pi n_2 y \\
 &\quad \cdot \sin \pi m x \sin \pi n y \, dx \, dy.
 \end{aligned} \tag{A.1}$$

These integrals could be easily calculated using the following formulas [36]:

$$\begin{aligned}
 \sin \alpha \sin \beta \sin \gamma &= \frac{1}{4} [\sin (\alpha + \beta - \gamma) + \sin (\beta + \gamma - \alpha) \\
 &\quad + \sin (\gamma + \alpha - \beta) - \sin (\alpha + \beta + \gamma)], \\
 \sin \alpha \cos \beta \cos \gamma &= \frac{1}{4} [\sin (\alpha + \beta - \gamma) - \sin (\beta + \gamma - \alpha) \\
 &\quad + \sin (\gamma + \alpha - \beta) - \sin (\alpha + \beta + \gamma)], \\
 \sin \alpha \sin \beta \cos \gamma &= \frac{1}{4} [-\cos (\alpha + \beta - \gamma) + \cos (\beta + \gamma - \alpha) \\
 &\quad + \cos (\gamma + \alpha - \beta) + \cos (\alpha + \beta + \gamma)], \\
 \cos \alpha \cos \beta \cos \gamma &= \frac{1}{4} [\cos (\alpha + \beta - \gamma) + \cos (\beta + \gamma - \alpha) \\
 &\quad + \cos (\gamma + \alpha - \beta) + \cos (\alpha + \beta + \gamma)].
 \end{aligned} \tag{A.2}$$

Disclosure

Jean Claude Ngenzi is on leave from École Normale Supérieure, ENS Burundi.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] M. M. Rashidi, A. Shooshtari, and O. Anwar Bég, "Homotopy perturbation study of nonlinear vibration of von Karman rectangular plates," *Computers & Structures*, vol. 106-107, pp. 46–55, 2012.
- [2] Y. A. Rossikhin and M. V. Shitikova, "Free damped nonlinear vibrations of a viscoelastic plate under the two-to-one internal resonance," *Materials Science Forum*, vol. 440-441, pp. 29–36, 2003.
- [3] Y. A. Rossikhin and M. V. Shitikova, "Analysis of free non-linear vibrations of a viscoelastic plate under the conditions of different internal resonances," *International Journal of Non-Linear Mechanics*, vol. 41, no. 2, pp. 313–325, 2006.
- [4] A. Shooshtari and S. E. Khadem, "A multiple scale method solution for the nonlinear vibration of rectangular plates," *Scientia Iranica*, vol. 14, no. 1, pp. 64–71, 2007.
- [5] M. Amabili, "Nonlinear vibrations of rectangular plates with different boundary conditions: theory and experiments," *Computers & Structures*, vol. 82, no. 31-32, pp. 2587–2605, 2004.
- [6] P. Ribeiro and M. Petyt, "Non-linear free vibration of isotropic plates with internal resonance," *International Journal of Non-Linear Mechanics*, vol. 35, no. 2, pp. 263–278, 2000.
- [7] A. H. Nayfeh and B. Balachandran, "Experimental investigation of resonantly forced oscillations of a two-degree-of-freedom structure," *International Journal of Non-Linear Mechanics*, vol. 25, no. 2-3, pp. 199–209, 1990.
- [8] M. Amabili, *Nonlinear Vibrations and Stability of Shells and Plates*, Cambridge University Press, Cambridge, UK, 2008.
- [9] I. D. Breslavsky, M. Amabili, and M. Legrand, "Physically and geometrically non-linear vibrations of thin rectangular plates," *International Journal of Non-Linear Mechanics*, vol. 58, pp. 30–40, 2014.
- [10] M. Sathyamoorthy, "Nonlinear vibrations of plates: an update of recent research developments," *Applied Mechanics Reviews*, vol. 49, no. 10, pp. S55–S62, 1996.
- [11] G. Anlas and O. Elbeyli, "Nonlinear vibrations of a simply supported rectangular metallic plate subjected to transverse harmonic excitation in the presence of a one-to-one internal resonance," *Nonlinear Dynamics*, vol. 30, no. 1, pp. 1–28, 2002.
- [12] D. A. Khodzaev and B. K. Éshmatov, "Nonlinear oscillations of a viscoelastic plate with concentrated masses," *Journal of Applied Mechanics and Technical Physics*, vol. 48, no. 6, pp. 905–914, 2007.
- [13] T.-W. Kim and J.-H. Kim, "Nonlinear vibration of viscoelastic laminated composite plates," *International Journal of Solids and Structures*, vol. 39, no. 10, pp. 2857–2870, 2002.
- [14] A. H. Nayfeh, *Nonlinear Interaction: Analytical, Computational, and Experimental Methods*, Wiley Series in Nonlinear Science, John Wiley & Sons, New York, NY, USA, 2000.
- [15] S. I. Chang, A. K. Bajaj, and C. M. Krousgrill, "Non-linear vibrations and chaos in harmonically excited rectangular plates with one-to-one internal resonance," *Nonlinear Dynamics*, vol. 4, no. 5, pp. 433–460, 1993.
- [16] A. A. Witt and G. A. Gorelik, "Vibrations of an elastic pendulum as an example of vibrations of two parametrically coupled linear systems," *Journal of Technical Physics*, vol. 2-3, pp. 294–307, 1933 (Russian).
- [17] A. M. Abdel-Ghaffar and G. W. Housner, "Ambient vibration tests of suspension bridge," *ASCE Journal of the Engineering Mechanics Division*, vol. 104, no. 5, pp. 983–999, 1978.
- [18] A. M. Abdel-Ghaffar and R. H. Scanlan, "Ambient vibration studies of Golden Gate Bridge. I: suspended structure," *Journal of Engineering Mechanics*, vol. 111, no. 4, pp. 463–482, 1985.
- [19] Y. A. Rossikhin and M. V. Shitikova, "Application of fractional calculus for analysis of nonlinear damped vibrations of suspension bridges," *ASCE Journal of Engineering Mechanics*, vol. 124, no. 9, pp. 1029–1036, 1998.
- [20] Y. A. Rossikhin and M. V. Shitikova, "Nonlinear free damped vibrations of suspension bridges with uncertain fractional damping," *Journal Europeen des Systemes Automatisés*, vol. 42, no. 6–8, pp. 879–894, 2008.
- [21] A. S. Volmir, *Nonlinear Dynamics of Plates and Shells*, Nauka, Moscow, Russia, 1972, (Russian).
- [22] Y. A. Rossikhin and M. V. Shitikova, "Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results," *Applied Mechanics Reviews*, vol. 63, no. 1, Article ID 010801, 2010.
- [23] R. W. Clough and J. Penzien, *Dynamics of Structures*, McGraw-Hill, New York, NY, USA, 1975.
- [24] Y. A. Rossikhin and M. V. Shitikova, "Analysis of damped vibrations of linear viscoelastic plates with damping modeled with fractional derivatives," *Signal Processing*, vol. 86, no. 10, pp. 2703–2711, 2006.
- [25] Y. A. Rossikhin and M. V. Shitikova, "Analysis of damped vibrations of thin bodies embedded into a fractional derivative viscoelastic medium," *Journal of the Mechanical Behavior of Materials*, vol. 21, no. 5-6, pp. 155–159, 2012.
- [26] T. J. Anderson, B. Balachandran, and A. H. Nayfeh, "Nonlinear resonances in a flexible cantilever beam," *Journal of Vibration and Acoustics, Transactions of the ASME*, vol. 116, no. 4, pp. 480–484, 1994.
- [27] S. A. Nayfeh and A. H. Nayfeh, "Energy transfer from high-to-low-frequency modes in a flexible structure via modulation," *Journal of Vibration and Acoustics*, vol. 116, no. 2, pp. 203–207, 1994.
- [28] P. Popovic, A. H. Nayfeh, K. Oh, and S. A. Nayfeh, "Experimental investigation of energy transfer from a high-frequency mode to a low-frequency mode in a flexible structure," *Journal of Vibration and Control*, vol. 1, no. 1, pp. 115–128, 1995.
- [29] A. H. Nayfeh, *Perturbation Methods*, John Wiley & Sons, New York, NY, USA, 1973.
- [30] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, English

translation by Gordon and Breach Science Publishers, Amsterdam, The Netherlands, 1993, Nauka i Tekhnika, Minsk, Belarus, 1988, (Russian).

- [31] Y. A. Rossikhin, M. V. Shitikova, and T. Shcheglova, "Forced vibrations of a nonlinear oscillator with weak fractional damping," *Journal of Mechanics of Materials and Structures*, vol. 4, no. 9, pp. 1619–1636, 2009.
- [32] Y. A. Rossikhin and M. V. Shitikova, "On fallacies in the decision between the Caputo and Riemann-Liouville fractional derivatives for the analysis of the dynamic response of a nonlinear viscoelastic oscillator," *Mechanics Research Communications*, vol. 45, pp. 22–27, 2012.
- [33] Yu. A. Rossikhin and M. V. Shitikova, "Nonlinear dynamic response of a thin plate in a fractional viscoelastic medium under internal resonance 1:1:2," *Applied Mechanics and Materials*, vol. 518, pp. 60–65, 2014.
- [34] M. Abramowitz and I. Stegun, Eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Tables*, vol. 55, Applied Mathematical Services, National Bureau of Standards USA, Washington, DC, USA, 1964.
- [35] Y. A. Rossikhin and M. V. Shitikova, "Nonlinear dynamic response of a thin plate embedded in a fractional viscoelastic medium under combinational internal resonances," *Applied Mechanics and Materials*, vol. 595, pp. 105–110, 2014.
- [36] I. N. Bronshtein and K. A. Semendyayev, *Handbook of Mathematics*, Springer, 3rd edition, 1985.