Research Article

Influence of Bearing Stiffness on the Nonlinear Dynamics of a Shaft-Final Drive System

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The bearing stiffness has a considerable influence on the nonlinear coupling vibration characteristics of the shaft-final drive system. A 14-DOF nonlinear coupled vibration model was established by employing the lumped mass method so as to identify the coupling effects of the bearing stiffness to the vibration response of the shaft-final drive system. The engine's torque ripple, the alternating load from the universal joint (U-joint), and the time-varying mesh parameters of hypoid gear of the shaft-final drive system were also considered for accurate quantitative analysis. The numerical analysis of the vibration response of the coupled system was performed and the experimental measurements were carried out for the validation test. Results show that, at the given driving speed, improving the bearing stiffness can reduce the vibration response of the given coupled system; however, when the bearing stiffness increases to a critical value, the effects of bearing stiffness on the vibration reduction become insignificant; when the driving speed changes, the resonance regions of the coupled system vary with the bearing stiffness. The results are helpful to determine the proper bearing stiffness and the optimum control strategy for the shaft-final drive system. It is hoped that the optimal shaft-final drive system can provide good vibration characteristics to achieve the energy saving and noise reduction for the vehicle application.

1. Introduction

It is a common case in vehicles that power is transmitted through shafts and gears which are not in a direct line; this layout is even more complex with all-wheel drive vehicles. Some precision components are inherently nonlinear; for example, the hypoid gear pair inside the gearbox is often coupled with bearing systems. On account of the increasing demand for high NVH performance, the research in the field of parameter matching on the transmission system becomes very important. As one of the major parameters in a shaft-final drive system, the bearing stiffness significantly affects the performance of a high-speed pinion shaft and leads to the coupled vibration of the rear axle assembly [1, 2].

In the transmission system, the dynamic mesh force excited by the transmission error is believed to be an important vibration source. Therefore, some considerable research on the gear system has been carried on in recent years. Lim and Singh [3] proposed and established the basic theory and coupling mechanism of vibration transmission through rolling element bearing. Donley et al. [4] studied

the three-dimensional stress of a gear pair with finite element method. Bibel et al. [5] considered the contact stress analysis of spiral bevel gears with the invariable mesh point and line-of-action. Gosselin et al. [6] derived a general formula and applied it to the load distribution and transmission error of spiral bevel gear. Theodossades et al. [7, 8] gave a model method to analyze the dynamic characteristic using time-varying stiffness and backlash nonlinearity. Wang et al. [9, 10] carried out the analysis and comparison from the gear pair to the gear system with the bearing. Cheng and Lim [11, 12] developed an analytical framework for analyzing both linear and nonlinear pure vibration characteristics of geared rotor system. Tao [13] further developed the coupled multibody dynamics and vibration model with the practical cases in his work. Yang and Lim [14] proposed a new capability to analyze the dynamic interaction of gear mesh characteristics and time-varying bearing stiffness.

The tapered roller bearing (TRB) is widely used to compensate the installation errors along the axial and radial directions and the displacement by the movement of transmission system; the bearing stiffness in rotor-bearing system

FIGURE 1: Mathematical model of a shaft-final drive system.

can significantly affect the dynamic characteristics including natural frequencies and dynamic response [15]. In recent years, a theoretical analysis of lateral response and the optimization algorithm via the enhanced genetic algorithm was presented in [16]. The running error conditions of the misalignment rotor systems connected by the gear coupling were indicated in [17]. In [18], a nonlinear dynamic model of a spur gear pair with linear models of shafts, bearings, and disks was developed. Considering the rotor dynamic balance, the flexible rotor bearing system was studied by numerical analysis elsewhere [19–21]. These researches mentioned above mainly focused on the internal incentives of gear or rotor system; the interactional relationships between transmission system and the influence of bearing stiffness on a shaft-final drive system were still not well defined [22]; few studies have been done on fluctuations of the system with transmission error in which the bearing stiffness changes with the speed in operation.

In this investigation, both nonuniformity of multiple U-joints and time-varying mesh parameters of hypoid gear are considered, a 14-DOF nonlinear model is established, and the numerical optimisation method is proposed to analyze the influence of bearing stiffness on the nonlinear dynamics of the shaft-final drive. Then the transmission system of a rear wheel drive vehicle is taken as an example to obtain the reasonable region of bearing stiffness; experiments are carried out to demonstrate the effectiveness of this method by simultaneously optimising system stiffness.

2. Mathematical Model

2.1. Nonlinear Dynamic Model. The shaft-final drive system model is shown in Figure 1. It mainly consists of the drive shaft, gears, and rear axle parts.

For an effective dynamic model, the lumped mass method is adopted to deal with the components and parts. It allocates the moment of inertia of the shaft to the U-joints and pinion, the driven gear and differential can be counted as an assembly with a concentrated mass, the flexible shaft can be described by torsion springs and dampers, and the paired mounting bearings of the reducer can be treated as a whole and simulated by a spring-damper system. To simplify the calculation, the friction and lubrication and some inertia of the small components such as the needle bearings in the U-joints are neglected [23].

2.2. Universal Joint Geometry. On the basis of U-joints kinematics, there is

$$\tan \theta_2 = \tan \theta_3 \cdot \cos \alpha_1, \quad (1)$$

where θ_2 and θ_3 are the input and output angle of U-joint, respectively, α_1 is the axial angle, and the angular speed and angular acceleration of main drive shaft are defined; thus,

$$\begin{aligned} \dot{\theta}_3 &= \frac{\cos \alpha_1}{(1 + \cos \alpha_1^2) / 2 - \cos 2\theta_2 \cdot (1 - \cos \alpha_1^2) / 2} \dot{\theta}_2, \\ \ddot{\theta}_3 &= \frac{\ddot{\theta}_2}{A - B \cos 2\theta_2} - \frac{2B \sin 2\theta_2}{(A - B \cos 2\theta_2)^2} (\dot{\theta}_2)^2, \end{aligned} \quad (2)$$

$$A = \frac{1 + \cos^2 \alpha_1}{2 \cos \alpha_1}, \quad B = \frac{1 - \cos^2 \alpha_1}{2 \cos \alpha_1}.$$

Similarly, θ_4 and θ_5 also share the aforementioned relationship.

2.3. Mesh Force Analysis. In order to make quantitative analysis, the uniform load of the pinion is replaced by the

FIGURE 2: Force analysis of the pinion.

concentrated force acting on midpoint of the face width; the simplified mechanical model is shown in Figure 2.

In process of the pinion mesh transmission, the normal force at the contact point and its components along x -, y -, and z -axes are of the following form:

$$\begin{aligned} F_n &= k_m(t) f(x_n) + c_m \dot{x}_n, \\ F_x &= -F_n \cos \alpha_n \sin \beta_1 \cos \delta_1 + F_n \sin \alpha_n \sin \delta_1, \\ F_y &= -F_n \cos \alpha_n \sin \beta_1 \cos \delta_1 - F_n \sin \alpha_n \cos \delta_1, \\ F_z &= -F_n \cos \alpha_n \cos \beta_1. \end{aligned} \quad (3)$$

The gear pair mesh stiffness function $k_m(t)$ is

$$k_m(t) = k_m + \sum_{i=1}^N k_{Ai} \cos(i\Omega_n t + \varphi_k). \quad (4)$$

The gear backlash function f can be expressed by the nonlinear combination:

$$f(x_n) = \begin{cases} x_n - b, & x_n \geq b, \\ 0, & -b \leq x_n \leq b, \\ x_n + b, & x_n \leq -b. \end{cases} \quad (5)$$

The relative displacement of the normal direction at the contact point x_n is

$$\begin{aligned} x_n &= a_1(x_p - x_g) + a_2(y_p - y_g) \\ &\quad + a_3(z_p - z_g + R_p\theta_p - R_g\theta_g) - e(t). \end{aligned} \quad (6)$$

The synthetic error $e(t)$, in consideration of manufacturing factors, can be described as

$$e(t) = e_0 + \sum_{i=1}^N e_{Ai} \cos(i\Omega_n t + \varphi_e). \quad (7)$$

Generally, with $e_0 = 0$, the coefficients in (6) can be expressed as follows:

$$\begin{aligned} a_1 &= \cos \alpha_n \sin \beta_1 \cos \delta_1 - \sin \alpha_n \sin \delta_1, \\ a_2 &= \cos \alpha_n \sin \beta_1 \sin \delta_1 + \sin \alpha_n \cos \delta_1, \\ a_3 &= \cos \alpha_n \cos \beta_1. \end{aligned} \quad (8)$$

3. Governing Differential Equations

3.1. Dynamic Differential Equations. Based on the kinetics of the shaft and the mesh force analysis of the hypoid gear, the differential equations of the shaft-final drive system are as follows:

$$\begin{aligned} J_1 \ddot{\theta}_1 + c_1'(\dot{\theta}_1 - \dot{\theta}_2) + K_1'(\theta_1 - \theta_2) &= T_1, \\ J_2 \ddot{\theta}_2 + c_2(\dot{\theta}_2 - \dot{\theta}_1) + K_2(\theta_2 - \theta_1) &= -T_2, \\ J_3 \ddot{\theta}_3 + c_3(\dot{\theta}_3 - \dot{\theta}_4) + K_3(\theta_3 - \theta_4) &= T_3, \\ J_4 \ddot{\theta}_4 + c_4(\dot{\theta}_4 - \dot{\theta}_3) + K_4(\theta_4 - \theta_3) &= -T_4, \\ J_5 \ddot{\theta}_5 - F_z R_5 &= T_5, \\ J_6 \ddot{\theta}_6 + F_z R_6 &= -T_6, \\ J_7 \ddot{\theta}_7 + c_7(\dot{\theta}_7 - \dot{\theta}_8) + K_7(\theta_7 - \theta_8) &= T_7, \\ J_8 \ddot{\theta}_8 + c_8(\dot{\theta}_8 - \dot{\theta}_7) + K_8(\theta_8 - \theta_7) &= -T_8, \\ m_{xp} \ddot{x}_p + c_{px} \dot{x}_p + K_{px} x_p &= F_x, \\ m_{yp} \ddot{y}_p + c_{py} \dot{y}_p + K_{py} y_p &= F_y, \\ m_{zp} \ddot{z}_p + c_{pz} \dot{z}_p + K_{pz} z_p &= F_z, \\ m_{xg} \ddot{x}_g + c_{gx} \dot{x}_g + K_{gx} x_g &= -F_x, \end{aligned}$$

$$\begin{aligned} m_{yg} \ddot{\bar{y}}_g + c_{gy} \dot{\bar{y}}_g + K_{gy} \bar{y}_g &= -F_y, \\ m_{zg} \ddot{\bar{z}}_g + c_{gz} \dot{\bar{z}}_g + K_{gz} \bar{z}_g &= -F_z. \end{aligned} \quad (9)$$

To simplify the model, the intermediate support bearing assembly can be treated as a K_0 - c_0 elastic system; the K'_1 and c'_1 are the equivalent values of the two components of intermediate support bearing assembly and intermediate shaft.

3.2. Nondimensionalization Parameters. The difference of parameter magnitude is obvious; for instance, the magnitude of stiffness is generally 10^7 to 10^9 , but the vibration response is about 10^{-6} , and some parameters in (9) are variable and nonlinear, so it is necessary for better solving to convert the dimension equations to nondimensionalization. The displacement is normalized by half of the gear backlash b and the frequency is normalized by the natural frequency of the gear pair ω_n ; the conversion parameters are of the following form:

$$\begin{aligned} m_i &= \frac{J_i}{R_i^2}, \\ k_i &= \frac{K_i}{(m_i \omega_n^2 R_i^2)}, \\ \varsigma_i &= \frac{c_i}{(2m_i \omega_n R_i^2)}, \\ x_i &= \frac{R_i \theta_i}{b}, \\ \dot{x}_i &= \frac{R_i \dot{\theta}_i}{(b \omega_n)}, \\ \ddot{x}_i &= \frac{R_i \ddot{\theta}_i}{(b \omega_n^2)}, \\ x_n &= \frac{\bar{x}_n}{b}, \\ \dot{x}_n &= \frac{\dot{\bar{x}}_g}{(b \omega_n)}, \\ \ddot{x}_n &= \frac{\ddot{\bar{x}}_g}{(b \omega_n^2)}, \\ x_{p(g)} &= \frac{\bar{x}_{p(g)}}{b}, \\ \dot{x}_{p(g)} &= \frac{\dot{\bar{x}}_{p(g)}}{(b \omega_n)}, \\ \ddot{x}_{p(g)} &= \frac{\ddot{\bar{x}}_{p(g)}}{(b \omega_n^2)}, \end{aligned}$$

$$\begin{aligned} k_{p(g)j} &= \frac{K_{p(g)j}}{(m_{p(g)j} \omega_n^2)}, \\ \varsigma_{p(g)j} &= \frac{c_{p(g)j}}{(2m_{p(g)j} \omega_n)}, \\ k_{mpj}(t) &= \frac{K_m(\bar{t})}{(m_{jp} \omega_n^2)}, \\ \varsigma_{mpj} &= \frac{c_m}{(2m_{jp} \omega_n)}, \\ k_m(t) &= 1 + \sum_{i=1}^N \frac{k_{Ai}}{k_m} \cos(i \omega_n t + \varphi_k), \\ \varsigma_m &= \frac{c_m}{(2m_e \omega_n)}, \\ f(x_n) &= \bar{f}(\bar{x}_n) = \begin{cases} f(x_n) - 1, & f(x_n) > 1, \\ 0, & -1 \leq f(x_n) \leq 1, \\ f(x_n) + 1, & f(x_n) < -1, \end{cases} \\ &\quad (i = 1, 2, \dots, 8; j = x, y, z). \end{aligned} \quad (10)$$

Substituting (10) into (9), the 14-DOF differential equations are transformed to the no-dimensional form, and the further simplification is possible. Combining (1), (2), and (10), the parameter x_3 can be altered as

$$\begin{aligned} x_3 &= \frac{R_3 \theta_3}{b} = \frac{R_3}{b} \arctan \frac{\tan b x_2 / R_2}{\cos \alpha}, \\ \dot{x}_3 &= \frac{R_3 \dot{\theta}_3}{b \omega_n} = \frac{R_3}{b \omega_n} \frac{1}{M_2} \dot{\theta}_2 = \frac{R_3 \dot{x}_2}{R_1 M_2}, \\ \ddot{x}_3 &= \frac{R_3 \ddot{\theta}_3}{b \omega_n^2} = \frac{R_3}{R_1 M_2} \ddot{x}_2 - \frac{2B \sin 2\theta_2}{M_2^2} \frac{R_3 b}{R_1^2} (\dot{x}_2)^2, \\ T_3 &= \frac{T_2 \dot{\theta}_2}{\dot{\theta}_3} = T_2 M_2. \end{aligned} \quad (11)$$

Similarly, the parameter x_4 shares the aforementioned relationship, so x_3 and x_4 can be expressed by x_2 and x_5 separately:

$$\begin{aligned} M_i &= A - B \cos 2\theta_i, \\ P_i &= 2B \sin 2\theta_i, \\ &\quad (i = 2, 5). \end{aligned} \quad (12)$$

The relationship between the driven gear and differential is given by

$$\begin{aligned} \theta_6 &= \theta_7, \\ T_6 &= 2T_7. \end{aligned} \quad (13)$$

Combining (10)–(13), (9) can be simplified to the following 11-DOF differential equations:

$$\begin{aligned}
& \ddot{x}_1 + 2\zeta_1 (\dot{x}_1 - \dot{x}_2) + k_1 (x_1 - x_2) - \frac{T_1}{b\omega_n^2 m_1 R_1} = 0, \\
& \left(m_2 R_1 + \frac{m_3 R_3^2}{R_1 M_2^2} \right) \ddot{x}_2 - \frac{m_3 P_2}{M_2^3} \frac{R_3^2 b}{R_1^2} (\dot{x}_2)^2 + 2 \left(m_2 R_1 \zeta_2 \right. \\
& \quad \left. + \frac{m_3 \zeta_3 R_3^2}{R_1 M_2^2} \right) \dot{x}_2 - 2m_2 R_1 \zeta_2 \dot{x}_1 - \frac{2m_3 \zeta_3 R_4 R_3}{R_5 M_5 M_2} \dot{x}_5 \\
& \quad + k_2 m_2 R_1 x_2 - k_2 m_2 R_1 x_1 \\
& \quad + \frac{k_3 m_3 R_3}{b M_2} \left[R_3 \arctan \frac{\tan(bx_2/R_2)}{\cos \alpha} \right. \\
& \quad \left. - R_4 \arctan \frac{\tan(bx_5/R_5)}{\cos \alpha} \right] = 0, \\
& \left(m_5 R_5 + \frac{m_4 R_4^2}{R_5 M_5^2} \right) \ddot{x}_5 + \frac{2m_4 \zeta_4 R_4}{M_5} \left(\frac{R_4}{R_5 M_5} \dot{x}_5 \right. \\
& \quad \left. - \frac{R_3}{R_1 M_2} \dot{x}_2 \right) + a_3 k_5(t) m_5 R_5 f(x_n) \\
& \quad + \frac{k_4 m_4 R_4}{b M_5} \left[R_4 \arctan \frac{\tan(bx_5/R_5)}{\cos \alpha} \right. \\
& \quad \left. - R_3 \arctan \frac{\tan(bx_2/R_2)}{\cos \alpha} \right] - \frac{m_4 P_5}{M_5^3} \frac{R_4^2 b}{R_5^2} (\dot{x}_5)^2 \\
& \quad + 2a_3 m_5 \zeta_5 \dot{x}_n = 0, \\
& \left(\frac{m_6 R_6^2 + 2m_7 R_7^2}{R_6} \right) \dot{x}_6 + 2m_7 R_7 \left[\zeta_7 \left(\frac{R_7}{R_6} \dot{x}_6 - \dot{x}_8 \right) \right.
\end{aligned}$$

$$\begin{aligned}
& \left. + k_7 \left(\frac{R_7}{R_6} x_6 - x_8 \right) \right] - a_3 m_6 R_6 [k_6 f(x_n) - 2\zeta_6 \dot{x}_n] \\
& = 0, \\
& m_8 \ddot{x}_8 + 2m_8 \zeta_8 \left(\dot{x}_8 - \frac{R_7}{R_6} \dot{x}_6 \right) + m_8 k_8 \left(x_8 - \frac{R_7}{R_6} x_6 \right) \\
& \quad + \frac{T_8}{b\omega_n^2 R_8} = 0, \\
& \ddot{x}_p + 2\zeta_{px} \dot{x}_p + k_{px} x_p + 2a_1 \zeta_{mpx} \dot{x}_n + a_1 k_{mpx}(t) f(x_n) \\
& = 0, \\
& \ddot{y}_p + 2\zeta_{py} \dot{y}_p + k_{py} y_p + 2a_2 \zeta_{mpy} \dot{x}_n + a_2 k_{mpy}(t) f(x_n) \\
& = 0, \\
& \ddot{z}_p + 2\zeta_{pz} \dot{z}_p + k_{pz} z_p + 2a_3 \zeta_{mpz} \dot{x}_n + a_3 k_{mpz}(t) f(x_n) \\
& = 0, \\
& \ddot{x}_g + 2\zeta_{gx} \dot{x}_g + k_{gx} x_g - 2a_1 \zeta_{mgx} \dot{x}_n - a_1 k_{mgx}(t) f(x_n) \\
& = 0, \\
& \ddot{y}_g + 2\zeta_{gy} \dot{y}_g + k_{gy} y_g - 2a_2 \zeta_{mgy} \dot{x}_n - a_2 k_{mgy}(t) f(x_n) \\
& = 0, \\
& \ddot{z}_g + 2\zeta_{gz} \dot{z}_g + k_{gz} z_g - 2a_3 \zeta_{mgz} \dot{x}_n - a_3 k_{mgz}(t) f(x_n) \\
& = 0.
\end{aligned} \tag{14}$$

4. Numerical Analysis

To solve the vibration response of the system, the state space variables $\{q\}$ are introduced:

$$\{q\} = \{q_1 \ q_2 \ q_3 \ q_4 \ q_5 \ q_6 \ q_7 \ q_8 \ q_9 \ q_{10} \ q_{11} \ q_{12} \ q_{13} \ q_{14} \ q_{15} \ q_{16} \ q_{17} \ q_{18} \ q_{19} \ q_{20} \ q_{21} \ q_{22}\}^T. \tag{15}$$

Select the state space variables:

$$\begin{aligned}
q_1 &= x_1, \\
q_2 &= \dot{x}_1, \\
q_3 &= x_2, \\
q_4 &= \dot{x}_2, \\
q_5 &= x_5, \\
q_6 &= \dot{x}_5, \\
q_7 &= x_6, \\
q_8 &= \dot{x}_6, \\
q_9 &= x_8,
\end{aligned}$$

$$\begin{aligned}
q_{10} &= \dot{x}_8, \\
q_{11} &= x_p, \\
q_{12} &= \dot{x}_p, \\
q_{13} &= y_p, \\
q_{14} &= \dot{y}_p, \\
q_{15} &= z_p, \\
q_{16} &= \dot{z}_p, \\
q_{17} &= x_g, \\
q_{18} &= \dot{x}_g,
\end{aligned}$$

$$\begin{aligned}
q_{19} &= y_g, \\
q_{20} &= \dot{y}_g, \\
q_{21} &= z_g, \\
q_{22} &= \dot{z}_g.
\end{aligned} \tag{16}$$

Combining (15) and (16), (14) can be altered as

$$\begin{aligned}
\dot{q}_1 &= q_2, \\
\dot{q}_2 &= \frac{T_1}{b\omega_n^2 m_1 R_1} - 2\zeta_1 (q_2 - q_4) - k_1 (q_1 - q_3), \\
\dot{q}_3 &= q_4, \\
\dot{q}_4 &= \left[\frac{m_3 P_2}{M_2^3} \frac{R_3^2 b}{R_1^2} q_4^2 - 2 \left(m_2 R_1 \zeta_2 + m_3 \zeta_3 \frac{R_3^2}{R_1 M_2^2} \right) q_4 \right. \\
&\quad + 2m_2 R_1 \zeta_2 q_2 + \frac{2m_3 \zeta_3 R_4 R_3}{R_5 M_5 M_2} q_6 - k_2 m_2 R_1 q_3 \\
&\quad + k_2 m_2 R_1 q_1 - \frac{k_3 m_3 R_3^2}{bM_2} \arctan \frac{\tan(bq_3/R_2)}{\cos \alpha} \\
&\quad + \left. \frac{k_3 m_3 R_4 R_3}{bM_2} \arctan \frac{\tan(bq_5/R_5)}{\cos \alpha} \right] \left(m_2 R_1 \right. \\
&\quad + \left. \frac{m_3 R_3^2}{R_1 M_2^2} \right)^{-1}; \\
\dot{q}_5 &= q_6, \\
\dot{q}_6 &= \left[\frac{m_4 P_5}{M_5^3} \frac{R_4^2 b}{R_5^2} q_6^2 - \frac{2m_4 \zeta_4 R_4}{M_5} \left(\frac{R_4}{R_5 M_5} q_6 \right. \right. \\
&\quad - \left. \left. \frac{R_3}{R_1 M_2} q_4 \right) - a_3 k_5 (t) m_5 R_5 f(x_n) \right. \\
&\quad - 2a_3 m_5 \zeta_5 [(q_{12} - q_{18}) a_1 + (q_{14} - q_{20}) a_2 \\
&\quad + (q_{16} - q_{22} + q_6 - q_8) a_3 - \dot{e}(t)] \\
&\quad - \frac{k_4 m_4 R_4}{bM_5} \left(R_4 \arctan \frac{\tan(bq_5/R_5)}{\cos \alpha} \right. \\
&\quad - \left. R_3 \arctan \frac{\tan(bq_3/R_2)}{\cos \alpha} \right) \left. \right] \left(m_5 R_5 + \frac{m_4 R_4^2}{R_5 M_5^2} \right)^{-1}; \\
\dot{q}_7 &= q_8, \\
\dot{q}_8 &= \left[-2m_7 \zeta_7 R_7 \left(\frac{R_7}{R_6} q_8 - q_{10} \right) - 2m_7 k_7 R_7 \left(\frac{R_7}{R_6} q_7 \right. \right. \\
&\quad - \left. \left. q_9 \right) + a_3 k_6 m_6 R_6 f(x_n) \right.
\end{aligned}$$

$$\begin{aligned}
&\quad + 2a_3 m_6 \zeta_6 R_6 [(q_{12} - q_{18}) a_1 + (q_{14} - q_{20}) a_2 \\
&\quad + (q_{16} - q_{22} + q_6 - q_8) a_3 - \dot{e}(t)] \left. \right] \left(m_6 R_6 \right. \\
&\quad + \left. \frac{2m_7 R_7^2}{R_6} \right)^{-1}, \\
\dot{q}_9 &= q_{10}, \\
\dot{q}_{10} &= \frac{T_8}{b\omega_n^2 m_8 R_8} - 2\zeta_8 \left(q_{10} - \frac{R_7}{R_6} q_8 \right) - k_8 \left(q_9 - \frac{R_7}{R_6} \right. \\
&\quad \cdot \left. q_7 \right), \\
\dot{q}_{11} &= q_{12}, \\
\dot{q}_{12} &= -2\zeta_{px} q_{12} - k_{px} q_{11} - a_1 k_{mpx}(t) f(x_n) \\
&\quad - 2a_1 k_{mpx} [(q_{12} - q_{18}) a_1 + (q_{14} - q_{20}) a_2 + (q_{16} \\
&\quad - q_{22} + q_6 - q_8) a_3 - \dot{e}(t)], \\
\dot{q}_{13} &= q_{14}, \\
\dot{q}_{14} &= -2\zeta_{py} q_{14} - k_{py} q_{13} - a_2 k_{mpy}(t) f(x_n) \\
&\quad - 2a_2 k_{mpy} [(q_{12} - q_{18}) a_1 + (q_{14} - q_{20}) a_2 + (q_{16} \\
&\quad - q_{22} + q_6 - q_8) a_3 - \dot{e}(t)], \\
\dot{q}_{15} &= q_{16}, \\
\dot{q}_{16} &= -2\zeta_{pz} q_{16} - k_{pz} q_{15} - a_3 k_{mpz}(t) f(x_n) \\
&\quad - 2a_3 k_{mpz} [(q_{12} - q_{18}) a_1 + (q_{14} - q_{20}) a_2 + (q_{16} \\
&\quad - q_{22} + q_6 - q_8) a_3 - \dot{e}(t)], \\
\dot{q}_{17} &= q_{18}, \\
\dot{q}_{18} &= -2\zeta_{gx} q_{18} - k_{gx} q_{17} + a_1 k_{mgx}(t) f(x_n) \\
&\quad + 2a_1 k_{mgx} [(q_{12} - q_{18}) a_1 + (q_{14} - q_{20}) a_2 + (q_{16} \\
&\quad - q_{22} + q_6 - q_8) a_3 - \dot{e}(t)], \\
\dot{q}_{19} &= q_{20}, \\
\dot{q}_{20} &= -2\zeta_{gy} q_{20} - k_{gy} q_{19} + a_2 k_{mgy}(t) f(x_n) \\
&\quad + 2a_2 k_{mgy} [(q_{12} - q_{18}) a_1 + (q_{14} - q_{20}) a_2 + (q_{16} \\
&\quad - q_{22} + q_6 - q_8) a_3 - \dot{e}(t)], \\
\dot{q}_{21} &= q_{22}, \\
\dot{q}_{22} &= -2\zeta_{gz} q_{22} - k_{gz} q_{21} + a_3 k_{mgz}(t) f(x_n) \\
&\quad + 2a_3 k_{mgz} [(q_{12} - q_{18}) a_1 + (q_{14} - q_{20}) a_2 + (q_{16} \\
&\quad - q_{22} + q_6 - q_8) a_3 - \dot{e}(t)].
\end{aligned} \tag{17}$$

TABLE 1: Parameters of the drive shaft.

Parameters	Shear modulus G (GPa)	Length of the shaft L (mm)	Outside diameter D (mm)	Inside diameter d (mm)
Shaft 1	79.4	508	63.5	59.9
Shaft 2	79.4	879	63.5	59.9
Left (right) half-shaft	82.8	719.5	31 (solid circular section)	

TABLE 2: Parameters of the pinion and gear.

	Pinion	Gear
Number of teeth	8	41
Transverse module		4.161
Average pressure angle (ALPHA)		21.25°
Pitch diameter (D_0) (mm)		170.6
Offset distance (mm)		25 (lower)
Mean spiral angle (BETA)	47.97°	28.27°
Pitch cone angle (DELTA-B)	12.75°	76.50°
Tooth width (BF) (mm)	33.02	28
Base circle diameter (D_b)	$D_b = D_0 * \cos(\text{ALPHA})$	

TABLE 3: Values of the torsional stiffness and damping.

Torsional stiffness (N·mm/rad)	Value	Torsional damping (N·mm-s/rad)	Value
K_1, K_2	51945.046	c_1, c_2	1310.286
K_3, K_4	30020.572	c_3, c_4	1310.286
K_7, K_8	652.11800	c_7, c_8	437.416

The vibration characteristics of the system can be obtained by solving (17) with Runge-Kutta method.

4.1. Parameter Setting. The main parameters of the drive shaft and gear pair are listed in Tables 1 and 2.

Based on the initial parameters, the torsional stiffness and damping of the shaft-final drive system are calculated in Table 3. The average torsional stiffness and damping of the gear pair are as follows:

$$k_m = k_{mb} \left(\frac{r_{b1}^2 + r_{b2}^2}{2r_{b1}^2 r_{b2}^2} \right), \quad (18)$$

$$c_m = 2\xi_m \sqrt{k_{mb} \left(\frac{1}{m_p} + \frac{1}{m_g} \right)},$$

where r_{b1} , r_{b2} are the pitch circle radius of the pinion and gear, m_p , m_g are the masses of the pinion and gear, k_{mb} is the gear mesh stiffness with a linear value: 1.84×10^3 N/mm, and ξ_m is

TABLE 4: Values of the equivalent moment of inertia and equivalent mass.

Equivalent moment of inertia (kg·mm ²)	Value	Equivalent mass (kg)	Value
J_1	952.242	m_{px}	1.012
J_2	2106.616	m_{py}	1.012
$J_3, \dots, J_4$	1979.326	m_{pz}	1.012
J_5	955.202	m_{gx}	5.115
J_6	17122.301	m_{gy}	5.115
J_7	10329.170	m_{gz}	5.115
J_8	124675.6		

the damping ratio with a value of 0.1 here. Then the equivalent polar moment of inertia of the first transmission shaft at Node 1 of the U-joint can be described as

$$J_1 = \frac{1}{2} (J_{szz1} + J_{z1}) + J_{cdc1}. \quad (19)$$

J_{szz1} is the first universal joint cross-shaft's equivalent polar moment of inertia about the first transmission axis, J_{z1} is the first transmission shaft's equivalent polar moment of inertia about its geometric centre, and J_{cdc1} is the first joint shaft fork's equivalent polar moment of inertia about the first transmission axis. Similarly, the equivalent moment of inertia and equivalent mass can be calculated by the 3D CAD model, and the values are listed in Table 4.

In the gear system, there are two parallel TRBs for supporting the pinion and the other two TRBs for supporting the gear. The equivalent method is an effective way to deal with the bearing stiffness; the equivalent stiffness and damping of the bearings along x-, y-, and z-axes are obtained as

$$K_{pj} = \frac{K_{pj1}}{2} + \frac{K_{pj2}}{2},$$

$$c_{pj} = \frac{c_{pj1}}{2} + \frac{c_{pj2}}{2}, \quad (20)$$

$$K_{gj} = \frac{K_{gj1}}{2} + \frac{K_{gj2}}{2},$$

$$c_{gj} = \frac{c_{gj1}}{2} + \frac{c_{gj2}}{2},$$

where K_{ij1} , K_{ij2} , c_{ij1} , c_{ij2} ($i = p, g$; $j = x, y, z$) are the four different bearing stiffnesses and dampings.

4.2. Results at the Given Speed. Considering the drive shaft at a speed of 2000 rpm, and the input torque $T_1 = 2.2 \times 10^5$ N·mm, the influence of bearing stiffness on the coupled flexural-torsional vibration of the shaft-final drive system can be analyzed. A bearing stiffness coefficient a is introduced in this paper. Figure 3 shows the output dynamic response of the shaft and driven gear. The speed fluctuation value first increases and then decreases, but the fluctuation value of

FIGURE 3: Influence of bearing stiffness on torsional vibration of the system.

driven gear is more sensitive to the different bearing stiffness coefficients. When the coefficient a is 0.7, the fluctuation maximum occurs; it is because different bearing stiffness can change the system natural frequency, and the system resonance region appears at a certain excitation frequency corresponding to $a = 0.7$. In cases where other parameters remain constant, the shaft and gear speed fluctuations are significantly reduced as the bearing stiffness coefficient $a > 0.7$; when it increases to a critical value, the reduction trend becomes less obvious.

Figures 4(a) and 4(b) show the bending vibration of the hypoid gear when the coefficient increases from 0.6 to 2.0. By comparison under the same conditions, a maximum displacement response occurs along z -axis, and the minimal fluctuation occurs along y -axis, the displacement response along x -axis and z -axis has little difference, and the response of pinion is stronger than gear. Considering the influence of different bearing stiffness on bending vibration system, the displacement response follows a similar trend to the torsional vibration in Figure 3, and the displacement fluctuation increases to a peak value and then moves downward to the steady state.

4.3. Results at the Different Speeds. For further analysis of the vibration characteristics, the gear transmission errors under different speeds are considered. Figure 5 shows the vibration response when the stiffness coefficient varies from 0.7 to 1.5 and the drive shaft input speed changes from 800 rpm to 5000 rpm, in which a speed variation coefficient n_v is used to describe the drive excitation.

Figure 5(a) shows the displacement fluctuation of the transmission error under different stiffness coefficients and different input regions, partial enlargements are shown in Figures 5(b) and 5(c), and the parameters in the resonance region of the system vary with the stiffness, but the resonance regions mainly occur from 1.45 to 1.65 and 2.00 to 2.20. In the regions of $1.45 \leq n_v \leq 1.65$, the maximum fluctuation occurs when a is 1.3, and the dimensionless value is 4.84×10^{-5} , but the fluctuation region is not obvious compared with other speed regions (see Figure 5(b)). In the regions

of $2.00 \leq n_v \leq 2.20$, the minimum fluctuation occurs when $a = 1.2$ and then follows by 0.8 and 1.0, but when a is 1.5, the fluctuation increases significantly (see Figure 5(c)). Overall, the transmission error has a nonlinear relationship with the stiffness coefficient, considering the different input speeds; when a is 0.8, 1.0, and 1.2, the fluctuations are relatively smaller; meanwhile the influence on transmission error fluctuation is not obvious with the other stiffness coefficients.

The bearing stiffness exerts a significant influence on the vibration characteristics of the shaft-final drive system. When the bearing stiffness increases at 2000 rpm, the speed fluctuation first increases and then decreases; when it increases to a critical value, the reduction trend is hardly changeable; then the resonance regions of the coupled system vary with the bearing stiffness under different speeds. Obviously, the system has a comprehensive minimum fluctuation at $a = 1.2$.

5. Dynamic Simulation

To investigate the dynamic response of the system under different bearing stiffness, a rigid-flexible coupling model is established in the ADAMS software, and then by introducing variables DV_1, DV_2, DV_3, and DV_4, where DV_1 and DV_2 are the damping and bearing stiffness of the pinion and DV_3 and DV_4 are the damping and bearing stiffness of the gear, the model adopts a Bushing connection to simulate the bearing, and the rotational stiffness and damping values are set to 0; then the measure functions of the virtual prototype model are established. The simulation is based on the following: the input speed is 2000 rpm, the load torque is 2.2×10^5 N-mm, and the speed fluctuation coefficient which comes from the engine experiments of the vehicle is 0.025.

As the bearing stiffness increases, the fluctuations in a certainty scope on output speed and torque of the system are nonlinear and complicated, and the output of drive shaft and driven gear could well describe the system performance, especially the influence of different bearing stiffness on coupling vibration of the shaft-final drive assembly. For the quantitative analysis, Trial 1~Trial 14 represent fourteen sets of data successively in Figure 6, corresponding to $0.6 \leq a \leq 1.9$; it means that Trial 1 represents $a = 0.6$, and Trial 14 represents $a = 1.9$. Figures 6(a) and 6(b) show the output speed and torque fluctuations of drive shaft. From Trial 5 to Trial 7 ($1.0 \leq a \leq 1.2$), the output speed fluctuations change slowly and the values are relatively small, but the output torque is not identical: the fluctuations are obvious in regions of $0.6 \leq a \leq 0.8$ and $1.6 \leq a \leq 1.9$, the minimum value occurs at $a = 1.5$, and it is relatively smooth and steady at $1.2 \leq a \leq 1.4$.

Similarly, Figure 7 shows the fluctuation curves of driven gear output speed and torque. Fourteen sets of data, Trial 1~Trial 14, are also selected as the variables corresponding to $0.6 \leq a \leq 1.9$. From Trial 5 to Trial 7, the output speed and torque fluctuations are obvious in the regions of $0.6 \leq a \leq 0.8$ or $1.6 \leq a \leq 1.9$. Figures 7(a) and 7(b) show the similar fluctuation trend under different bearing stiffness and change slowly at $1.2 \leq a \leq 1.5$.

The simulation results in Figures 6 and 7 show that the dynamics of shaft-final drive system is affected by the bearing

FIGURE 4: Influence of bearing stiffness on bending vibration of the hypoid gear.

FIGURE 5: Gear transmission error x_n .

FIGURE 6: Output speed and torque of the drive shaft.

FIGURE 7: Output speed and torque fluctuations of the driven gear.

stiffness. Considering the output fluctuations of drive shaft and driven gear, and the influence of gear transmission error, the fluctuation values of drive shaft are lower than that of driven gear. Compared with the initial speed and torque, the fluctuations are relatively smaller. From the fluctuation trend, there exists a reasonable region of bearing stiffness coefficient $1.2 \leq a \leq 1.5$ in which the dynamic fluctuations of the shaft-final drive system keep a smaller and steady state.

In fact, the speed and torque fluctuations are closely related to engine output dynamics and transmission ratio. When running in the 3rd gear, the power matching parameters in detail are as shown in Table 5.

The curves of seven different colours from a_1 to a_7 which represent the output fluctuations under different speeds are shown in Figure 8, corresponding to seven groups of bearing stiffness. The selected stiffness reflects the key dynamics of this shaft-final drive system as it originates from the base points: 0.6, 0.7, 0.8, 1.0, 1.2, 1.3, and 1.5 in turn. Figures 8(a) and 8(b) are the output speed fluctuation curves of drive shaft and driven gear.

In Figure 8(a), the output fluctuations of drive shaft increase as the input speed increases at $1.0 \leq a \leq 1.5$, but the fluctuations increase sharply above 3570 rpm from 0.6 to 0.8. Figure 8(b) shows that the output fluctuations are not obvious under different bearing stiffness at the low input speeds, and

TABLE 5: Drive shaft input speed/torque matching parameters.

Engine output speed (rpm)	2500	3000	3500	4000	4500	5000	5500
Engine output torque $\times 10^3$ (N·mm)	99	104	108	110	109	106	104
Output speed in third gear (rpm)	1785.7	2142.9	2500	2857.1	3214.3	3571.4	3928.6
Output torque in third gear $\times 10^3$ (N·mm)	138.6	145.6	151.2	154	152.6	148.4	145.6

it is minimal from 1800 to 2400 rpm at $a = 0.8$. However, the curves show an obvious difference above 3500 rpm, the difference is minimal at $a = 1.2$ and then follows by 1.3 and 1.5, and the fluctuations of driven gear have a sudden change at $0.6 \leq a \leq 0.8$.

6. Experimental Verification

From the above theoretical analysis, the influence of bearing stiffness on the nonlinear dynamics of a shaft-final drive

FIGURE 8: Speed fluctuations of the drive shaft and driven gear.

FIGURE 9: Layout and installation of two sensors.

system is obvious, and the different bearing stiffness coefficients lead to the resonance parameters of the system changing. In the installation process, bearing preload adjustment is a simple and effective method to change the bearing stiffness, but it is still not enough for the improved system. To obtain suitable preload adjustment, the initial bearing pairs are replaced by other ones with the improved stiffness. Comparison test of the initial ($a = 1.0$) and optimized bearing stiffness ($a = 1.2$) of the drive system is designed to analyze the rear axle vibration behaviour and interior noise under different conditions.

6.1. Shaft-Final Drive System Vibration Behaviour. For obtaining better test results, the self-developed software of *Portable vibration noise performance testing system* is designed by Wuhan University of Technology (WUT) to detect the relationship between shaft speed and rear axle vibration in vehicle driving. Its hardware part consists of computer, sensors, DAQ Card, preamplifier, and so forth and the key structure components are shown in Figure 9.

There are two different types of sensors: one is photoelectric sensor, which is used to detect the speed; its magnetic base is fixed on flat chassis, and it is installed to vertical alignment of reflective tape pasted in the drive shaft keeping

20 mm distance; the other is the vibration acceleration sensor, which is used to rear axle vibration behaviour. The two sensors are fixed on the reducer along vertical and horizontal axis, the signal lines connect two channels of instrument box, and the sensitivity values are 4.77 pc/s^2 and 4.94 pc/s^2 , respectively.

In order to analyze the relative error between the simulation and experiment, comparative analysis is used to research the system response. The speed fluctuations of driven gear which reflect the system output characteristics are obvious compared to that of drive shaft; it can be tested in the shaft-final drive assembly with the *Rear axle noise test platform* in SAIC-GM-Wuling Automobile Co., Ltd. (SGMW). The statistical data are listed in Table 6.

From the comparative analysis in Table 6, the results of dynamic simulation match with the experiment data in the commonly used speed range of 1500 rpm–3000 rpm, the relative errors of the initial and optimized driven gear speed fluctuations are less than 3%, the relative error maximum is 2.36% at $a = 1.0$ with the input speed of 1500 rpm, and the fluctuations at $a = 1.2$ are less than that at $a = 1.0$; it can be interpreted as the excitation frequency is away from the system natural frequency at $a = 1.2$, which can provide good vibration characteristics. Overall, the relative errors of the

FIGURE 10: Test curves along vertical and horizontal axis.

TABLE 6: Comparative analysis on the speed fluctuations of the driven gear.

	Input speed (rpm)	Simulation (rad/s)	Experiment data (rad/s)	Relative error (%)
$a = 1.0$	1500	2053.29	2101.80	2.36
	2000	2014.77	2028.05	0.66
	2500	1953.95	1975.70	1.11
	3000	1899.10	1931.92	1.72
$a = 1.2$	1500	2034.05	2074.88	2.01
	2000	2005.67	2015.74	0.50
	2500	1947.22	1966.38	0.98
	3000	1890.26	1918.56	1.49

simulation and experiment results follow the same trend; the dynamic simulation can meet the system test accuracy.

When the field test layout is finished, the vibration responses under different conditions are obtained in the test, and the initial and improved vibration signals along vertical and horizontal axis are shown in Figure 10.

In Figure 10, the vertical fluctuations are larger from 3200 rpm to 3500 rpm and stronger than those along horizontal axis obviously. The dynamic behaviour is slightly different under different conditions, considerable fluctuation mainly occurs from 3500 rpm to 4200 rpm in Figure 10(a1), and it has an abrupt horizontal vibration in 5th gear of WOT in

Figure 10(b1). Compared with the initial test, the improved vibration signals shown in Figures 10(a2) and 10(b2) are more stable, and the vertical fluctuation amplitudes are below the target; this trend is obvious from 3500 rpm to 4200 rpm.

6.2. Interior Noise in Vehicle Driving. LMS Test. Lab is widely adopted for data acquisition and processing in vehicle NVH performance evaluation. In the test road, the repetitive tests of the Wide Open Throttle (WOT) and Coast down with 5th gear are finished with the test system, and then the initial and improved vertical vibration signals are shown in Figure 11.

In Figure 11(a), the interior noise mostly ranges from 42.5 dB to 67.5 dB in WOT and 40 dB to 57.5 dB in Coast down; the peak occurs at 2900 rpm and 3500 rpm nearby; when the bearing stiffness is improved in Figure 11(b), the overall interior noise decreases obviously by comparison, especially in the peak region from 2900 rpm to 3500 rpm. When running at a low speed, the average decibel value is larger than the initial test, but the noise level is mostly below the target, which reflects the consistency to the theoretical analysis on the nonlinear dynamics of the shaft-final drive system.

7. Conclusions

A 14-DOF coupled vibration model of a shaft-final drive system is established; the influence of nonlinear factors was considered. Based on the deduced dynamic differential equations, numerical analysis method is adopted to analyze the

FIGURE 11: Interior noise test under different conditions.

dynamic response with the introduced stiffness coefficient. Results show that improving the bearing stiffness can reduce the vibration response at the given speed; when the bearing stiffness increases to a critical value, the reduction becomes insignificant, and the resonance regions of the coupled system vary with the bearing stiffness at the different driving speeds.

Influence of different bearing stiffness on the nonlinear system was simulated to investigate the dynamic response at the given speed and different speeds. When the stiffness coefficient a is 1.2, compared with other coefficients, the output fluctuations of the drive shaft and gear can be decreased, and the system performance improves significantly. Then the experiments on the system vibration behaviour and interior noise in vehicle driving were carried out, and the self-developed software of Portable vibration noise performance testing system was designed for performance detection of the vibration system. By comparative analysis of the initial and improved system, the vibration behaviour and interior noise can be improved when a is 1.2; meanwhile, the relative error analysis of the simulation and experiment demonstrates the correction of numerical results.

Nomenclature

- θ_i : Angular displacement of transmission components (where $i = 1, \dots, 10$)
- x_i, y_i, z_i : Displacement of hypoid gear pair along x -, y -, and z -axes (where $i = p, g$)
- c_m : Average damping of hypoid gear
- α_n : Normal pressure angle
- β_1 : Mean spiral angle
- δ_1 : Pitch cone angle
- x_n : Relative displacement of normal direction at the contact point
- a : Bearing stiffness coefficient (ratio of selected stiffness and initial bearing stiffness)
- n_v : Speed variation coefficient (ratio of selected speed and given speed)

- k_m : Mean mesh stiffness
- k_{Ai} : Fluctuation amplitude in i th order
- Ω_h : Mesh frequency of hypoid gear
- φ_k : Initial phase of dynamic mesh stiffness
- R_i : Radius of pinion and gear at the contact point (where $i = p, g$)
- b : Half of gear backlash
- φ_e : Initial phase of static transmission error
- e_0 : Mean transmission error at the contact point
- e_{Ai} : Transmission error in i th order at the contact point
- J_i : Equivalent moment of inertia of transmission components (where $i = 1, \dots, 8$)
- m_{ji} : Equivalent lumped mass of pinion and gear (where $i = p, g; j = x, y, z$)
- $m_{xp} = m_{yp} = m_{zp}, m_{xg} = m_{yg} = m_{zg}$
- K_i : Equivalent stiffness of shaft (where $i = 1, \dots, 4$) and half-shaft (where $i = 7, 8$)
- c_i : Damping of shaft (where $i = 1, \dots, 4$) and half-shaft (where $i = 7, 8$)
- K_{ij} : Equivalent stiffness along x -, y -, and z -axes (where $i = p, g; j = x, y, z$)
- c_{ij} : Equivalent damping along x -, y -, and z -axes (where $i = p, g; j = x, y, z$).

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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