Research Article

Free Vibration Analysis of Circular Cylindrical Shells with Arbitrary Boundary Conditions by the Method of Reverberation-Ray Matrix

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An analytical procedure for free vibration analysis of circular cylindrical shells with arbitrary boundary conditions is developed with the employment of the method of reverberation-ray matrix. Based on the Flügge thin shell theory, the equations of motion are solved and exact solutions of the traveling wave form along the axial direction and the standing wave form along the circumferential direction are obtained. With such a unidirectional traveling wave form solution, the method of reverberation-ray matrix is introduced to derive a unified and compact form of equation for natural frequencies of circular cylindrical shells with arbitrary boundary conditions. The exact frequency parameters obtained in this paper are validated by comparing with those given by other researchers. The effects of the elastic restraints on the frequency parameters are examined in detail and some novel and useful conclusions are achieved.

1. Introduction

Circular cylindrical shells are widely used in engineering structures, such as rockets, aircrafts, submarines, and pipelines. Due to the extensive application, the vibration characteristics of circular cylindrical shells have been a topic of major interest to many researchers. A comprehensive review of various thin shell theories developed before 1973 was presented by Leissa [1], in which a few solution techniques have been developed for some types of boundary conditions as well.

A large number of researches on vibration characteristics of circular cylindrical shells with various methods are available in the literature. Chang and Greif [2] studied the vibrations of a multisegment cylindrical shell with a common mean radius based on the component mode method coupled with Fourier series and Lagrange multipliers. Chung [3] presented a general analytical method for evaluating the free vibration characteristics of a circular cylindrical shell with classical boundary conditions of any type based on Sanders' shell equations. Lam and Loy [4] used a formulation based on Love's first approximation theory and with beam functions

used as axial modal functions in the Ritz procedure to study the effects of various classical boundary conditions on the free vibration characteristics for a multilayered cylindrical shell. Mirza and Alizadeh [5] investigated the effects of detached base length on the natural frequencies and modal shapes of cylindrical shells. Zhou and Yang [6] proposed a distributed transfer function method for static and dynamic analysis of general cylindrical shells. Loy et al. [7] presented the free vibration analysis of cylindrical shells using an improved version of the differential quadrature method. Yim et al. [8] extended the receptance method to a clamped-free cylindrical shell with a plate attached in the shell at an arbitrary axial position. An analytical procedure to study the free vibration characteristics of thin circular cylindrical shells was presented by Naeem and Sharma [9], in which Ritz polynomial functions are assumed to model the axial modal dependence and the Rayleigh-Ritz variational approach is employed to formulate the general eigenvalue problem. Based on Love's equations and an infinite length model, Wang and Lai [10] introduced a novel wave approach to predict the natural frequencies of finite length circular cylindrical shells with different boundary conditions without simplifying the

equations of motion. Ghoshal et al. [11] and Zhang et al. [12, 13] presented the vibration analysis of cylindrical shells using a wave propagation method. El-Mously [14] presented a comparative study of three approximate explicit formulae for estimating the fundamental natural frequency of a thin cylindrical shell and its associated fundamental mode number. Karczub [15] presented algebraic expressions for direct evaluation of structural wave number in cylindrical shells based on the Flügge equations of motion. Mofakhami et al. [16] developed a general solution to analyze the vibration of finite circular cylinders utilizing the infinite circular cylinders solution based on the technique of variables separation. Sakka et al. [17] presented an analytical investigation to the free vibration response of moderately thick shear flexible isotropic cylindrical shells with all edges clamped. Based on the Flügge thin shell theory, Li [18] obtained the free vibration frequencies of a thin circular cylindrical shell using a new method which introduces a general displacement representation and develops a type of coupled polynomial eigenvalue problem. Pellicano [19] presented a method for analyzing linear and nonlinear vibrations of circular cylindrical shells having different boundary conditions based on the Sanders-Koiter theory. Based on the Flügge thin shell theory, Zhang and Xiang [20] presented exact solutions for the vibration of circular cylindrical shells with stepwise thickness variations in the axial direction. Li [21] presented a wave propagation approach for free vibration analysis of circular cylindrical shell based on Flügge classical thin shell theory.

Recently, Brischetto and Carrera [22] carried out the free vibration analysis of one-layered and two-layered metallic cylindrical shell panels for both thermomechanical and pure mechanical problems. Based on the Flügge thin shell theory, Zhou et al. [23] investigated the free vibrations of cylindrical shells with elastic boundary conditions by the method of wave propagations. Chen et al. [24] presented a wave based method to analyze the free vibration characteristics of ring-stiffened cylindrical shell with intermediate large frame ribs for arbitrary boundary conditions. Chen et al. [25] investigated the free vibrations of cylindrical shell with nonuniform elastic boundary constraints using the improved Fourier series method. Dai et al. [26] obtained an exact series solution for the vibration analysis of circular cylindrical shells with arbitrary boundary conditions by using the elastic equations based on the Flügge thin shell theory. Qu et al. [27, 28] presented a domain decomposition technique for solving vibration problems of uniform and stepped cylindrical shells as well as isotropic and composite cylindrical shells with arbitrary boundary conditions. Based on the Donnell-Mushtari shell theory, Xing et al. [29] presented exact solutions of a simple and compact form for free vibration of circular cylindrical shells with classical boundary conditions. Chen et al. [30] presented an analytic method to analyze free and forced vibration characteristics of ring-stiffened combined conical-cylindrical shells with arbitrary boundary conditions. Li et al. [31] investigated the influence on modal parameters of thin cylindrical shell under bolt looseness boundary. Yao et al. [32] carried out the free vibration analysis of open circular cylindrical shells with either the two straight edges or the two curved edges simply supported and the remaining two edges

FIGURE 1: Geometry and the dual local coordinate system for a circular cylindrical shell.

supported by arbitrary classical boundary conditions based on the Donnell-Mushtari-Vlasov thin shell theory.

As far as the researches of circular cylindrical shells are concerned, most of the existing works are limited to classical boundary conditions. Some efforts are recently made to study the vibration characteristics of circular cylindrical shells with arbitrary boundary conditions. Based on the Flügge thin shell theory, this paper presents a unified and compact formulation for free vibration analysis of circular cylindrical shells with arbitrary boundary conditions including classical ones and nonclassical ones using the method of reverberation-ray matrix (MRRM). With the analytical solution of the traveling wave form along the axial direction and the standing wave form along the circumferential direction obtained by Yao et al., MRRM is introduced to derive the unified equation for natural frequencies of the circular cylindrical shell with arbitrary boundary conditions. To validate the method presented in this paper, some results for both classical boundary conditions and nonclassical boundary conditions are compared with those found in the published literature. The effects of the elastic restraints on the natural frequencies are examined in detail and some novel and useful conclusions are achieved at the end of this paper.

2. Formulation

Consider an isotropic circular cylindrical shell with length L , uniform thickness h , middle surface radius R , Young's modulus E , Poisson's ratio μ , and mass density ρ as shown in Figure 1. The axial, circumferential, and radial displacements of the middle surface of the circular cylindrical shell with reference to the coordinate system are denoted as $u(x, \theta, t)$, $v(x, \theta, t)$, and $w(x, \theta, t)$. The problem at hand is to determine the natural frequencies of the circular cylindrical shell with arbitrary boundary conditions.

2.1. Force and Moment Resultants. The force and moment resultants acting on the cross section perpendicular to the axial direction in a circular cylindrical shell, shown in Figure 2, are expressed in terms of the displacement components u , v , and w as

$$N_x = C \left[\frac{\partial u}{\partial x} + \mu \frac{1}{R} \frac{\partial v}{\partial \theta} - \lambda R \frac{\partial^2 w}{\partial x^2} + \mu \frac{1}{R} w \right], \quad (1)$$

FIGURE 2: Force and moment resultants in a circular cylindrical shell.

$$N_{x\theta} = \frac{1-\mu}{2}C \left[\frac{1}{R} \frac{\partial u}{\partial \theta} + (1+\lambda) \frac{\partial v}{\partial x} - \lambda \frac{\partial^2 w}{\partial x \partial \theta} \right], \quad (2)$$

$$M_x = D \left[\frac{1}{R} \frac{\partial u}{\partial x} + \mu \frac{1}{R^2} \frac{\partial v}{\partial \theta} - \frac{\partial^2 w}{\partial x^2} - \mu \frac{1}{R^2} \frac{\partial^2 w}{\partial \theta^2} \right], \quad (3)$$

$$M_{x\theta} = (1-\mu)D \left(\frac{1}{R} \frac{\partial v}{\partial x} - \frac{1}{R} \frac{\partial^2 w}{\partial x \partial \theta} \right), \quad (4)$$

$$Q_{xz} = D \left[\frac{1}{R} \frac{\partial^2 u}{\partial x^2} - (1-\mu) \frac{1}{2} \frac{1}{R^3} \frac{\partial^2 u}{\partial \theta^2} + \frac{1+\mu}{2} \frac{1}{R^2} \frac{\partial^2 v}{\partial x \partial \theta} - \frac{\partial^3 w}{\partial x^3} - \frac{1}{R^2} \frac{\partial^3 w}{\partial x \partial \theta^2} \right], \quad (5)$$

$$F_{x\theta} = \frac{1-\mu}{2}C \left[\frac{1}{R} \frac{\partial u}{\partial \theta} + (1+3\lambda) \frac{\partial v}{\partial x} - 3\lambda \frac{\partial^2 w}{\partial x \partial \theta} \right], \quad (6)$$

$$V_{xz} = D \left[\frac{1}{R} \frac{\partial^2 u}{\partial x^2} - (1-\mu) \frac{1}{2} \frac{1}{R^3} \frac{\partial^2 u}{\partial \theta^2} + \frac{3-\mu}{2} \frac{1}{R^2} \frac{\partial^2 v}{\partial x \partial \theta} - \frac{\partial^3 w}{\partial x^3} - (2-\mu) \frac{1}{R^2} \frac{\partial^3 w}{\partial x \partial \theta^2} \right], \quad (7)$$

where u , v , and w denote the displacement components in the axial (x), circumferential (θ), and radial (z) directions. $C = Eh/(1-\mu^2)$ and $D = Eh^3/12(1-\mu^2)$ represent the middle surface stiffness and the bending stiffness of the shell. N_x and $N_{x\theta}$ denote the in-plane normal force and in-plane shear force. M_x and $M_{x\theta}$ represent the bending moment and torsional moment. Q_{xz} denotes the out-of-plane shear force. $F_{x\theta} = N_{x\theta} + M_{x\theta}/R$ and $V_{xz} = Q_{xz} + \partial M_{x\theta}/R \partial \theta$, respectively, represent the in-plane and out-of-plane resultant shear forces.

2.2. Governing Differential Equations. Based on the Flügge thin shell theory, the governing differential equations for free vibration of a circular cylindrical shell are written as

$$R^2 \frac{\partial^2 u}{\partial x^2} + (1+\lambda) \frac{1-\mu}{2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1+\mu}{2} R \frac{\partial^2 v}{\partial x \partial \theta} - \lambda R^3 \frac{\partial^3 w}{\partial x^3} + \lambda \frac{1-\mu}{2} R \frac{\partial^3 w}{\partial x \partial \theta^2} + \mu R \frac{\partial w}{\partial x} \quad (8)$$

$$- R^2 \frac{\rho(1-\mu^2)}{E} \frac{\partial^2 u}{\partial t^2} = 0,$$

$$\frac{1+\mu}{2} R \frac{\partial^2 u}{\partial x \partial \theta} + (1+3\lambda) \frac{1-\mu}{2} R^2 \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial \theta^2} \quad (9)$$

$$- \lambda \frac{3-\mu}{2} R^2 \frac{\partial^3 w}{\partial x^2 \partial \theta} + \frac{\partial w}{\partial \theta} - R^2 \frac{\rho(1-\mu^2)}{E} \frac{\partial^2 v}{\partial t^2} = 0,$$

$$- \lambda R^3 \frac{\partial^3 u}{\partial x^3} + \lambda \frac{1-\mu}{2} R \frac{\partial^3 u}{\partial x \partial \theta^2} + \mu R \frac{\partial u}{\partial x} - \frac{3-\mu}{2} \lambda R^2 \frac{\partial^3 v}{\partial x^2 \partial \theta} + \frac{\partial v}{\partial \theta} + \lambda R^4 \frac{\partial^4 w}{\partial x^4} \quad (10)$$

$$+ 2\lambda R^2 \frac{\partial^4 w}{\partial x^2 \partial \theta^2} + \lambda \frac{\partial^4 w}{\partial \theta^4} + 2\lambda \frac{\partial^2 w}{\partial \theta^2} + (1+\lambda) w$$

$$+ R^2 \frac{\rho(1-\mu^2)}{E} \frac{\partial^2 w}{\partial t^2} = 0,$$

where $\lambda = h^2/12R^2$ is a dimensionless parameter related with the ratio of the shell thickness to shell radius.

The Fourier transform of an arbitrary physical quantity $X(t)$ is defined by

$$\bar{X}(\omega) = \int_{-\infty}^{+\infty} X(t) e^{-i\omega t} dt, \quad (11)$$

where ω is the circular frequency, and a tilde over a symbol represents the corresponding quantity expressed in the frequency domain.

Taking the Fourier transforms of (8)~(10) and eliminating the other two displacement components from the Fourier transforms, three independent equations of motion, respectively, in terms of the axial, circumferential, and radial displacement components are obtained as

$$[(L_4 L_6 - L_3 L_3)(\mu_1 L_8 - L_5 L_7) + (L_1 L_5 - L_4 L_7)(L_2 L_7 - L_5 L_6)] \frac{\partial \bar{u}}{R \partial \theta} = 0, \quad (12)$$

$$[(L_4 L_6 - L_3 L_3)(\mu_1 L_8 - L_5 L_7) + (L_1 L_5 - L_4 L_7)(L_2 L_7 - L_5 L_6)] \frac{\partial \bar{v}}{\partial x} = 0, \quad (13)$$

$$[(L_4 L_6 - L_3 L_3)(\mu_1 L_8 - L_5 L_7) + (L_1 L_5 - L_4 L_7)(L_2 L_7 - L_5 L_6)] \frac{\partial^2 \bar{w}}{R \partial x \partial \theta} = 0, \quad (14)$$

where L_j ($j = 1 \sim 6$) are linear differential operators defined in Appendix A and μ_1 is a dimensionless parameter defined in Appendix B.

It can be observed from (12)~(14) that the differences among the linear differential operators operating on the axial, circumferential, and radial displacement components lie in the parts outside the braces, which represent zero wave number solutions or, in other words, the rigid-body motions. As normal in free vibration analysis, the zero wave number solutions are neglected throughout this paper.

2.3. Solutions for the Circular Cylindrical Shell. Solutions to the equations of motion of the circular cylindrical shell are generally assumed in the following forms:

$$\begin{aligned}\tilde{u}(x, \theta) &= \sum_{n=0}^{\infty} U(x) \cos(n\theta), \\ \tilde{v}(x, \theta) &= \sum_{n=0}^{\infty} V(x) \sin(n\theta), \\ \tilde{w}(x, \theta) &= \sum_{n=0}^{\infty} W(x) \cos(n\theta),\end{aligned}\quad (15)$$

where n is the circumferential mode number and $U(x)$, $V(x)$, and $W(x)$ are axial, circumferential, and radial displacement component functions of the axial variable. Assuming that the axial, circumferential, and radial displacement components can be expressed by exponential functions of the axial variable, a common axial wave number equation for the axial, circumferential, and radial displacement components is obtained as

$$\begin{aligned}&(\lambda_2 \mu_2 k_x^4 + \zeta_1^2 k_x^2 + \zeta_2^2 \zeta_3^2)(\lambda_3 k_x^4 + \zeta_4^2 k_x^2 + \zeta_5^4) \\ &+ \frac{1}{\lambda} (2\lambda \mu_2 k_x^4 + \zeta_6^2 k_x^2 + \zeta_2^2 \zeta_7^2)(\lambda \lambda_2 \mu_2 k_x^4 + \zeta_8^2 k_x^2 + \zeta_9^4) \\ &= 0,\end{aligned}\quad (16)$$

where μ_2 , λ_2 , and λ_3 are dimensionless parameters and ζ_j ($j = 1 \sim 9$) are parameters of the same dimension to the axial wave number. They are presented in detail in Appendix B.

By solving (16), eight axial wave numbers are obtained as

$$k_x = \pm k_{x1}, \pm k_{x2}, \pm k_{x3}, \pm k_{x4}. \quad (17)$$

Therefore, the solutions for the displacement components of an isotropic circular cylindrical shell can be explicitly rewritten as

$$\tilde{u}(x, \theta) = \sum_{n=0}^{\infty} \sum_{j=1}^4 \alpha_j (a_j e^{ik_{xj}x} - d_j e^{-ik_{xj}x}) \cos(n\theta), \quad (18)$$

$$\tilde{v}(x, \theta) = \sum_{n=0}^{\infty} \sum_{j=1}^4 \beta_j (a_j e^{ik_{xj}x} + d_j e^{-ik_{xj}x}) \sin(n\theta), \quad (19)$$

$$\tilde{w}(x, \theta) = \sum_{n=0}^{\infty} \sum_{j=1}^4 (a_j e^{ik_{xj}x} + d_j e^{-ik_{xj}x}) \cos(n\theta), \quad (20)$$

where α_j and β_j ($j = 1, 2, 3, 4$) are ratios of the axial and circumferential wave amplitudes to the radial wave amplitude. The expressions of α_j and β_j are defined as follows:

$$\begin{aligned}\alpha_j &= \frac{iRk_{xj}(\lambda \lambda_2 \mu_2 k_{xj}^4 + \zeta_8^2 k_{xj}^2 + \zeta_9^4)}{(\lambda_2 \mu_2 k_{xj}^4 + \zeta_1^2 k_{xj}^2 + \zeta_2^2 \zeta_3^2)}, \\ \beta_j &= -\frac{n(2\lambda \mu_2 k_{xj}^4 + \zeta_6^2 k_{xj}^2 + \zeta_2^2 \zeta_7^2)}{(\lambda_2 \mu_2 k_{xj}^4 + \zeta_1^2 k_{xj}^2 + \zeta_2^2 \zeta_3^2)}.\end{aligned}\quad (21)$$

The rotation of the normal to the middle surface about the circumferential direction is defined as

$$\varphi_x = -\frac{\partial w}{\partial x}. \quad (22)$$

Substituting (20) into the Fourier transform of (22) yields the frequency domain expression of the rotation

$$\tilde{\varphi}_x = \sum_{n=0}^{\infty} \sum_{j=1}^4 -ik_{xj} (a_j e^{ik_{xj}x} - d_j e^{-ik_{xj}x}) \cos(n\theta). \quad (23)$$

Substituting (18)~(20) into the Fourier transforms of (1), (3), (6), and (7) yields the frequency domain expressions of the force and moment resultants of the circular cylindrical shell:

$$\begin{aligned}\tilde{N}_x &= \frac{C}{R} \sum_{n=0}^{\infty} \sum_{j=1}^4 (\mu + \lambda R^2 k_{xj}^2 + iRk_{xj} \alpha_j + \mu n \beta_j) \\ &\cdot (a_j e^{ik_{xj}x} + d_j e^{-ik_{xj}x}) \cos(n\theta),\end{aligned}\quad (24)$$

$$\begin{aligned}\tilde{F}_{x\theta} &= \mu_2 \frac{C}{R} \sum_{n=0}^{\infty} \sum_{j=1}^4 (3\lambda n i R k_{xj} - n \alpha_j + \lambda_2 i R k_{xj} \beta_j) \\ &\cdot (a_j e^{ik_{xj}x} - d_j e^{-ik_{xj}x}) \sin(n\theta),\end{aligned}\quad (25)$$

$$\begin{aligned}\tilde{V}_{xz} &= \frac{D}{R^3} \sum_{n=0}^{\infty} \sum_{j=1}^4 [(2 - \mu) n^2 i R k_{xj} + i R^3 k_{xj}^3 \\ &+ (\mu_2 n^2 - R^2 k_{xj}^2) \alpha_j + \mu_3 n i R k_{xj} \beta_j] (a_j e^{ik_{xj}x} \\ &- d_j e^{-ik_{xj}x}) \cos(n\theta),\end{aligned}\quad (26)$$

$$\begin{aligned}\tilde{M}_x &= \frac{D}{R^2} \sum_{n=0}^{\infty} \sum_{j=1}^4 (\mu n^2 + R^2 k_{xj}^2 + iRk_{xj} \alpha_j + \mu n \beta_j) \\ &\cdot (a_j e^{ik_{xj}x} + d_j e^{-ik_{xj}x}) \cos(n\theta).\end{aligned}\quad (27)$$

2.4. Solutions Expressed in Matrix Form. For an arbitrary circumferential mode number n , (18)~(20) and (23) can be expressed in matrix form as

$$\mathbf{W}_d = \mathbf{H}_n(\theta) \mathbf{W}_d^*, \quad (28)$$

where $\mathbf{W}_d$ denotes the displacement vector of the shell, $\mathbf{H}_n(\theta)$ represents the circumferential mode matrix, and $\mathbf{W}_d^*$ denotes the wave vector of the displacement with respect to the axial variable. They are presented in detail as follows:

$$\mathbf{W}_d = \{\tilde{u} \quad \tilde{v} \quad \tilde{w} \quad \tilde{\varphi}_x\}^T, \quad (29)$$

$$\begin{aligned} \mathbf{H}_n(\theta) \\ = \text{diag} \{ \cos(n\theta) \quad \sin(n\theta) \quad \cos(n\theta) \quad \cos(n\theta) \}, \end{aligned} \quad (30)$$

$$\mathbf{W}_d^* = \mathbf{A}_d \mathbf{P}_h(x) \mathbf{a} + \mathbf{D}_d \mathbf{P}_h(-x) \mathbf{d} \quad (31)$$

in which $\mathbf{P}_h(x)$ denotes the phase matrix, $\mathbf{A}_d$ and $\mathbf{D}_d$ are coefficient matrices of fourth order, and $\mathbf{a}$ and $\mathbf{d}$ are amplitude vectors of the arriving wave and the departing wave corresponding to the displacement vector. They are presented in detail as follows:

$$\mathbf{P}_h(x) = \text{diag} \{ e^{ik_{x1}x} \quad e^{ik_{x2}x} \quad e^{ik_{x3}x} \quad e^{ik_{x4}x} \},$$

$$\mathbf{a} = \{a_1 \quad a_2 \quad a_3 \quad a_4\}^T,$$

$$\mathbf{d} = \{d_1 \quad d_2 \quad d_3 \quad d_4\}^T,$$

$$\mathbf{A}_d(1, j) = -\mathbf{D}_d(1, j) = \alpha_j,$$

$$\mathbf{A}_d(2, j) = \mathbf{D}_d(2, j) = \beta_j,$$

$$\mathbf{A}_d(3, j) = \mathbf{D}_d(3, j) = 1,$$

$$\mathbf{A}_d(4, j) = -\mathbf{D}_d(4, j) = -ik_{xj},$$

(32)

where $j = 1, 2, 3, 4$.

In the same manner, (24)~(27) are expressed in matrix form as

$$\mathbf{W}_f = \mathbf{H}_n(\theta) \mathbf{W}_f^*, \quad (33)$$

where $\mathbf{H}_n(\theta)$ is defined in (30). $\mathbf{W}_f$ denotes the force vector of the shell, and $\mathbf{W}_f^*$ denotes the wave vector of the force and moment resultants with respect to the axial variable. They are presented in detail as follows:

$$\mathbf{W}_f = \{\tilde{N}_x \quad \tilde{F}_{x\theta} \quad \tilde{V}_{xz} \quad \tilde{M}_x\}^T, \quad (34)$$

$$\mathbf{W}_f^* = \mathbf{A}_f \mathbf{P}_h(x) \mathbf{a} + \mathbf{D}_f \mathbf{P}_h(-x) \mathbf{d}$$

in which the physical significances and expressions of $\mathbf{P}_h(x)$, $\mathbf{a}$, and $\mathbf{d}$ are the same to those presented in (31). $\mathbf{A}_f$ and $\mathbf{D}_f$ are coefficient matrices of the arriving wave and the departing wave corresponding to the force and moment resultants of the shell. They are presented in detail as follows:

$$\begin{aligned} \mathbf{A}_f(1, j) &= \mathbf{D}_f(1, j) = \frac{(\mu + \lambda R^2 k_{xi}^2 + iRk_{xi}\alpha_i + \mu n\beta_i)C}{R}, \\ \mathbf{A}_f(2, j) &= -\mathbf{D}_f(2, j) = \frac{(3\lambda n i R k_{xi} - n\alpha_i + \lambda_2 i R k_{xi}\beta_i)\mu_2 C}{R}, \\ \mathbf{A}_f(3, j) &= -\mathbf{D}_f(3, j) = \frac{[(2 - \mu)n^2 i R k_{xi} + iR^3 k_{xi}^3 + (\mu_2 n^2 - R^2 k_{xi}^2)\alpha_i + \mu_3 n i R k_{xi}\beta_i]D}{R^3}, \\ \mathbf{A}_f(4, j) &= \mathbf{D}_f(4, j) = \frac{(\mu n^2 + R^2 k_{xi}^2 + iRk_{xi}\alpha_i + \mu n\beta_i)D}{R^2}, \end{aligned} \quad (35)$$

where $j = 1, 2, 3, 4$.

2.5. Equation of Natural Frequencies. Taking advantage of the unidirectional traveling wave solutions of the circular cylindrical shell obtained in the preceding subsections, the MRRM is introduced to derive the equation of natural frequencies. Since the scattering matrix is related to boundary conditions of the shell, it will be discussed in the first step. Then, the phase matrix and permutation matrix, which are independent of the boundary conditions, are derived. Finally, the reverberation-ray matrix and the equation for natural frequencies of the circular cylindrical shell are obtained. The formulation mentioned above is presented in detail in the following discussions.

2.5.1. Scattering Matrix for Various Boundary Conditions. In various boundary conditions including classical ones and nonclassical ones, for instance, the elastic-support boundary conditions can be expressed in a unified compact form at the local coordinate $(ox\theta z)$ ¹² as

$$\mathbf{K}_f^1 \mathbf{W}_f^{12} = \mathbf{K}_d^1 \mathbf{W}_d^{12}, \quad (36)$$

where $\mathbf{K}_f^1$ and $\mathbf{K}_d^1$ are coefficient matrices of fourth order. For common boundary conditions, the coefficient matrices $\mathbf{K}_f^1$ and $\mathbf{K}_d^1$ are presented in Table 1.

TABLE 1: Coefficient matrices for common boundary conditions.

BCs	$\mathbf{K}_f^1$	$\mathbf{K}_d^1$
CE	diag {0 0 0 0}	diag {1 1 1 1}
FE	diag {1 1 1 1}	diag {0 0 0 0}
SS	diag {1 0 0 1}	diag {0 1 1 0}
ES	diag {1 1 1 1}	diag { k_u k_v k_w k_φ }
E ^I	diag {1 0 0 0}	diag { k_u 0 0 0}
E ^{II}	diag {0 1 0 0}	diag {0 k_v 0 0}
E ^{III}	diag {1 1 0 0}	diag { k_u k_v 0 0}

Substituting (28) and (33) into (36) yields

$$\begin{aligned} & \mathbf{K}_f^1 [\mathbf{A}_f \mathbf{P}_h(0) \mathbf{a}^{12} + \mathbf{D}_f \mathbf{P}_h(0) \mathbf{d}^{12}] \\ &= \mathbf{K}_d^1 [\mathbf{A}_d \mathbf{P}_h(0) \mathbf{a}^{12} + \mathbf{D}_d \mathbf{P}_h(0) \mathbf{d}^{12}]. \end{aligned} \quad (37)$$

Since the phase matrix turns into a unit matrix at the origin of the local coordinate, which is obvious from its definition, (37) can be rewritten as

$$\mathbf{d}^1 = \mathbf{S}^1 \mathbf{a}^1, \quad (38)$$

where $\mathbf{d}^1 = \mathbf{d}^{12} = \{d_1^{12} d_2^{12} d_3^{12} d_4^{12}\}^T$ and $\mathbf{a}^1 = \mathbf{a}^{12} = \{a_1^{12} a_2^{12} a_3^{12} a_4^{12}\}^T$ are amplitude vectors of the departing wave and the arriving wave and $\mathbf{S}^1 = -(\mathbf{K}_d^1 \mathbf{D}_d - \mathbf{K}_f^1 \mathbf{D}_f)^{-1} (\mathbf{K}_d^1 \mathbf{A}_d - \mathbf{K}_f^1 \mathbf{A}_f)$ is the scattering matrix at one end of the circular cylindrical shell.

Similarly, the scattering relation for the other end of the circular cylindrical shell can be obtained as

$$\mathbf{d}^2 = \mathbf{S}^2 \mathbf{a}^2, \quad (39)$$

where $\mathbf{d}^2 = \mathbf{d}^{21} = \{d_1^{21} d_2^{21} d_3^{21} d_4^{21}\}^T$ and $\mathbf{a}^2 = \mathbf{a}^{21} = \{a_1^{21} a_2^{21} a_3^{21} a_4^{21}\}^T$ are amplitude vectors of the departing wave and the arriving wave and $\mathbf{S}^2 = -(\mathbf{K}_d^2 \mathbf{D}_d - \mathbf{K}_f^2 \mathbf{D}_f)^{-1} (\mathbf{K}_d^2 \mathbf{A}_d - \mathbf{K}_f^2 \mathbf{A}_f)$ is the scattering matrix at the other end.

Assembling the local scattering equations at both ends of the circular cylindrical shell by stacking $\mathbf{d}^1, \mathbf{d}^2$ and $\mathbf{a}^1, \mathbf{a}^2$ into two column vectors $\mathbf{d}$ and $\mathbf{a}$, the global scattering equation is obtained as

$$\mathbf{d} = \mathbf{S} \mathbf{a}, \quad (40)$$

where $\mathbf{d} = \{(\mathbf{d}^1)^T (\mathbf{d}^2)^T\}^T$ and $\mathbf{a} = \{(\mathbf{a}^1)^T (\mathbf{a}^2)^T\}^T$ are global amplitude vectors of the departing wave and the arriving wave and $\mathbf{S} = \text{diag}\{\mathbf{S}^1 \mathbf{S}^2\}$ is the global scattering matrix.

2.5.2. Phase Matrix and Permutation Matrix. The phase relations of harmonic waves in the dual local coordinate system of MRRM provide additional equations for solving the unknown amplitude vectors. Note that the departing wave from one end of the circular cylindrical shell is exactly the arriving wave to the other end, and vice versa. Therefore, the

amplitudes of the departing wave and the arriving wave differ with each other by a phase factor. The relations between the amplitudes of the departing wave and the arriving wave are presented as

$$\begin{aligned} a_1^{12} &= -e^{-ik_{x1}L} d_1^{21}; \\ a_2^{12} &= -e^{-ik_{x2}L} d_2^{21}; \\ a_3^{12} &= -e^{-ik_{x3}L} d_3^{21}; \\ a_4^{12} &= -e^{-ik_{x4}L} d_4^{21}; \\ a_1^{21} &= -e^{-ik_{x1}L} d_1^{12}; \\ a_2^{21} &= -e^{-ik_{x2}L} d_2^{12}; \\ a_3^{21} &= -e^{-ik_{x3}L} d_3^{12}; \\ a_4^{21} &= -e^{-ik_{x4}L} d_4^{12} \end{aligned} \quad (41)$$

which can be rewritten in matrix form as

$$\begin{aligned} \mathbf{a}^1 &= -\mathbf{P}_h(-L) \mathbf{d}^2, \\ \mathbf{a}^2 &= -\mathbf{P}_h(-L) \mathbf{d}^1. \end{aligned} \quad (42)$$

Assembling both local phase equations results in the global phase equation

$$\mathbf{a} = \mathbf{P} \mathbf{d}^*, \quad (43)$$

where $\mathbf{a}$ is defined in (40), $\mathbf{d}^* = \{(\mathbf{d}^2)^T (\mathbf{d}^1)^T\}^T$ is a rearranged global amplitude vector of the departing wave, and $\mathbf{P} = -\text{diag}\{\mathbf{P}_h(-L) \mathbf{P}_h(-L)\}$ is the global phase matrix.

A comparison of the global amplitude vectors of the departing waves $\mathbf{d}$ and $\mathbf{d}^*$ indicates that the two amplitude vectors contain the same scalar state variables arranged in different sequential orders. The relation between $\mathbf{d}$ and $\mathbf{d}^*$ is

$$\mathbf{d}^* = \mathbf{U} \mathbf{d}, \quad (44)$$

where $\mathbf{U} = [\mathbf{0}_4 \mathbf{I}_4; \mathbf{I}_4 \mathbf{0}_4]$ is the permutation matrix from $\mathbf{d}$ to $\mathbf{d}^*$, in which $\mathbf{0}_4$ and $\mathbf{I}_4$ are, respectively, the zero matrix and the unit matrix of fourth order.

2.5.3. Equation for Natural Frequencies. Substituting (43) and (44) into (40) yields

$$(\mathbf{I} - \mathbf{R}) \mathbf{d} = \mathbf{0}, \quad (45)$$

where $\mathbf{R} = \mathbf{S} \mathbf{P} \mathbf{U}$ is defined as the reverberation-ray matrix.

To obtain a nontrivial solution of the global amplitude vector of the departing wave $\mathbf{d}$, the determinant of $(\mathbf{I} - \mathbf{R})$ must be zero; namely,

$$\det(\mathbf{I} - \mathbf{R}) = 0 \quad (46)$$

which is the equation of natural frequencies of the circular cylindrical shell.

TABLE 2: Comparison of the frequency parameter $\Omega = \omega R[(1 - \mu^2)\rho/E]^{1/2}$ for a circular cylindrical shell with some classical boundary conditions ($R = 1$ m, $L/R = 20$, $h/R = 0.01$, $m = 1$).

n	SS-SS				CE-SS				CE-CE			
	Reference [7]	Reference [28]	Reference [13]	Present	Reference [7]	Reference [28]	Reference [13]	Present	Reference [7]	Reference [28]	Reference [13]	Present
1	0.016101	0.016103	0.016101	0.016101	0.023974	0.024065	0.024721	0.023930	0.032885	0.033176	0.034879	0.032792
2	0.009382	0.009388	0.009382	0.009382	0.011225	0.011238	0.011281	0.011215	0.013932	0.013970	0.014052	0.013902
3	0.022105	0.022108	0.022105	0.022105	0.022310	0.022314	0.022335	0.022309	0.022672	0.022677	0.022725	0.022667
4	0.042095	0.042097	0.042095	0.042095	0.042139	0.042141	0.042166	0.042139	0.042208	0.042210	0.042271	0.042208
5	0.068008	0.068008	0.068008	0.068008	0.068024	0.068026	0.068054	0.068025	0.068046	0.068048	0.068116	0.068046
6	0.099730	0.099730	0.099731	0.099731	0.099738	0.099739	0.099771	0.099739	0.099748	0.099749	0.099823	0.099749
7	0.137239	0.137239	0.137240	0.137240	0.137244	0.137244	0.137279	0.137245	0.137249	0.137250	0.137328	0.137251
8	0.180527	0.180528	0.180527	0.180530	0.180531	0.180531	0.180569	0.180533	0.180535	0.180535	0.180617	0.180536
9	0.229594	0.229594	0.229596	0.229596	0.229596	0.229596	0.229636	0.229599	0.229599	0.229600	0.229684	0.229601
10	0.284435	0.284436	0.284438	0.284439	0.284437	0.284438	0.284478	0.284440	0.284439	0.284441	0.284526	0.284442

2.6. Mode Shape Calculation. By substituting one of the natural frequencies obtained by solving (46) in the preceding subsection into (45), the global amplitude vector of the departing wave $\mathbf{d}$ is obtained. By normalization, the global amplitude vector of the departing wave is determinate. Subsequently, by substituting the global amplitude vector of the departing wave into (43) and (44), the global amplitude vector of the arriving wave $\mathbf{a}$ is determined.

Finally, substituting the global amplitude vector of the departing wave and the arriving wave into the expressions of the displacement components of the circular cylindrical shell, the mode shape corresponding to the natural frequency is obtained.

3. Numerical Results and Discussions

To begin with, the coefficient matrices $\mathbf{K}_f^1$ and $\mathbf{K}_d^1$ for common boundary conditions are defined in Table 1, in which k_u , k_v , and k_w , respectively, denote the translational stiffness coefficients for the springs in axial, circumferential, and radial directions, and k_ϕ represents the rotational stiffness coefficient for the spring around the circumferential direction. In order to simplify the presentation, CE, FE, SE, and ES represent clamped edge, free edge, simply supported edge, and elastic-support edge, respectively. To make a comparison with the results obtained by Qu et al. [28], three types of elastic-support edges indicated by symbols E^I , E^{II} , and E^{III} are introduced. E^I type edge is considered to be axially elastic only (i.e., $N_x = k_u u$, $v = w = \phi_x = 0$), support type E^{II} only allows elastically restrained displacement in the circumferential direction (i.e., $F_{x\theta} = k_v v$, $u = w = \phi_x = 0$), and E^{III} type edge indicates both axial and circumferential displacement components are elastically restrained (i.e., $N_x = k_u u$, $F_{x\theta} = k_v v$, $w = \phi_x = 0$).

3.1. Validation of the Present Method. In this subsection, a few well-studied classical boundary conditions and some nonclassical boundary conditions frequently encountered in practice are taken as calculation examples. The method

proposed in this paper is validated by comparing the present numerical results with those previously published in the literature.

Frequency parameters calculated for the three classical boundary conditions are presented in Table 2 together with those previously given by Loy et al. [7], Qu et al. [28], and Zhang et al. [13]. It can be observed from Table 2 that the present results agree very well with the reference solutions, which indicates that the method presented in this paper is suitable and of high accuracy for free vibration analysis of circular cylindrical shells with classical boundary conditions.

Subsequently, a comparison of the frequency parameters obtained by the method presented in this paper and those presented by Qu et al. [28] for the combination of the three types of elastic-support edges is shown in Table 3, which shows that the frequency parameters obtained by the method presented in this paper agree well with those from Qu et al. [28]. The slight differences in the results may be attributed to the different shell theories and solution approaches adopted in the literature and in this paper. It should be reminded that the present analysis is based on the Flügge thin shell theory, whereas Qu et al. [28] employed the Reissner-Naghdi thin shell theory.

Tables 2 and 3 also show that the boundary conditions affect the frequency parameters of the circular cylindrical shell more significantly for small mode numbers than large mode numbers. This will be further proved in the next discussions. Since it is impossible to undertake an all-encompassing survey of the free vibrations for every pair of mode numbers, only those mode numbers, for which the frequency parameters are strongly influenced by the boundary conditions, are chosen to be investigated for the circular cylindrical shells. Therefore, in the following analysis, the effects of the elastic-support stiffness on frequency parameters of the circular cylindrical shell are to be analyzed for small mode number $m \leq 3$ and $n \leq 3$.

3.2. Effect of Elastic-Support Stiffness on Frequency Parameters. In what follows, the effects of each of the axial, circumferential, radial, and torsional stiffness of the elastic-support on

TABLE 3: Comparison of the frequency parameter $\Omega = \omega R[(1 - \mu^2)\rho/E]^{1/2}$ for a circular cylindrical shell with some elastic-support boundary conditions ($R = 1$ m, $L/R = 20$, $h/R = 0.01$, $m = 1$).

n	E^I-E^I		E^I-E^{II}		E^I-E^{III}		$E^{II}-E^{II}$		$E^{II}-E^{III}$		$E^{III}-E^{III}$	
	Reference [28]	Present	Reference [28]	Present	Reference [28]	Present	Reference [28]	Present	Reference [28]	Present	Reference [28]	Present
1	0.021812	0.020328	0.027104	0.025721	0.021669	0.020163	0.033157	0.031534	0.026984	0.025467	0.021527	0.020004
2	0.011674	0.010301	0.012695	0.011816	0.011471	0.010301	0.013957	0.013881	0.012457	0.011816	0.011272	0.010301
3	0.022463	0.022211	0.022558	0.02239	0.02244	0.022209	0.022673	0.022667	0.022531	0.022387	0.022418	0.022206
4	0.042178	0.04212	0.042193	0.042156	0.042175	0.042118	0.042209	0.042208	0.04219	0.042154	0.042172	0.042117
5	0.068038	0.067936	0.068043	0.068031	0.068037	0.067936	0.068048	0.068046	0.068042	0.06803	0.068037	0.068016
6	0.099745	0.099736	0.099747	0.099742	0.099745	0.099735	0.099748	0.099749	0.099746	0.099741	0.099745	0.099735
7	0.137248	0.137243	0.137249	0.137247	0.137248	0.137243	0.13725	0.137251	0.137249	0.137246	0.137248	0.137243
8	0.180534	0.180532	0.180535	0.180534	0.180534	0.180532	0.180535	0.180536	0.180535	0.180534	0.180534	0.180531
9	0.229599	0.229598	0.229599	0.229599	0.229599	0.229598	0.229599	0.229601	0.229599	0.229599	0.229599	0.229598
10	0.284439	0.28444	0.284439	0.284441	0.284439	0.28444	0.284441	0.284442	0.284439	0.284441	0.284439	0.284440

the frequency parameters for the clamped, free, and elastic-support boundary conditions are, respectively, investigated in the following discussions. The clamped, free, and elastic-support boundary conditions, respectively, indicate the other three degrees of freedom are clamped, free, and elastic-supported.

3.2.1. The Clamped Boundary Conditions. Firstly, each of the axial, circumferential, radial, and torsional stiffness is taken as from 10 to 10^{12} N/m (or N/rad for torsional stiffness) while the other three degrees of freedom are assumed to be clamped. The frequency parameters against the nondimensional stiffness $\log(k/k_0)$ for mode numbers $(m, n) = (1\sim 3, 1)$ and $(1, 1\sim 3)$ are plotted in Figure 3, in which k is a general appellation of the axial, circumferential, radial, and torsional stiffness and the reference stiffness k_0 is taken as 1 N/m (or N/rad).

It can be observed from Figure 3 that, for the clamped boundary conditions, only axial and circumferential stiffness exert a significant influence on the frequency parameters of the circular cylindrical shell. Figures 3(a) and 3(b) indicate that the axial and circumferential stiffness mainly affect the frequency parameters in the range of $10^7 \sim 10^{11}$ N/m.

Besides, Figure 3(a) shows that the increase of the axial stiffness affects the frequency parameters in almost the same extents for different axial mode numbers; however, it exerts a more evident influence for small circumferential mode numbers than it does for large circumferential mode numbers. Figure 3(b) shows that the effect of the circumferential stiffness on the frequency parameters increases with the increase of the axial mode number while decreasing with the increase of the circumferential mode number.

3.2.2. The Free Boundary Conditions. Subsequently, each of the axial, circumferential, radial, and torsional stiffness is taken as from 10 to 10^{12} N/m (or N/rad) while the other three degrees of freedom are assumed to be free. The frequency parameters against the nondimensional stiffness $\log(k/k_0)$ for mode numbers $(m, n) = (1\sim 3, 1)$ and $(1, 1\sim 3)$ are plotted in Figure 4.

Figure 4 indicates that, except for the torsional stiffness, the stiffness of all the other three degrees of freedom exerts dramatic influence on the frequency parameters of the circular cylindrical shell with a sensitive stiffness range of $10^5 \sim 10^{10}$ N/m.

To put it specifically, Figure 4(a) shows that the increase of the axial stiffness affects the frequency parameters for the free boundary conditions in the same manner as it does for the clamped boundary conditions. However, as shown in Figure 4(b), the effect of the circumferential stiffness on the frequency parameters becomes complicated. In the sensitive stiffness range, there exists switch stiffness, at which the frequency parameters switch into another variation curve which is quite inconsistent with the original one. The mode shape analysis indicates that the free modes turn into the clamped ones at the switch stiffness. To make it intuitive with the free mode shapes and the clamped mode shapes, some selected mode shapes are presented in Figure 5, from which it can be found that the most remarkable difference between the two types of mode shapes is that the free mode shapes contain free edge modes while the clamped mode shapes do not.

It can be observed from Figure 4(b)-(1) that, for circumferential mode number $n = 1$, all the switch stiffness for different axial mode numbers is the same to be 10^8 N/m, at which there are both free modes and clamped modes for mode numbers $m = 2$ and 3 . Figure 4(b)-(2) shows that, as the axial mode number $m = 1$, the free modes turn into the clamped modes with the switch stiffness 10^8 N/m for circumferential mode number $n = 1$. However, as the circumferential mode numbers $n = 2$ and 3 , the free modes disappear at circumferential stiffness $k_\nu = 10^7$ N/m while the clamped modes appear at the stiffness of 10^6 N/m, which indicates that there exist both free modes and clamped modes in the stiffness range of $10^6 \sim 10^7$ N/m.

Figure 4(c)-(1) shows that, as the circular cylindrical shell is free in the axial, circumferential, and torsional directions, the frequency parameters vary against the radial stiffness with only free modes for small radial stiffness. However, as the radial stiffness increases to a certain value, for instance,

(a) Frequency parameters versus axial stiffness

(b) Frequency parameters versus circumferential stiffness

(c) Frequency parameters versus radial stiffness

(d) Frequency parameters versus torsional stiffness

FIGURE 3: Effect of elastic-support stiffness on the frequency parameters for the clamped boundary conditions.

FIGURE 4: Effect of elastic-support stiffness on the frequency parameters for the free boundary conditions.

FIGURE 5: Some selected free mode shapes and clamped mode shapes.

10^7 N/m for $(m, n) = (1, 1)$ and 10^8 N/m for $(m, n) = (2, 1)$, the clamped modes appear with no disappearance of the free modes, which indicates that the free modes and the clamped modes coexist with each other for large radial stiffness. However, this only applies to small mode numbers like $(m, n) = (1, 1)$ and $(2, 1)$. Figure 4(c)-(2) shows that there are only free modes for small radial stiffness and only clamped modes for large radial stiffness. The free modes disappear at the radial stiffness of 10^6 N/m while the clamped modes appear at the radial stiffness of 10^5 N/m.

Besides, it can be observed from Figure 4(d) that there is hardly any influence of the torsional stiffness on the frequency parameters for the free boundary conditions.

3.2.3. The Elastic-Support Boundary Conditions. Then, with the other three degrees of freedom elastic-supported, the curves of the frequency parameters against the axial, circumferential, radial, and torsional stiffness are, respectively, presented in Figures 6(a)~6(d).

It can be found from Figure 6(a) that the sensitive range of the axial stiffness is $10^6 \sim 10^{11}$ N/m for the elastic-support boundary conditions. There are both free modes and clamped modes for mode numbers $(m, n) = (1, 1)$ and $(2, 1)$. The effect of the axial stiffness on the frequency parameters corresponding to the clamped modes decreases with the increase of the axial mode numbers, while it increases with the increase of the axial mode number on those corresponding to the free modes. However, only free modes exist for the circumferential mode number $n = 1$ and axial mode numbers $m \geq 3$, and only clamped modes appear for the axial mode number $m = 1$ and circumferential mode numbers $n \geq 2$. In this instance, the effect of the axial stiffness on the frequency parameters decreases with the increase of the circumferential mode number.

Figure 6(b) shows that the circumferential stiffness mainly affects the frequency parameters in the range of $10^5 \sim 10^{10}$ N/m. For mode numbers $(m, n) = (1, 1)$ and $(2, 1)$, the free modes and the clamped modes coexist as the circumferential stiffness is small. However, with the increase of the circumferential stiffness, for instance, to 10^7 N/m for mode numbers $(m, n) = (1, 1)$ and 10^8 N/m for mode numbers $(m, n) = (2, 1)$, the free modes disappear while the clamped modes are reserved. For the circumferential mode number $n = 1$ and axial mode numbers $m \geq 3$, there is only one type of modes for each circumferential stiffness except for $k_v = 10^8$ N/m, at which the free modes turn into the clamped modes with the increase of the circumferential stiffness. For the axial mode number $m = 1$ and circumferential mode numbers $n \geq 2$, only clamped modes exist and the circumferential stiffness exerts little influence on the frequency parameters.

It can be observed from Figure 6(c) that the radial stiffness mainly affects the frequency parameters in the range of $10^5 \sim 10^{10}$ N/m. For mode number $(m, n) = (1, 1)$, both the free modes and the clamped modes are present for small radial stiffness; however, the free modes disappear as $k_w \geq 10^8$ N/m. The frequency parameters corresponding to the free modes increase rapidly while those corresponding to the clamped modes increase slightly with the increase of

the radial stiffness in the sensitive stiffness range. For mode number $(m, n) = (2, 1)$, the free modes and the clamped modes coexist for all the radial stiffness. The curves of the frequency parameters corresponding to the two types of modes are of the same pattern while in different heights. As for mode number $(m, n) = (3, 1)$, only free modes are present for small radial stiffness $k_w \leq 10^8$ N/m; however, as it increases to be larger than 10^9 N/m, both the free modes and the clamped modes are present. Besides, the effect of the radial stiffness on the frequency parameters corresponding to the clamped modes increases with the increase of the axial mode numbers while it decreases with the increase of the axial mode numbers with respect to the free modes. However, for the axial mode number $m = 1$ and circumferential mode numbers $n \geq 2$, both the free modes and the clamped modes are present for small radial stiffness; as it increases to be larger than 10^6 N/m, only the clamped modes are reserved and the free modes disappear. The increase of the radial stiffness can hardly affect the frequency parameters corresponding to the clamped modes; however, it increases those corresponding to the free modes in the sensitive stiffness range.

Figure 6(d) indicates that, for the elastic-support boundary conditions, the torsional stiffness can hardly affect the frequency parameters of the circular cylindrical shell.

3.2.4. Comparison of the Three Types of Boundary Conditions. Finally, curves of frequency parameters varying with the axial, circumferential, radial, and torsional stiffness for different mode numbers of the circular cylindrical shell with the free, clamped, and elastic-support boundary conditions are shown in Figure 7.

It can be observed from Figure 7(a) that the increase of the axial stiffness will not cause mode switch for all of the three types of boundary conditions. For mode number $(m, n) = (1, 1)$, the frequency parameters for the free boundary conditions are much greater than those for the clamped boundary conditions. There exist both the free modes and the clamped modes for the elastic-support boundary conditions. The frequency parameters corresponding to the free modes of the elastic-support boundary conditions are greater than those for the free boundary conditions while those corresponding to the clamped modes are smaller than those for the clamped boundary conditions. The increase of the axial stiffness enlarges the disparity of the frequency parameters corresponding to the two clamped modes while reducing that of the two free modes. For the axial mode number $m = 1$ and circumferential mode numbers $n \geq 2$, only the clamped modes are present for the elastic-support boundary conditions, and the frequency parameters are very close to those for the clamped boundary conditions. With the increase of the axial stiffness, the disparity of the frequency parameters corresponding to the two clamped modes increases. However, for axial mode numbers $m \geq 2$ and the circumferential mode number $n = 1$, the frequency parameters for the elastic-support boundary conditions are close to those for the free boundary conditions, and the disparity of the frequency parameters corresponding to the two free modes decreases with the increase of the axial stiffness.

FIGURE 6: Effect of elastic-support stiffness on the frequency parameters for the elastic-support boundary conditions (elastic-support stiffness $k = 1 \times 10^7$ N/m or N/rad).

(a) Frequency parameters versus axial stiffness

(b) Frequency parameters versus circumferential stiffness

FIGURE 7: Continued.

FIGURE 7: Effect of elastic-support stiffness on the frequency parameters for the free, the clamped, and the elastic-support boundary conditions (elastic-support stiffness $k = 1 \times 10^7$ N/m or N/rad).

What is particular is that there are also clamped modes for mode number $(m, n) = (2, 1)$ and the frequency parameters are almost independent of the axial stiffness.

Figure 7(b) shows that, with the increase of the circumferential stiffness, mode switch occurs for all mode numbers for the free boundary conditions while appearing only for mode numbers $(m, n) = (1\sim 3, 1)$ for the elastic-support boundary conditions. However, as for elastic-support boundary conditions, only the clamped modes are present for the axial mode number $m = 1$ and circumferential mode numbers $n \geq 2$ and the frequency parameters are almost independent of the circumferential stiffness. There is no mode switch for the clamped boundary conditions. For the free boundary conditions, the free modes are present for small circumferential stiffness while the clamped modes are present for large circumferential stiffness. The frequency parameters corresponding to the free modes of the elastic-support boundary conditions are greater than those of the free boundary conditions. However, the frequency parameters corresponding to the clamped modes for the elastic-support boundary conditions are very close to those of the free boundary conditions. Both of them are much smaller than those of the clamped boundary conditions.

Figure 7(c) indicates that the effects of the boundary conditions of the other three degrees of freedom on the frequency parameters depend on mode numbers and the radial stiffness. For small radial stiffness (e.g., $k_w \leq 10^8$ N/m for $(m, n) = (1, 1)$ and $(2, 1)$, $k_w \leq 10^6$ N/m for $(m, n) = (1, 2)$ and $(1, 3)$), both free modes and clamped modes are present for the elastic-support boundary conditions, while only free modes are present for the free boundary conditions and only clamped modes are present for the clamped boundary conditions. The frequency parameters corresponding to the free modes of the elastic-support boundary conditions are greater than those of the free boundary conditions and the frequency parameters corresponding to the clamped modes of the elastic-support boundary conditions are smaller than those of the clamped boundary conditions. As for axial mode numbers $m \geq 3$ and the circumferential mode number $n = 1$, only free modes are reserved for small radial stiffness.

However, for large radial stiffness, as for mode number $(m, n) = (1, 1)$, the free modes of the elastic-support boundary conditions disappear and the clamped modes of free boundary conditions appear. The frequency parameters corresponding to the free modes of the free boundary conditions are much greater than those of the clamped boundary conditions and subsequently greater than those corresponding to the clamped modes of the free boundary conditions and the elastic-support boundary conditions, and the latter two are very close to each other. For mode numbers $(m, n) = (1, 2)$ and $(1, 3)$, only the clamped modes are present for the three types of boundary conditions. The frequency parameters for the clamped boundary conditions are greater than those for the free boundary conditions and the elastic-support boundary conditions, and the latter two are very close to each other. For mode number $(m, n) = (2, 1)$, both the free modes and the clamped modes are present for the free boundary conditions and the elastic-support boundary

conditions. The frequency parameters corresponding to the free modes of the elastic-support boundary conditions are greater than those of the free boundary conditions, and those of the clamped boundary conditions are greater than those corresponding to the clamped modes of the elastic-support boundary conditions and subsequently greater than those corresponding to the clamped modes of the free boundary conditions. For mode number $(m, n) = (3, 1)$, the free modes are present for the free boundary conditions and the elastic-support boundary conditions, while the clamped modes are present for the clamped boundary conditions and the elastic-support boundary conditions. The frequency parameters corresponding to the free modes of the elastic-support boundary conditions are much greater than those of the free boundary conditions, and those corresponding to the clamped modes of the elastic-support boundary conditions are much smaller than those of the clamped boundary conditions.

Figure 7(d) shows that, for mode numbers $(m, n) = (1, 1)$ and $(2, 1)$, both the free modes and the clamped modes are present for the elastic-support boundary conditions, while only the free modes are present for the free boundary conditions and the clamped modes are present for the clamped boundary conditions. The frequency parameters corresponding to the free modes of the elastic-support boundary conditions are greater than those of the free boundary conditions, and those corresponding to the clamped modes of the elastic-support boundary conditions are smaller than those of the clamped boundary conditions. For mode numbers $(m, n) = (1, 2)$ and $(1, 3)$, there is only one type of modes for the free, the clamped, and the elastic-support boundary conditions. The frequency parameters corresponding to the free boundary conditions are greater than those of the clamped boundary conditions and subsequently much greater than those of the elastic-support boundary conditions. For mode number $(m, n) = (3, 1)$, there is also only one type of modes for each of the three types of boundary conditions. However, the frequency parameters corresponding to the three types of boundary conditions are arranged from the highest to the lowest order of elastic-support, free, and clamped boundary conditions.

4. Conclusions

An analytical procedure is developed to analyze the free vibration characteristics of circular cylindrical shells with arbitrary boundary conditions. Based on the Flügge thin shell theory, exact solutions of the traveling wave form along the axial direction and the standing wave form along the circumferential direction are obtained. MRRM is introduced to derive the equation of the natural frequencies of a unified and compact form. The agreement of the comparisons in the frequency parameters obtained by MRRM and those presented in the published literature proves the suitability and accuracy of applying MRRM to the study of free vibration of circular cylindrical shells with arbitrary boundary conditions.

Free vibration characteristics of circular cylindrical shells with elastic-support boundary conditions are investigated by MRRM. Results show that the elastic-support stiffness effectively affects the frequency parameters in the range of $10^5 \sim 10^{11}$ N/m, which is comparable to the bending rigidity of a

shell. The effects of the elastic-support stiffness on the frequency parameters vary with different mode numbers. The axial and circumferential stiffness of the elastic-support have a dramatic influence on the frequency parameters in the sensitive stiffness range. The radial stiffness affects the frequency parameters in certain circumstances. However, the torsional stiffness can hardly affect the frequency parameters of the circular cylindrical shell. The free-clamped mode switch frequently occurs for the free boundary conditions and the free-clamped mode coexistence is usually present for the elastic-support boundary conditions at small mode numbers. Generally, the elastic-support can effectively increase the frequency parameters corresponding to the free modes and decrease those corresponding to the clamped modes.

Appendices

A. Linear Differential Operators L_j ($j = 1 \sim 6$)

Consider the following:

$$\begin{aligned}
 L_1 &= \mu_1 R^2 \frac{\partial^2}{\partial x^2}, \\
 L_2 &= \mu_1 \frac{\partial^2}{\partial \theta^2}, \\
 L_3 &= \mu_1 R \frac{\partial^2}{\partial x \partial \theta}, \\
 L_4 &= R^2 \frac{\partial^2}{\partial x^2} + \lambda_1 \mu_2 \frac{\partial^2}{\partial \theta^2} + R^2 k_L^2, \\
 L_5 &= -\lambda R^2 \frac{\partial^2}{\partial x^2} + \lambda \mu_2 \frac{\partial^2}{\partial \theta^2} + \mu, \\
 L_6 &= \lambda_2 \mu_2 R^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2} + R^2 k_L^2, \\
 L_7 &= 1 - \lambda \mu_3 R^2 \frac{\partial^2}{\partial x^2}, \\
 L_8 &= \lambda \left(R^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2} + 1 \right)^2 \\
 &\quad - \left(2\lambda R^2 \frac{\partial^2}{\partial x^2} + R^2 k_L^2 - 1 \right),
 \end{aligned} \tag{A.1}$$

where $k_L = \omega[(1 - \mu^2)\rho/E]^{1/2}$ is the longitudinal wave number and $\gamma = \{[R^2 k_L^2 - (1 - \mu)/(1 + \mu)]/\lambda R^4\}^{1/4}$ is a parameter of the same dimension with the axial wave number.

B. Dimensional Parameters ζ_j ($j = 1 \sim 6$)

Consider the following:

$$\begin{aligned}
 \zeta_1^2 &= \frac{(1 - \mu_1^2 + \lambda_1 \lambda_2 \mu_2^2) n^2}{R^2} - (1 + \lambda_2 \mu_2) k_L^2, \\
 \zeta_2^2 &= \frac{\lambda_1 \mu_2 n^2}{R^2} - k_L^2,
 \end{aligned}$$

$$\begin{aligned}
 \zeta_3^2 &= \frac{n^2}{R^2} - k_L^2, \\
 \zeta_4^2 &= \frac{[(2\mu_1 + \lambda \mu_2 \mu_3) n^2 - (1 + \mu \mu_3)]}{R^2}, \\
 \zeta_5^4 &= \frac{[\mu_1 (n^2 - 1)^2 + \mu_2 n^2 + (\mu_2 - \mu_1 R^2 k_L^2)/\lambda]}{R^4}, \\
 \zeta_6^2 &= \frac{[(1 - \mu \mu_1) + \lambda \mu_2 (\mu_1 + \lambda_1 \mu_3) n^2 - \lambda \mu_3 R^2 k_L^2]}{R^2}, \\
 \zeta_7^2 &= \frac{1}{R^2}, \\
 \zeta_8^2 &= \frac{[\lambda_2 \mu_2 \mu + \lambda (1 - \lambda_2 \mu_2^2 - \mu_1 \mu_3) n^2 - \lambda R^2 k_L^2]}{R^2}, \\
 \zeta_9^4 &= \frac{(\lambda \mu_2 n^2 R^2 k_L^2 - \mu R^2 k_L^2 - \mu_2 n^2 - \lambda \mu_2 n^4)}{R^4},
 \end{aligned} \tag{B.1}$$

where $\mu_1 = (1 + \mu)/2$, $\mu_2 = (1 - \mu)/2$, $\mu_3 = (3 - \mu)/2$, and $\lambda_1 = 1 + \lambda$, $\lambda_2 = 1 + 3\lambda$, $\lambda_3 = \mu_1 - \mu_3 \lambda$ are dimensionless parameters.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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