Research Article

Free Transverse Vibration Analysis of Axially Functionally Graded Tapered Euler-Bernoulli Beams through Spline Finite Point Method

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A new model for the free transverse vibration of axially functionally graded (FG) tapered Euler-Bernoulli beams is developed through the spline finite point method (SFPM) by investigating the effects of the variation of cross-sectional and material properties along the longitudinal directions. In the proposed method, the beam is discretized with a set of uniformly scattered spline nodes along the beam axis instead of meshes, and the displacement field is approximated by the particularly constructed cubic B-spline interpolation functions with good adaptability for various boundary conditions. Unlike traditional discretization and modeling methods, the global structural stiffness and mass matrices for beams of the proposed model are directly generated after spline discretization without needing element meshes, generation, and assembling. The proposed method shows the distinguished features of high modeling efficiency, low computational cost, and convenience for boundary condition treatment. The performance of the proposed method is verified through numerical examples available in the published literature. All results demonstrate that the proposed method can analyze the free vibration of axially FG tapered Euler-Bernoulli beams with various boundary conditions. Moreover, high accuracy and efficiency can be achieved.

1. Introduction

Functionally graded (FG) materials were initially proposed by material scientists from Japan in 1984 as thermal barrier materials; since then, the development of engineering application and theoretical research on structures with FG materials has made a great breakthrough because of the particular properties of these materials, such as high stiffness and thermal resistance [1–3]. FG materials are multiphase composites with the volume fraction of phase varying through any desired direction [4]. FG material gradation along an arbitrary direction can be expressed with power (polynomial) laws and exponential laws, which are the most renowned theories for FG material description. The gradient variation for FG beams may also be oriented in the thickness direction, longitudinal direction, and bidirection (oriented in both the thickness and lengthwise directions).

During the past decade, the problems of the stability, dynamic response, and linear/nonlinear free vibration behavior of the FG structures with material properties varying along the thickness direction have been thoroughly investigated by researchers around the world, and lots of numerical and analytical solutions have been obtained [5–11]. However, given the existence of variable coefficients on the governing equation, the free vibration analysis of beams with nonuniform cross sections and variable material properties along the longitudinal direction becomes more complicated than the beams with uniform cross sections and homogenous materials [3]. And so far, few analytical solutions are found for the free vibration of the axially FG tapered beams due to the difficulty of mathematical treatment of the problem; for example, Elishakoff et al. [12, 13] proposed the close-form solutions for axially FG tapered beam with material gradation that follows the polynomial law using the semi-inverse

method; however, the proposed semi-inverse method cannot be applied for graded beams of any axial nonhomogeneity. Therefore, a large number of numerical methods have been proposed in the reported literatures by other researchers, according to which lots of numerical models, based on different beam theories, namely, the Euler-Bernoulli beam theory (EBT) [2, 3, 14–20] and the Timoshenko beam theory (TBT) [21–25], have been constructed to investigate the free vibration of the tapered beams considering the material property homogeneity or nonhomogeneity. However, it is well understood that the effects of the shear deformation and rotary inertia of the cross section for the long slender beams can be neglected; thus the EBT models are capable of analyzing the vibration problems of the long slender beams and the reliable results with high accuracy can be achieved. In this paper, the authors mainly engage in establishing a new EBT model for the free vibration of axially tapered beams (long slender beams).

Based on EBT, Shahba et al. [2] firstly investigated the stability and free vibration of the axially FG Euler beams considering cross-sectional breadth and height both linearly tapered along the longitudinal direction by using the finite element method (FEM); then, in order to improve computational efficiency and modify the convergence conditions, Shahba and Rajasekaran [3] developed other models by using the differential transform element method (DTEM) and differential quadrature element method of the lowest order (DQEL) to investigate the same problems. Results show that the DTEM and DQEL models obtained better convergence and had fewer elements than the FEM model. However, with the increase of the taper ratio, all models proposed by Shahba et al. needed to solve a larger scale of eigenvalue equations in order to achieve a stable accuracy. Huang and Li [14] studied free vibration problems of axially FG tapered Euler beams through a new approach that transformed the governing equation into the Fredholm integral equation with consideration of the case of material properties in polynomial and exponential variations along longitudinal direction. Li et al. [15] developed the exact frequency equation for axially exponentially FG Euler beams, according to which the results showed that the natural frequencies noticeably depend on exponential gradient parameter and boundary conditions. Akgöz and Civalek [16] developed a model combined with the EBT and modified coupled stress theory to study the free vibration of the axially FG micro-Euler beams, in which the Rayleigh-Ritz solution method was utilized to obtain the natural transverse frequencies. Mao and Pietrzko [17] studied the free vibration of tapered Euler beams with a continuously exponential variation of breadth and a constant height along the length through Adomian decomposition method. Hsu et al. [18] developed a model to investigate the free vibration of homogenous tapered Euler beams with elastic supports through Adomian modified decomposition method. Rajasekaran [4] and Özdemir and Kaya [19] applied differential transform method (DTM) to study the free bending vibration of rotating tapered Euler beams by considering material nonhomogeneity and homogeneity along the longitudinal direction. Banerjee et al. [20] applied the dynamic stiffness method (DSM) to analyze the free bending vibration

of the rotating homogeneous tapered Euler beams; though the high accuracy was obtained, the modeling process was not convenient. Other models have been investigated based on TBT; Shahba et al. [21] developed the FEM model to study the free vibration of the axially FG taper Timoshenko beams. Rajasekaran and Norouzzadeh Tochaie [22, 23] developed different models through DTM and DQEL to study axially FG Timoshenko beams by considering the effects of rotating and nonrotating. Ozgumus and Kaya [24] investigated model through DTM for the free bending vibration of rotating homogeneous tapered Timoshenko beams by considering the effects of taper ratio, rotating speed, and slenderness ratio. Yardimoglu [25] investigated a new FEM model with displacement functions derived from the coupled displacement field of transverse displacement and cross-sectional rotation to study the free bending vibration of the rotating Timoshenko beams.

Among the aforementioned numerical models for the axially tapered beams, such as FEM, DTM, DTEM, and DSM models, the disadvantages are obvious. (1) For the FEM models, the element generation and assembling are both needed in the modeling process; the displacement interpolation functions of FEM models were unable to fully capture the variation of the cross section and material properties along the longitudinal direction due to the homogenous uniform beams elements were utilized, and with the increase of the taper ratio and gradient parameter, the results' error and elements numbers increase noticeably [2, 21, 25]. (2) For the DTM and DTEM models, although both models obtained high accuracy, these two numerical models needed to solve a large scale of eigenvalue equations, thereby leading to a high computational cost; what is more, the DTM model encountered convergence problems in some cases [3, 4, 19]. (3) For the DSM models, the most common disadvantages were their complicated modeling process [20]. Therefore, the aforementioned numerical models need to be improved, and a new model with convenience modeling process, high accuracy, and low computational cost will benefit the analysis of the axially tapered Euler beams.

Spline finite point method (SFPM) was firstly proposed by Professor Qin in 1979 [26, 27], who proposed a cubic B-spline function-based method for structural analysis. Given that the interpolation of structural displacement is constructed through a set of scattered spline nodes along one direction instead of meshes, the element generations and assembling are both unnecessary for the modeling process of SFPM; thus, the SFPM contains the excellent feature of no mesh distortion. The process of modeling and computer programming can be achieved easily, which leads to high computation accuracy and efficiency because the interpolation functions used in SFPM have explicit forms. The performances of SFPM on the structures with uniform cross section and material property that keeps constant along the longitudinal direction were validated by reported works on the FGM beams/plates, laminates, and other structures [28–32]. Li et al. [28] investigated the free vibration of piezoelectric smart FGM plates using SFPM. Then, Li et al. [29] investigated a new model through bidirectional B-spline finite point method (B-SFPM) to analyze the displacement and stress

distribution of a piezoelectric laminate composite plate combined with the problems of material parameter identification; Zhou and Li [30] studied the free vibrations of asymmetrical sandwich plates with laminated faces and orthotropic core through SFPM. Qin et al. [31, 32] developed different models through SFPM to study the nonlinear behaviors of composite plates and the seismic behavior of a box arch bridge. Other researchers applied the spline strip method to the free vibration of rectangular composite laminates [33, 34], which also validated the advantages of application of the cubic B-spline functions. However, to the best of the authors' knowledge, the SFPM have not been applied to analyze the free vibration of the axially FG tapered beams before, and similar works or models have been rarely found in the reported literatures as well. In this study, the authors introduce the SFPM to analyze the vibration behavior of the axially FG tapered Euler-Bernoulli beams for the first time, which provide a novelty approach for solving such problems.

The main objective of this study is to develop new simple models to study the free vibrations of axially FG tapered Euler-Bernoulli beams by using SFPM. The new models are able to achieve faster modeling process, higher accuracy, and lower computational cost compared with those of reported numerical models. On the basis of SFPM, the axially FG tapered beam is discretized with a set of uniformly scattered spline nodes along the beam axis instead of meshes, and the displacement field is approximated by particularly constructed cubic B-spline interpolation functions, which possess good adaptability for various boundary conditions. Both the global structural stiffness and mass matrices for beams are directly generated after spline discretization, thereby making the modeling convenient. Because of the employment of presented displacement interpolation functions, the derived structural stiffness and mass matrices being in the explicit forms are capable of directly taking account of any variations of the material properties and cross-section profile for the axially FG beams directly. A suitable spline node number is discussed and selected to maintain high computational accuracy based on the convergence analysis. The processes of boundary condition treatment and derivation of the structural matrices are introduced in detail. The effects of the variation of cross-sectional and material properties along the longitudinal direction are investigated. The performance of the proposed method is verified with numerical examples available in the published literature. All results demonstrate that the proposed method can analyze the free vibration of axially FG tapered Euler-Bernoulli beams with various boundary conditions with quick convergence and high accuracy.

2. Structural Model

Figure 1 shows the axially FG tapered beams in length L . The beam with cross-sectional parameters, such as the cross-sectional area $A(x)$ and the second moment of inertia $I(x)$, and with material properties, such as Young's modulus $E(x)$ and the mass density $\rho(x)$, continuously vary along the lengthwise direction, which are the functions of longitudinal coordinate x .

FIGURE 1: Schematic of an axially FG tapered beam.

According to EBT, the following assumptions are made: (1) the cross section of FG beams remains plane after deformation; (2) the slenderness ratio (L/h_0) is big enough that the effects of rotary inertia and shear deformation in x - z plane can be ignored; (3) the transverse deformation is small enough that only the linear vibration response is considered. Given that the shear deformation and rotary inertia are not considered, the axial and transverse displacement fields for axially FG tapered beams are presented, respectively, as follows:

$$\begin{aligned} u &= u(x, y, z, t) = -z \frac{\partial w(x, t)}{\partial x}, \\ w &= w(x, y, z, t) = w(x, t), \end{aligned} \quad (1)$$

where x , y , and z are the spatial coordinates and w is the transverse displacement. The axial and shear strains are, respectively, obtained as follows by using (1):

$$\begin{aligned} \epsilon_{xx} &= -z \frac{\partial^2 w}{\partial x^2}, \\ \gamma_{xz} &= 0. \end{aligned} \quad (2)$$

The stresses can be given as

$$\begin{aligned} \sigma_{xx} &= E(x) \epsilon_{xx}, \\ \tau_{xz} &= 0, \end{aligned} \quad (3)$$

where σ_{xx} is the axial stress. The shear stress τ_{xz} is equal to zero because shear deformation is not considered in the model. The strain and kinetic energies of the axially FG tapered Euler beams can be written, respectively, as

$$\begin{aligned} U &= \frac{1}{2} \int_0^L \int_{A(x)} \sigma_{xx} \epsilon_{xx} dA dx \\ &= \frac{1}{2} \int_0^L E(x) I(x) \left(\frac{\partial^2 w}{\partial x^2} \right)^2 dx, \\ T &= \frac{1}{2} \int_0^L \int_{A(x)} \rho(x) \left(\frac{\partial w}{\partial t} \right)^2 dA dx \\ &= \frac{1}{2} \int_0^L \rho(x) A(x) \left(\frac{\partial w}{\partial t} \right)^2 dx. \end{aligned} \quad (4)$$

The following can be obtained by applying the Hamilton principle:

$$\delta \Pi = \delta \int_{t_1}^{t_2} (U - T) dt = 0, \quad (5)$$

FIGURE 2: Variation value of function $\varphi_3^{(3)}(x)$.

where Π is the total potential energy function for the axially FG tapered beam, which can be derived by some manipulations from (4) and (5). The derivation of Π can be seen in a reported work [19] and can be expressed by

$$\Pi = \frac{1}{2} \int_0^L \left[E(x) I(x) \left(\frac{\partial^2 w}{\partial x^2} \right)^2 - 2w \left(-\rho(x) A(x) \frac{\partial^2 w}{\partial t^2} \right) \right] dx. \quad (6)$$

The governing equations of motion for an axially FG tapered Euler beam undergoing transvers free vibration are derived as follows through the variation principle:

$$\frac{\partial^2}{\partial x^2} \left(E(x) I(x) \frac{\partial^2 w}{\partial x^2} \right) - \rho(x) A(x) \frac{\partial^2 w}{\partial t^2} = 0. \quad (7)$$

3. SFPM Formulation

3.1. Definition of Cubic B-Spline Function. The cubic B-spline function $\varphi_3(x)$ is adopted in this study and is defined by the following expressions [27]:

$$\varphi_3(x) = \sum_{k=0}^4 (-1)^k \binom{4}{k} \frac{(x - x_k)_+^3}{3!} \quad x \in [-2, 2], \quad (8)$$

$$(x - x_k)_+^3 = \begin{cases} (x - x_k)^3 & x - x_k \geq 0 \\ 0 & x - x_k < 0 \end{cases} \quad x_k = k - 2. \quad (9)$$

According to the characteristics of $\varphi_3(x)$, its derivative and integration can be expressed by the following:

$$\varphi_3^{(j)}(x) = \sum_{k=0}^4 (-1)^k \binom{4}{k} \frac{(x - x_k)_+^{3-j}}{(3-j)!} \quad x \in [-2, 2], \quad (10)$$

where $\varphi_3^{(j)}(x)$ denotes the j th derivative ($j > 0$) or j th integration ($j < 0$) for $\varphi_3(x)$ with the variation of “ j ”; if $j = 0$, $\varphi_3(x)$ is obtained; if $j > 3$, $\varphi_3^{(j)}(x) = 0$. Moreover, if $j = 3$, $\varphi_3^{(3)}(x)$ is maintained as a stepped constant value within the interval $[-2, 2]$ as shown in Figure 2.

3.2. SFPM Modeling. Figure 3 illustrates that the beam is modeled via SFPM considering an axially FG tapered Euler

FIGURE 3: Spline discretization of axially FG tapered beams.

beam described in Figure 1. The beam is discretized along the beam axis through uniformly scattered spline nodes on the entire beam. The tapered beam can be divided by the interval h_x in x direction, and the local coordinates of the corresponding nodes in the axis of the beam can be presented by

$$x_i = x_0 + ih_x \quad (i = 1, \dots, N), \quad (11)$$

$$h_x = \frac{L}{N}, \quad (12)$$

$$0 = x_0 < x_1 < x_2 < \dots < x_{N-1} < x_N = L, \quad (13)$$

where N stands for the number of subdivisions of beams in x direction, which is also called the maximum number of spline nodes; x_0 denotes the x coordinate of left end of beam; and h_x is the length between two spline nodes.

In accordance with the modeling process for SFPM, the linear combination of cubic B-spline functions defined in (8), also called spline interpolation functions $\phi_i(x)$, is applied to approximate the transverse displacement $w(x, t)$ of the beam, which can be expressed by

$$w = w(x, t) = [\phi_i(x)] \{a_i(t)\} = [\phi] \{a\}, \quad (14)$$

$$[\phi] = [\phi_{-1} \ \phi_0 \ \phi_1 \ \dots \ \phi_N \ \phi_{N+1}]_{1 \times (N+3)}, \quad (15)$$

$$\{a\} = [a_{-1} \ a_0 \ a_1 \ \dots \ a_N \ a_{N+1}]_{(N+3) \times 1}^T, \quad (16)$$

where $[\phi]$ denotes the spline interpolation functions formed by the linear combination of cubic B-spline functions and $\{a\}$ represents the vector of unknown generalized displacement parameters.

In this paper, the spline interpolation functions $[\phi]$, which are given in the literature [27], can be constructed by

$$[\phi] = [\varphi_{3,i}] [Q_s], \quad (17)$$

where $[Q_s]$ is $(N+3) \times (N+3)$ transformation matrix and $[\varphi_{3,i}]$ are a group of cubic B-spline functions, which is expressed by

$$[Q_s] = \text{diag}([g], [I], [h]), \quad (18)$$

$$[\varphi_{3,i}] = [\varphi_{3,-1} \ \varphi_{3,0} \ \dots \ \varphi_{3,i} \ \dots \ \varphi_{3,N} \ \varphi_{3,N+1}] \\ = [\varphi_{3,i}]_{1 \times (N+3)}, \quad (19)$$

$$\varphi_{3,i} = \varphi_3 \left(\frac{x - x_0}{h_x} - i \right) \quad i = -1, 0, \dots, N, N+1, \quad (20)$$

where $[I]$ is $(N - 3) \times (N - 3)$ identity matrix and

$$\begin{aligned} [g] &= \begin{bmatrix} 1 & -4 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}, \\ [h] &= \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 1 & -4 & 1 \end{bmatrix}. \end{aligned} \quad (21)$$

In this study, spline interpolation functions $[\phi]$ are characterized as the following boundary conditions:

$$\begin{aligned} \phi'_i(0) &= 0, \quad (i \neq -1, 0), \\ \phi_i(0) &= 0, \quad (i \neq -1), \\ \phi_i(L) &= 0, \quad (i \neq N + 1), \\ \phi'_i(L) &= 0, \quad (i \neq N, N + 1), \end{aligned} \quad (22)$$

where the subscript “ $'$ ” denotes the derivative of ϕ_i with respect to x .

With (14) substituted into (6), the total potential energy function Π for the free transverse vibration of the axially FG tapered Euler beams can be expressed by

$$\Pi = \frac{1}{2} \{a\}^T [K] \{a\} - \{a\}^T (-[M] \{\ddot{a}\}), \quad (23)$$

in which

$$\begin{aligned} [K] &= \int_0^L E(x) I(x) [\phi'']^T [\phi''] dx, \\ [M] &= \int_0^L \rho(x) A(x) [\phi]^T [\phi] dx, \end{aligned} \quad (24)$$

where the subscript double dot of $\{\ddot{a}\}$ denotes the differentiation with respect to time; $[K]$ and $[M]$ are, respectively, the $(N + 3) \times (N + 3)$ stiffness and mass matrices; and $\{\ddot{a}\}$ stands for the $(N + 3) \times 1$ vector of unknown generalized acceleration parameters.

The following is obtained through the variation principle:

$$\delta \Pi = \frac{\partial \Pi}{\partial \{a\}} = 0. \quad (25)$$

Moreover, the free vibration equation for axially FG tapered beam can be expressed as

$$[M] \{\ddot{a}\} + [K] \{a\} = \{0\}. \quad (26)$$

The following eigenvalue equation must be solved to perform free vibration analysis:

$$([K] - \omega^2 [M]) \{\phi\} = \{0\}, \quad (27)$$

where ω is the natural frequency and $\{\phi\}$ is the mode shape parameter vectors. However, the boundary conditions must be manipulated first before solving (27).

3.3. Calculations of Stiffness and Mass Matrices

3.3.1. Cross-Sectional Variation. As shown in Figure 1, the axially FG tapered beam with a rectangular cross section, whose breadth and height are tapered linearly, is considered. Thus, the cross-sectional area $A(x)$ and second moment of inertia $I(x)$ vary along the beam axis:

$$\begin{aligned} A(x) &= A_0 \left(1 - \frac{c_b x}{L}\right) \left(1 - \frac{c_h x}{L}\right), \\ I(x) &= I_0 \left(1 - \frac{c_b x}{L}\right) \left(1 - \frac{c_h x}{L}\right)^3. \end{aligned} \quad (28)$$

In (28), c_b and c_h are the breadth and height taper ratios, respectively; and A_0 and I_0 are the cross-sectional area and second moment of inertia, respectively, at $x = 0$. The tapered beam becomes uniform when $c_b = c_h = 0$, and the tapered beam theoretically tapers to be a point at $x = L$ when $c_b = c_h = 1$.

After some manipulations, (28) can be simplified as

$$\begin{aligned} A(x) &= \sum_{i=0}^2 \tilde{A}_i x^i, \\ I(x) &= \sum_{i=0}^4 \tilde{I}_i x^i, \end{aligned} \quad (29)$$

in which

$$\begin{aligned} \tilde{A}_0 &= A_0, \\ \tilde{A}_1 &= -\frac{(c_b + c_h)}{L} A_0, \\ \tilde{A}_2 &= \frac{c_b c_h}{L^2} A_0, \\ \tilde{I}_0 &= I_0, \\ \tilde{I}_1 &= -\frac{(c_b + 3c_h)}{L} I_0, \\ \tilde{I}_2 &= \frac{3(c_b c_h + c_h^2)}{L^2} I_0, \\ \tilde{I}_3 &= -\frac{(3c_b c_h^2 + c_h^3)}{L^3} I_0, \\ \tilde{I}_4 &= \frac{c_b c_h^3}{L^4} I_0, \end{aligned} \quad (30)$$

where $\tilde{A}_i$ ($i = 0, 1, 2$) and $\tilde{I}_i$ ($i = 0, \dots, 4$) are the constant coefficients for the cross-sectional area and second moment of inertia, respectively, which are equal to the polynomial coefficients of the expanded forms of $A(x)$ and $I(x)$.

3.3.2. Material Property Variation. In this study, two laws are considered for the variation of material property along the lengthwise direction of the beam in accordance with the reported literature [3, 14].

(i) *Material Type I: Power Law or Polynomial Law.* The material properties of the axially FG beam are considered to

vary along the lengthwise direction following the power law. The graded functions are simplified into a generalized form [3]:

$$\begin{aligned} E(x) &= \sum_{i=0}^{n1} \tilde{E}_i x^i; \\ \rho(x) &= \sum_{i=0}^{n2} \tilde{\rho}_i x^i, \end{aligned} \quad (31)$$

where $\tilde{E}_i$ ($i = 1, \dots, n1$) and $\tilde{\rho}_i$ ($i = 1, \dots, n2$) stand for the constant coefficients derived from the polynomial expansion of $E(x)$ and $\rho(x)$ and $n1$ and $n2$ represent the maximum power number for the expansion of $E(x)$ and $\rho(x)$. For homogenous beams, only the items $\tilde{E}_0 = E_0$ and $\tilde{\rho}_0 = \rho_0$ are not zero in (31).

(ii) *Material Type II: Exponential Law.* The material properties of the axially FG beam are considered to vary along the lengthwise direction following the exponential law, which can be simplified as

$$Y(x) = \tilde{Y}_1 + \tilde{Y}_2 e^{\alpha x/L}, \quad (32)$$

where $\tilde{Y}_1$ and $\tilde{Y}_2$ correspond to the coefficient of the simplified form of the exponential functions, such as Young's modulus $E(x)$ and mass density $\rho(x)$, respectively. α is the gradient parameter that describes the volume fraction change of both constituents involved. For homogenous beams, only the item $\tilde{Y}_1$ is not zero in (32).

3.3.3. Stiffness and Mass Matrices. According to (24), the calculation of $[K]$ and $[M]$ depends on the variation of the cross section and the graded functions of the material property.

(i) The cross sections are considered to be defined by (28), whereas the material properties are defined by (31) (polynomial law). Thus, the following is obtained:

$$\begin{aligned} E(x) I(x) &= \sum_{i=0}^{n1+4} \tilde{k}_i x^i; \\ \rho(x) A(x) &= \sum_{i=0}^{n2+2} \tilde{m}_i x^i, \end{aligned} \quad (33)$$

where $\tilde{k}_i$ ($i = 0, \dots, n1 + 4$) and $\tilde{m}_i$ ($i = 1, \dots, n2 + 2$) are constant coefficients of the polynomial expansion for the product of $E(x)I(x)$ and $\rho(x)A(x)$, respectively.

Then, (33) is substituted into (24). The stiffness and mass matrices are obtained:

$$\begin{aligned} [K] &= \int_0^L E(x) I(x) [\phi'']^T [\phi''] dx = \sum_{i=0}^{n1+4} \tilde{k}_i [A_{x,i}], \\ [M] &= \int_0^L \rho(x) A(x) [\phi]^T [\phi] dx = \sum_{i=0}^{n2+2} \tilde{m}_i [F_{x,i}], \end{aligned} \quad (34)$$

in which

$$[A_{x,m}] = \int_0^L x^m [\phi'']^T [\phi''] dx = [Q_s]^T A_{x,m} [Q_s],$$

$$[F_{x,n}] = \int_0^L x^n [\phi]^T [\phi] dx = [Q_s]^T F_{x,n} [Q_s],$$

$$A_{x,m} = \int_0^L x^m [\varphi_{3,i}'']^T [\varphi_{3,k}''] dx \quad (m = 0, 1, \dots, n1 + 4),$$

$$F_{x,n} = \int_0^L x^n [\varphi_{3,i}]^T [\varphi_{3,k}] dx \quad (n = 0, 1, \dots, n2 + 2), \quad (35)$$

$$\varphi_{3,i}'' = \varphi_3'' \left(\frac{x}{h_x} - i \right),$$

$$\varphi_{3,k}'' = \varphi_3'' \left(\frac{x}{h_x} - k \right),$$

$$\varphi_{3,i} = \varphi_3 \left(\frac{x}{h_x} - i \right),$$

$$\varphi_{3,k} = \varphi_3 \left(\frac{x}{h_x} - k \right)$$

$$(i, k = -1, 1, 0, \dots, N, N + 1).$$

(ii) The cross sections are considered to be defined by (28), and the material properties are defined by (32) (exponential law). Thus, the following is obtained:

$$E(x) I(x) = \sum_{i=0}^4 \tilde{k}_{11,i} x^i + \sum_{i=0}^4 \tilde{k}_{12,i} x^i e^{\alpha x/L}, \quad (36)$$

$$\rho(x) A(x) = \sum_{i=0}^2 \tilde{m}_{11,i} x^i + \sum_{i=0}^2 \tilde{m}_{12,i} x^i e^{\alpha x/L},$$

where α represents the gradient index of the exponentially FG materials. Moreover $\tilde{k}_{11,i}, \tilde{k}_{12,i}$ ($i = 0, \dots, 4$) and $\tilde{m}_{11,i}, \tilde{m}_{12,i}$ ($i = 0, 1, 2$) stand for the constant coefficients of the expanded form for the product of $E(x)I(x)$ and $\rho(x)A(x)$, respectively, with which the polynomial items and exponential items should be distinguished.

Then, (36) is substituted into (24), and the stiffness and mass matrices are obtained:

$$\begin{aligned} [K] &= \int_0^L E(x) I(x) [\phi'']^T [\phi''] dx \\ &= \sum_{i=0}^4 \tilde{k}_{11,i} [A_{x,i}] + \sum_{i=0}^4 \tilde{k}_{12,i} [A_{x,\exp,i}], \\ [M] &= \int_0^L \rho(x) A(x) [\phi]^T [\phi] dx \\ &= \sum_{i=0}^2 \tilde{m}_{11,i} [F_{x,i}] + \sum_{i=0}^2 \tilde{m}_{12,i} [F_{x,\exp,i}], \end{aligned} \quad (37)$$

in which

$$\begin{aligned}
 [A_{x,p}] &= \int_0^L x^p [\phi'']^T [\phi''] dx \\
 &= [Q_s]^T A_{x,p} [Q_s], \\
 [F_{x,q}] &= \int_0^L x^q [\phi]^T [\phi] dx = [Q_s]^T F_{x,q} [Q_s], \\
 A_{x,p} &= \int_0^L x^p [\varphi_{3,i}']^T [\varphi_{3,k}'] dx \\
 &\quad (p = 0, 1, \dots, 4), \\
 F_{x,q} &= \int_0^L x^q [\varphi_{3,i}]^T [\varphi_{3,k}] dx \quad (q = 0, 1, 2), \\
 [A_{x,\exp,p}] &= \int_0^L x^p e^{\alpha x/L} [\phi'']^T [\phi''] dx \\
 &= [Q_s]^T A_{x,\exp,p} [Q_s], \\
 [F_{x,\exp,q}] &= \int_0^L x^q e^{\alpha x/L} [\phi]^T [\phi] dx \\
 &= [Q_s]^T F_{x,\exp,q} [Q_s], \\
 A_{x,\exp,p} &= \int_0^L x^p e^{\alpha x/L} [\varphi_{3,i}']^T [\varphi_{3,k}'] dx \\
 &\quad (p = 0, 1, \dots, 4), \\
 F_{x,\exp,q} &= \int_0^L x^q e^{\alpha x/L} [\varphi_{3,i}]^T [\varphi_{3,k}] dx \\
 &\quad (q = 0, 1, 2), \\
 &\quad (i, k = -1, 1, 0, \dots, N, N+1).
 \end{aligned} \tag{38}$$

The matrices defined as (35) and (38) are presented in detail in Appendix A, which are the basic matrices for the derivation of stiffness and mass matrices. The integration methods are also proposed.

3.4. Boundary Conditions. Boundary conditions should be imposed on $[K]$ and $[M]$ matrices before solving the eigenvalue equation of (27) to obtain the natural transverse frequencies for the beams. According to (14) and (22), the end constraint of the transverse displacement $w = w(x, t)$ for beams in various boundary conditions can be expressed by $\{a\}$ in the proposed model as follows:

- (i) The following is obtained for pinned-pinned (P-P) beams:

$$\begin{aligned}
 w_{x=0} &= w_{x=L} = 0, \\
 \phi_i(0) &= 0 \quad (i \neq -1), \\
 \phi_i(L) &= 0 \quad (i \neq N+1)
 \end{aligned}$$

$\Downarrow$

$$\begin{aligned}
 a_{-1} &= 0, \\
 a_{N+1} &= 0.
 \end{aligned} \tag{39}$$

- (ii) The following is obtained for the clamped-clamped (C-C) beams:

$$\begin{aligned}
 w_{x=0} &= w'_{x=0} = w_{x=L} = w'_{x=L} = 0, \\
 \phi'_i(0) &= 0 \quad (i \neq -1, 0), \\
 \phi_i(0) &= 0 \quad (i \neq -1), \\
 \phi_i(L) &= 0 \quad (i \neq N+1), \\
 \phi'_i(L) &= 0 \quad (i \neq N, N+1)
 \end{aligned} \tag{40}$$

$\Downarrow$

$$\begin{aligned}
 a_{-1} &= a_0 = 0, \\
 a_N &= a_{N+1} = 0.
 \end{aligned}$$

- (iii) The following is obtained for clamped-pinned (C-P) beams:

$$\begin{aligned}
 w_{x=0} &= w'_{x=0} = w_{x=L} = 0, \\
 \phi'_i(0) &= 0 \quad (i \neq -1, 0), \\
 \phi_i(0) &= 0 \quad (i \neq -1) \\
 \phi_i(L) &= 0 \quad (i \neq N+1)
 \end{aligned} \tag{41}$$

$\Downarrow$

$$\begin{aligned}
 a_{-1} &= a_0 = 0, \\
 a_{N+1} &= 0.
 \end{aligned}$$

- (iv) The following is obtained for clamped-free (C-F) beams:

$$\begin{aligned}
 w_{x=0} &= w'_{x=0} = 0, \\
 \phi'_i(0) &= 0 \quad (i \neq -1, 0), \\
 \phi_i(0) &= 0 \quad (i \neq -1)
 \end{aligned} \tag{42}$$

$\Downarrow$

$$\begin{aligned}
 a_{-1} &= 0, \\
 a_0 &= 0,
 \end{aligned}$$

where a_{-1} , a_0 , a_N , and a_{N+1} are the parameters that denote the 1st, 2nd, $(N+2)$ th, and $(N+3)$ th element of $\{a\}$, which correspond to the elements on the 1st, 2nd, $(N+2)$ th, and $(N+3)$ th row, and the column on the $[K]$ and $[M]$ matrices according to the free vibration equation of (26). For the free vibration problem, reducing the scale of structural $[K]$

and $[M]$ matrices effectively prevents ill matrices to solve the eigenvalue equation of (27). Thus, the way of boundary conditions treatment for the beams of the proposed model can be readily achieved by (39)–(42) by deleting the elements of the rows and columns of the $[K]$ and $[M]$ matrices that correspond to parameters a_{-1} , a_0 , a_N , and a_{N+1} . Take the S-S

beam, for example; the 1st and $(N + 3)$ th rows and columns of the $[K]$ and $[M]$ matrices need to be deleted according to (39) to meet the boundary conditions. Therefore, the new forms of $[K]$ and $[M]$ matrices for beams in various boundary conditions after applying boundary constraints can be expressed as follows:

(i) The following is obtained for P-P beams:

$$[km]_{P-P} = \begin{bmatrix} km_{2,2} & km_{2,3} & \cdots & km_{2,(N+1)} & km_{2,(N+2)} \\ km_{3,2} & km_{3,3} & \cdots & km_{3,(N+1)} & km_{3,(N+2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ km_{(N+1),2} & km_{(N+1),3} & \cdots & km_{(N+1),(N+1)} & km_{(N+1),(N+2)} \\ km_{(N+2),2} & km_{(N+2),3} & \cdots & km_{(N+2),(N+1)} & km_{(N+2),(N+2)} \end{bmatrix}_{(N+1) \times (N+1)}. \quad (43)$$

(ii) The following is obtained for C-C beams:

$$[km]_{C-C} = \begin{bmatrix} km_{3,2} & km_{3,3} & \cdots & km_{3,N} & km_{3,(N+1)} \\ km_{4,2} & km_{4,4} & \cdots & km_{4,N} & km_{4,(N+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ km_{N,2} & km_{N,3} & \cdots & km_{N,N} & km_{N,(N+1)} \\ km_{(N+1),2} & km_{(N+1),3} & \cdots & km_{(N+1),N} & km_{(N+1),(N+1)} \end{bmatrix}_{(N-1) \times (N-1)}. \quad (44)$$

(iii) The following is obtained for C-P beams:

$$[km]_{C-P} = \begin{bmatrix} km_{3,2} & km_{3,3} & \cdots & km_{3,(N+1)} & km_{3,(N+1)} \\ km_{4,2} & km_{4,4} & \cdots & km_{4,(N+1)} & km_{4,(N+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ km_{(N+1),2} & km_{(N+1),3} & \cdots & km_{(N+1),(N+1)} & km_{(N+1),(N+2)} \\ km_{(N+2),2} & km_{(N+2),3} & \cdots & km_{(N+2),(N+1)} & km_{(N+2),(N+2)} \end{bmatrix}_{(N) \times (N)}. \quad (45)$$

(iv) The following is obtained for C-F beams:

$$[km]_{C-F} = \begin{bmatrix} km_{3,2} & km_{3,3} & \cdots & km_{3,N} & km_{3,(N+3)} \\ km_{4,2} & km_{4,4} & \cdots & km_{4,N} & km_{4,(N+3)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ km_{(N+2),2} & km_{(N+2),3} & \cdots & km_{(N+2),(N+2)} & km_{(N+2),(N+3)} \\ km_{(N+3),2} & km_{(N+3),3} & \cdots & km_{(N+3),(N+2)} & km_{(N+3),(N+3)} \end{bmatrix}_{(N+1) \times (N+1)}, \quad (46)$$

where $[km]_{P-P}$, $[km]_{C-C}$, $[km]_{C-P}$, and $[km]_{C-F}$ are the new forms of matrices for $[K]$ and $[M]$ after applying the boundary conditions with regard to the four defined boundary

conditions, namely, P-P, C-C, C-P, and C-F, respectively. These matrices are in the scale of $(N + 1) \times (N + 1)$, $(N - 1) \times (N - 1)$, $(N) \times (N)$, and $(N + 1) \times (N + 1)$. Moreover,

TABLE 1: First five nondimensional transverse natural frequencies $\bar{\omega}_i$ for uniform beam; boundary condition: C-F.

$\bar{\omega}_i$	SFPM					Exact [35]	Reference [14]
	$N = 5$	$N = 10$	$N = 15$	$N = 20$	$N = 25$		
$\bar{\omega}_1$	3.516	3.516	3.516	3.516	3.516	3.516	3.516
$\bar{\omega}_2$	22.049	22.035	22.035	22.035	22.035	22.034	22.0345
$\bar{\omega}_3$	62.11	61.716	61.701	61.698	61.698	61.701	61.6972
$\bar{\omega}_4$	124.526	121.062	120.929	120.91	120.905	120.909	120.9019
$\bar{\omega}_5$	205.179	200.702	199.994	199.899	199.875	199.855	—

TABLE 2: First five nondimensional transverse natural frequencies $\bar{\omega}_i$ for uniform beams with various boundary conditions.

$\bar{\omega}_i$	C-C		C-P		P-P	
	SFPM	Exact [35]	SFPM	Exact [35]	SFPM	Exact [35]
$\bar{\omega}_1$	22.373	22.372	15.418	15.413	9.87	9.869
$\bar{\omega}_2$	61.674	61.669	49.965	49.956	39.479	39.478
$\bar{\omega}_3$	120.912	120.912	104.253	104.244	88.828	88.826
$\bar{\omega}_4$	199.898	199.855	178.297	178.275	157.921	157.913
$\bar{\omega}_5$	298.69	298.529	272.132	272.019	246.769	246.739

$km_{i,j}$ ($i, j = 1, \dots, N + 3$) are the elements located in the i th row and j th column of the original structural $[K]$ and $[M]$ matrices of the proposed model, which are $(N + 3) \times (N + 3)$ matrices mentioned before. The boundary condition treatment process can be easily achieved in a computer program through (43)–(46).

After applying the boundary conditions, the stiffness and mass matrices are substituted into (27). The natural frequencies ω_i and the corresponding generalized mode shape parameters $\{\varphi_i\}$ are determined by solving (27) with the standard computer subroutines. Then, the mode displacement at the arbitrary location along the beam axis can be calculated by substituting $\{\varphi_i\}$ into (14).

3.5. Flowchart for Analysis of the Axially FG Tapered Beam via SFPM. The flowchart for free vibration of axially FG tapered beam via SFPM is shown in Figure 4 to state the proposed method visually. It is obvious that the proposed SFPM model is convenient for studying the vibration problems of the axially tapered Euler beams.

4. Numerical Examples

The nondimensional natural transverse frequencies are defined as $\bar{\omega} = \omega \sqrt{\rho_0 A_0 L^4 / E_0 I_0}$, where A_0, I_0, ρ_0 , and E_0 represent the cross-sectional and material properties of the beam at $x = 0$. And the geometrical properties for the beams studied in this paper are given as follows: $b_0 \times h_0 = 0.01 \text{ m} \times 0.03 \text{ m}$, $A_0 = 3 \times 10^{-2} \text{ m}^2$, $I_0 = 2.25 \times 10^{-8} \text{ m}^4$, and $L = 1 \text{ m}$; the material properties for all beams, except for the beams with material properties that vary in the exponential law (Section 4.3), are as follows: $\rho_0 = 7900 \text{ kg/m}^3$ and $E_0 = 2.1 \times 10^{11} \text{ N/m}^2$, and both the cross-sectional and material properties only vary along the longitudinal direction.

4.1. A Comparison of the Results for Uniform Homogeneous Beams. As the first example, we consider the case of

a homogeneous beam with uniform cross sections. For this case Young's modulus ($E(x) = E_0$), mass density ($\rho(x) = \rho_0$), cross-sectional area ($A(x) = A_0$), and second moment of inertia ($I(x) = I_0$) are not changed. We calculate the nondimensional natural transverse frequencies, $\bar{\omega}$, of cantilever beams (C-F) by choosing a different spline node number (N) in (12) to check the convergence of the presented method (i.e., SFPM). The first four nondimensional natural frequencies, $\bar{\omega}_i$, are tabulated in Table 1 and compared with the exact solutions [35] and other numerical methods [14]. The comparison indicated that the SFPM solution has rapid convergence. As N increases from 5 to 25, the first three nondimensional natural frequencies are identical to the exact solutions. This finding indicates that the present method is efficient except for the case of $N = 5$, in which the fourth and fifth nondimensional natural frequencies do not match the exact solution because of the insufficient order of the eigenvalue equation, that is, (27). However, the proposed model does not have any convergence problems. As N increases from 15 to 25, computational time and cost increase, but the results of the natural frequencies via SFPM vary slightly. Thus, we take spline node number $N = 15$ in the following examples. Moreover, the first five nondimensional natural transverse frequencies for C-C, C-P, and P-P beams are calculated and tabulated in Table 2 with $N = 15$. This finding verifies that SFPM can be applied to analyze the free vibration problem of beams with various boundary conditions.

4.2. Effect of the Variable Cross Section. The second example is devoted to the Euler-Bernoulli beam with a rectangular cross section and has a breadth and height that taper linearly, as shown in Figure 2. Thus, the cross-sectional area $A(x)$ and the second moment of inertia $I(x)$ that vary along axis are defined as (28), as mentioned earlier.

Firstly, the natural transverse frequencies of the tapered homogenous beams with various boundary conditions

FIGURE 4: Flowchart of free vibration analysis of axially FG tapered beam via SFPM.

FIGURE 5: FEM model for homogenous tapered beams by using ANSYS.

(C-F, C-C, and P-P) are calculated by using SFPM, for all beams, $N = 15$ is selected, and two cases of cross-sectional variation, namely, the case of only the height tapered linearly ($c_h = c_0$) and the case of both breadth and height tapered linearly ($c_b = c_h = c_0$), are both investigated. In order to validate the performance of the proposed method, FEM models (3D), which are implemented in ANSYS (see in Figure 5), are employed to investigate the free vibration of

the tapered homogenous beams. In the FEM models, 3D solid element SOLID64 is employed, and all beams are meshed into 1600 elements to ensure the computational accuracy for fundamental and high-order frequencies and to avoid producing the abnormal element due to the increase of taper ratios. The nondimensional natural transverse frequencies for C-F, C-C, and P-P beams are tabulated in Tables 3–5, which indicate that the results of SFPM model agree very well with those of FEM models and DTM model reported by Rajasekaran [4]. The taper ratio (c_0) plays an important role in the tapered homogenous beams. For all beams, the natural frequencies decrease with the increase of the taper ratio (c_0) except for the fundamental frequencies of C-F beams, which increase with c_0 ; for C-F beams, it is obvious that the frequencies of the case of height tapered linearly ($c_h = c_0$) are smaller than those of the case of both height and breadth tapered linearly ($c_b = c_h = c_0$), while, for C-C and P-P beams, the frequencies of the case of height tapered linearly ($c_h = c_0$) are equal to those of the case of both height and breadth tapered linearly ($c_b = c_h = c_0$). However, in this study, given $N = 15$ is employed, the SFPM model shows a considerable decrease of the dimensions of the eigenvalue equations compared with the FEM and DTM models [4], leading to remarkable decrease in computational cost. Furthermore, FEM models implemented in ANSYS are unable to analyze the free vibration of tapered

TABLE 3: Nondimensional transverse natural frequencies $\bar{\omega}_i$ for tapered homogenous beam; boundary condition: C-F.

c_0		$c_b = 0, c_h = c_0$			$c_b = c_h = c_0$		
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$
0.2	SFPM	3.6083	20.6212	56.1954	3.8551	21.0569	56.6335
	Reference [4]	3.6083	20.621	56.1923	3.8551	21.0567	56.6303
	FEM	3.6113	20.5759	55.8008	3.8580	21.0108	56.2356
0.4	SFPM	3.7371	19.114	50.3567	4.31878	20.0501	51.3378
	Reference [4]	3.7371	19.1138	50.3537	4.3188	20.05	51.3346
	FEM	3.7398	19.0857	50.0932	4.3213	20.0187	51.0670
0.6	SFPM	3.93428	17.488	44.0279	5.00904	19.0651	45.7417
	Reference [4]	3.9343	17.4879	44.0248	5.009	19.0649	45.7384
	FEM	3.9366	17.4717	43.8706	5.0110	19.0463	45.5747
0.8	SFPM	4.2925	15.7429	36.8881	6.19639	18.3858	39.8376
	Reference [4]	4.2925	15.7427	36.8852	6.1964	18.3855	39.8339
	FEM	4.2946	15.7366	36.8135	6.1978	18.3751	39.7517

TABLE 4: Nondimensional transverse natural frequencies $\bar{\omega}_i$ for tapered homogenous beam; boundary condition: C-C.

c_0		$c_b = 0, c_h = c_0$			$c_b = c_h = c_0$		
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$
0.2	SFPM	20.0784	55.3432	108.5080	20.0968	55.3683	108.5357
	FEM	20.0525	54.9593	106.9451	20.0708	54.9832	106.9690
0.4	SFPM	17.6357	48.5873	95.2438	17.7203	48.7027	95.3717
	FEM	17.6321	48.3680	94.2691	17.7151	48.4791	94.3873
0.6	SFPM	14.9624	41.1721	80.6681	15.1914	41.4862	81.0196
	FEM	14.9753	41.0829	80.1634	15.2019	41.3883	80.4955
0.8	SFPM	11.8486	32.5079	63.6182	12.3970	33.2794	64.5189
	FEM	11.8684	32.4963	63.4193	12.4054	33.2308	64.2326

TABLE 5: Nondimensional transverse natural frequencies $\bar{\omega}_i$ for tapered homogenous beam; boundary condition: P-P.

c_0		$c_b = 0, c_h = c_0$			$c_b = c_h = c_0$		
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$
0.2	SFPM	8.8462	35.4461	79.7399	8.8246	35.4665	79.7721
	FEM	8.8370	35.2994	78.9997	8.8153	35.3191	79.0321
0.4	SFPM	7.7295	31.1945	70.0980	7.6314	31.2881	70.2453
	FEM	7.7234	31.0975	69.6109	7.6251	31.1904	69.7559
0.6	SFPM	6.4667	26.6014	59.6076	6.2087	26.8534	60.0036
	FEM	6.4631	26.5453	59.3230	6.2050	26.7957	59.7141
0.8	SFPM	4.9201	21.3485	47.5048	4.3536	21.9447	48.4357
	FEM	4.9184	21.3218	47.3703	4.3515	21.9142	48.2878

beams with material nonhomogeneity along the longitudinal direction.

Then, we study the free vibration of axially FG tapered beam with Young's modulus, and the material densities along the axis are defined as

$$\begin{aligned}
 E(x) &= E_0 \left(1 + \frac{x}{L} \right), \\
 \rho(x) &= \rho_0 \left(1 + \frac{x}{L} + \left(\frac{x}{L} \right)^2 \right),
 \end{aligned} \tag{47}$$

where E_0 and ρ_0 are Young's modulus and mass density, respectively, at $x = 0$.

The material properties defined by (47) follow the polynomial law distribution, and (47) can be simplified as (31), with $n1 = 1$ and $n2 = 2$:

$$\begin{aligned}
 E(x) &= \sum_{i=0}^1 \tilde{E}_i x^i, \\
 \rho(x) &= \sum_{i=0}^2 \tilde{\rho}_i x^i,
 \end{aligned} \tag{48}$$

FIGURE 6: Convergence of natural frequencies for axially FG tapered cantilever beams with respect to spline node number N ($c_b = c_h = 0.8$).

in which

$$\begin{aligned} \tilde{E}_i &= \frac{E_0}{L^i} \quad (i = 0, 1); \\ \tilde{\rho}_i &= \frac{\rho_0}{L^i} \quad (i = 0, 1, 2). \end{aligned} \quad (49)$$

Using (48), the structural stiffness and mass matrices can be calculated by (34). In Figure 6, the convergence of natural frequencies for axially FG tapered cantilever beams ($c_b = c_h = 0.8$) (material gradation following the polynomial law) with respect to the spline node number N is discussed; it is noticeable that the natural frequencies achieve a convergence value when N increase from 5 to 15 (about 4-digit precision), and when N increase from 15 to 25, the frequencies vary a little but the computational cost increased remarkably. Therefore, $N = 15$ is a suitable choice for the proposed SFPM models to ensure both the accuracy and efficiency for the free vibration of the axially FG tapered beams with different taper ratios.

The natural transverse frequencies for axially FG tapered beams with different breadth taper ratio (c_b) and height taper ratio (c_h) are calculated by using the proposed method,

and the results are tabulated in Tables 6–8. The results of SFPM agree with the results of DTEM reported by Shahba and Rajasekaran [3]. This observation indicates that the proposed method can be used to investigate the free vibration of axially FG tapered beams with high accuracy. The variations of the first two nondimensional natural frequencies of the axially FG beams with various boundary conditions for different values of taper ratios (c_b, c_h) are illustrated in Figures 7–9.

Figures 7–9 illustrate that the varying trend of the natural frequencies is distinguished from beams with different boundary conditions. In general, the natural transverse frequencies are considerably affected by the variation of height taper ratio c_h . Figure 7 and Table 6 indicate that the increase of taper ratios c_h and c_b leads to the increase of the fundamental natural frequencies for C-F beams, whose second frequencies decrease with c_h and slightly increase with c_b . For C-C and P-P beams, Figures 8 and 9 and Tables 7 and 8 indicate that all frequencies remarkably decrease with c_h , whereas the variations with the increase of c_b are not monotonic. Thus, predicting the natural frequencies for axially FG tapered beams of different

TABLE 6: Nondimensional transverse natural frequencies $\bar{\omega}_i$ for axially FG tapered beam; boundary condition: C-F.

c_h	c_b	0		0.2		0.4		0.6		0.8	
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$
0	SFPM	2.4256	18.6042	2.6054	19.0042	2.8508	19.5304	3.2137	20.2959	3.8310	21.6760
	Reference [3]	2.4256	18.6041	2.6054	19.0041	2.8508	19.5303	3.2137	20.2958	3.8310	21.6759
0.2	SFPM	2.5051	17.3802	2.6863	17.7501	2.9336	18.2379	3.2993	18.9501	3.9219	20.2432
	Reference [3]	2.5051	17.3802	2.6863	17.7501	2.9336	18.2379	3.2994	18.9501	3.9220	20.2432
0.4	SFPM	2.6155	16.0705	2.7987	16.4092	3.0486	16.8571	3.4181	17.5139	4.0471	18.7164
	Reference [3]	2.6155	16.0705	2.7988	16.4092	3.0486	16.8571	3.4181	17.5139	4.0471	18.7164
0.6	SFPM	2.7835	14.6508	2.9699	14.9567	3.2237	15.3628	3.5985	15.9616	4.2355	17.0694
	Reference [3]	2.7836	14.6508	2.9699	14.9567	3.2237	15.3627	3.5985	15.9616	4.2355	17.0694
0.8	SFPM	3.0871	13.1142	3.2794	13.3850	3.5401	13.7466	3.9232	14.2848	4.5695	15.2955
	Reference [3]	3.0871	13.1142	3.2794	13.3849	3.5401	13.7466	3.9232	14.2848	4.5695	15.2954

TABLE 7: Nondimensional transverse natural frequencies $\bar{\omega}_i$ for axially FG tapered beam; boundary condition: C-C.

c_h	c_b	0		0.2		0.4		0.6		0.8	
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$
0	SFPM	20.4721	56.5492	20.4152	56.4716	20.2883	56.2981	20.0186	55.9206	19.3845	54.9713
	Reference [3]	20.4721	56.5482	20.4151	56.4706	20.2883	56.2971	20.0186	55.9195	19.3844	54.9699
0.2	SFPM	18.2170	50.4796	18.1996	50.4560	18.1286	50.3593	17.9436	50.1011	17.4565	49.3718
	Reference [3]	18.2170	50.4792	18.1995	50.4556	18.1286	50.3589	17.9436	50.1006	17.4564	49.3711
0.4	SFPM	15.8282	44.0246	15.8498	44.0553	15.8350	44.0371	15.7367	43.9027	15.4025	43.4066
	Reference [3]	15.8281	44.0236	15.8497	44.0542	15.8349	44.0359	15.7366	43.9013	15.4021	43.4043
0.6	SFPM	13.2294	36.9654	13.2896	37.0509	13.3319	37.1137	13.3238	37.1105	13.1529	36.8680
	Reference [3]	13.2291	36.9636	13.2894	37.0490	13.3316	37.1116	13.3234	37.1079	13.1521	36.8635
0.8	SFPM	10.2236	28.7498	10.3231	28.8919	10.4257	29.0416	10.5170	29.1851	10.5343	29.2419
	Reference [3]	10.2217	28.7406	10.3211	28.8822	10.4234	29.0311	10.5143	29.1728	10.5301	29.2239

TABLE 8: Nondimensional transverse natural frequencies $\bar{\omega}_i$ for axially FG tapered beam; boundary condition: P-P.

c_h	c_b	0		0.2		0.4		0.6		0.8	
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$
0	SFPM	9.0286	36.3718	9.0599	36.3418	9.0867	36.3153	9.0994	36.2969	9.0685	36.2769
	Reference [3]	9.0285	36.3715	9.0599	36.3415	9.0867	36.3151	9.0994	36.2966	9.0685	36.2766
0.2	SFPM	8.1341	32.5236	8.1462	32.5123	8.1498	32.5079	8.1336	32.5164	8.0646	32.5326
	Reference [3]	8.1341	32.5234	8.1462	32.5121	8.1498	32.5077	8.1336	32.5161	8.0646	32.5323
0.4	SFPM	7.1531	28.4747	7.1455	28.4822	7.1254	28.5003	7.0794	28.5370	6.9703	28.5928
	Reference [3]	7.1531	28.4744	7.1455	28.4819	7.1254	28.5001	7.0794	28.5367	6.9703	28.5924
0.6	SFPM	6.0357	24.1101	6.0082	24.1371	5.9638	24.1791	5.8868	24.2469	5.7351	24.3497
	Reference [3]	6.0357	24.1098	6.0082	24.1367	5.9637	24.1787	5.8868	24.2465	5.7351	24.3491
0.8	SFPM	4.6520	19.1315	4.6046	19.1804	4.5355	19.2510	4.4264	19.3591	4.2284	19.5302
	Reference [3]	4.6520	19.1304	4.6045	19.1792	4.5354	19.2496	4.4263	19.3576	4.2281	19.5281

boundary conditions with respect to the taper ratios is impossible.

Notice that the proposed model needs considerably less computational cost and modeling time compared with the DTM and DTEM models developed by Rajasekaran [4] and Shahba et al. [2], respectively. In the DTM and DTEM models, 132 simple points and 12 elements are

used, and the eigenvalue equation in the dimension of 578×578 needs to be solved. However, in the proposed model, as $N = 15$ is selected based on the convergence study, the eigenvalue equation is only in the dimension of 18×18 , which shows that the proposed method has higher remarkable ascending efficiency than the reported methods.

FIGURE 7: The variation of the first two nondimensional natural frequencies of axially FG beams with respect to taper ratios; boundary condition: C-F.

FIGURE 8: The variation of the first two nondimensional natural frequencies of axially FG beams with respect to taper ratios; boundary condition: C-C.

Furthermore, the natural transverse frequencies for tapered beams with consideration of material nonhomogeneity and homogeneity are distinguished by comparing the results in Table 3 with those in Table 4. The material nonhomogeneity defined by (47) decreases the natural frequencies for tapered C-F beams compared with homogenous beams. Thus, the material properties obviously affect the dynamic characteristics of the tapered beams.

4.3. Effect of Axial Gradient Parameter. Finally, the free vibration for tapered rectangular beams with axially exponentially graded material is investigated to verify the adaptability of the proposed method to different gradient functions. Similar

works have been conducted by Huang and Li [14]. The material properties are defined as follows:

$$Y(x) = Y_1 \left(1 - \frac{e^{\alpha x/L} - 1}{e^\alpha - 1} \right) + Y_2 \frac{e^{\alpha x/L} - 1}{e^\alpha - 1}, \quad \text{if } \alpha \neq 0, \quad (50)$$

$$Y(x) = Y_1 \left(1 - \frac{x}{L} \right) + Y_2 \frac{x}{L}, \quad \text{if } \alpha = 0,$$

in which Y_1 and Y_2 correspond to the material properties, such as Young's modulus $E(x)$ and mass density $\rho(x)$ at $x = 0$ and $x = L$, respectively, and α is the gradient parameter that

FIGURE 9: The variation of the first two nondimensional natural frequencies of axially FG beams with respect to taper ratios; boundary condition: P-P.

FIGURE 10: Variation of gradation material properties defined by (50) with $Y_1 = 3Y_2$.

describes the volume fraction change of both constituents involved. The variation of $Y(x)$ against x is shown in Figure 10 for $Y_1 = 3Y_2$.

The material properties defined by (50) can be classified into two types of distribution that depends on the value of gradient index α ; one is the polynomial type that is defined by (31) ($\alpha = 0$) with $n_1 = n_2 = 1$ and the other is exponential type that is defined by (32) ($\alpha \neq 0$), which can be simplified into the following equations:

$$\begin{aligned} Y(x) &= \tilde{Y}_1 + \tilde{Y}_2 e^{\alpha x/L}, & \text{if } \alpha \neq 0, \\ Y(x) &= \tilde{Y}_1 + \tilde{Y}_2 x, & \text{if } \alpha = 0, \end{aligned} \quad (51)$$

where

$$\begin{aligned} \tilde{Y}_1 &= \left(Y_1 - \frac{Y_2 - Y_1}{e^\alpha - 1} \right), \\ \tilde{Y}_2 &= \left(\frac{Y_2 - Y_1}{e^\alpha - 1} \right), & \text{if } \alpha \neq 0, \\ \tilde{Y}_1 &= Y_1, \\ \tilde{Y}_2 &= \frac{Y_2 - Y_1}{L}, & \text{if } \alpha = 0. \end{aligned} \quad (52)$$

Using (51), the stiffness and mass matrices can be calculated by (34) and (37) for $\alpha = 0$ and $\alpha \neq 0$, respectively. In the following calculations, the two materials chosen are aluminum (Al) and zirconia (ZrO_2). The material properties are presented by [14]

$$\begin{aligned} \text{ZrO}_2: E_z &= 2.0 \times 10^{11} \text{ N/m}^2, \\ \rho_z &= 7900 \text{ kg/m}^3, \\ \text{Al: } E_a &= 7.0 \times 10^{10} \text{ N/m}^2, \\ \rho_a &= 2702 \text{ kg/m}^3. \end{aligned} \quad (53)$$

The nondimensional frequencies are defined by

$$\bar{\omega} = \omega \sqrt{\frac{\rho_a A_0 L^4}{E_a I_0}}. \quad (54)$$

TABLE 9: Nondimensional fundamental natural transverse frequencies $\bar{\omega}_i$ for axially FG beams with respect to the gradient index.

α		P-P		C-C		C-P		C-F	
		Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
-10	Reference [14]	11.4532	9.9358	24.0576	24.7949	16.4775	17.2993	3.5656	4.1800
	SFPM	11.4560	9.9366	24.0660	24.8068	16.4111	17.3037	3.5340	4.1801
-3	Reference [14]	11.2443	10.3669	23.9456	24.9375	16.0307	17.8701	3.1421	4.8317
	SFPM	11.2443	10.3669	23.9446	24.9366	16.0262	17.8701	3.1410	4.8317
0	Reference [14]	10.8663	10.8663	24.3752	24.3752	15.8734	17.9147	2.9256	5.0156
	SFPM	10.8664	10.8664	24.3756	24.3756	15.8735	17.9148	2.9256	5.0156
3	Reference [14]	10.3669	11.2443	24.9375	23.9456	15.7171	17.8867	2.8544	4.8466
	SFPM	10.3669	11.2443	24.9366	23.9446	15.7169	17.8874	2.8545	4.8466
10	Reference [14]	9.9358	11.4532	24.7949	23.9456	15.4930	17.9050	3.0431	4.4629
	SFPM	9.9366	11.4560	24.8068	24.0660	15.4958	17.9155	3.0431	4.4629

TABLE 10: Nondimensional higher-order transverse natural frequencies $\bar{\omega}_i$ for axially FG beams with respect to the gradient index.

α		P-P		C-C		C-P		C-F	
		Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
-10	$\bar{\omega}_2$	45.6887	40.0700	68.1328	66.2401	54.9276	53.9917	23.7075	24.4362
	$\bar{\omega}_3$	102.6601	90.5367	135.4215	127.6871	116.4430	110.4619	68.1642	66.2609
	$\bar{\omega}_4$	182.4345	161.3146	225.7493	209.1345	201.0000	186.8845	135.4187	127.6897
	$\bar{\omega}_5$	285.1358	252.4647	339.2851	310.8793	308.7099	283.5206	225.7516	209.1407
-3	$\bar{\omega}_2$	44.8550	41.9691	67.7638	67.1081	54.4731	54.9320	23.0531	25.4924
	$\bar{\omega}_3$	100.8244	94.5153	134.4815	130.2588	115.5467	112.8131	67.4843	67.7907
	$\bar{\omega}_4$	179.1822	168.0682	223.7508	214.4078	199.1799	191.6794	134.1969	130.8653
	$\bar{\omega}_5$	280.0397	262.7506	335.7204	319.7813	305.4568	291.7087	223.4883	214.9894
0	$\bar{\omega}_2$	43.6649	43.6649	67.5918	67.5918	53.9610	55.9731	22.3541	26.4650
	$\bar{\omega}_3$	98.1296	98.1296	132.8706	132.8706	113.7904	115.8588	66.1736	70.2707
	$\bar{\omega}_4$	174.3787	174.3787	220.0032	220.0032	195.4598	197.5581	131.4575	135.6415
	$\bar{\omega}_5$	272.5370	272.5370	329.1761	329.1761	299.1037	301.2218	218.6135	222.8467
3	$\bar{\omega}_2$	41.9691	44.8550	67.1081	67.7638	52.8041	56.9021	21.4951	27.1414
	$\bar{\omega}_3$	94.5153	100.8244	130.2588	134.4815	110.6243	118.4554	63.6755	72.7816
	$\bar{\omega}_4$	168.0682	179.1822	214.4078	223.7508	189.4610	202.4128	126.6073	140.3662
	$\bar{\omega}_5$	262.7506	280.0397	319.7813	335.7204	289.4704	308.9241	210.6519	230.2512
10	$\bar{\omega}_2$	40.0700	45.6887	66.2401	68.1328	50.7956	57.7893	20.7361	27.0569
	$\bar{\omega}_3$	90.5367	102.6601	127.6871	135.4215	106.3245	120.4638	60.1871	74.3559
	$\bar{\omega}_4$	161.3146	182.4345	209.1345	225.7493	182.1851	205.9533	120.0294	144.0360
	$\bar{\omega}_5$	252.4647	285.1358	310.8793	339.2851	278.4567	314.4208	200.4314	236.3716

In consideration of the two cases in this paper, one is Al rich near the end $x = 0$ and ZrO_2 rich near the end at $x = L$ (Case 1), whereas the other is Al rich near the end at $x = L$ and ZrO_2 rich near the end at $x = 0$ (Case 2).

The free vibration of axially FG beams with uniform cross-sectional ($c_b = c_h = 0$) and material properties defined as (50) is calculated to verify the proposed method (i.e., SFPM). The results for the fundamental nondimensional natural transverse frequencies are tabulated in Table 9, and the high-order frequencies calculated with SFPM are tabulated in Table 10. Figure 11 gives the convergence of the natural frequencies for axially exponentially FG cantilever

beams (C-F) ($\alpha = -10$) with respect to spline node number N ; the variations of the first three nondimensional natural frequencies of axially FG beams with respect to gradient parameter α are depicted in Figures 12–15.

Figure 11 indicates that convergence solutions of natural frequencies with high accuracy will be obtained for the proposed model of the axially exponentially FG beams at $N = 15$. Therefore, for axially FG beams with different gradient index investigated in this section, let $N = 15$ be the suitable choice for all models to achieve the demands for both the high accuracy and low computational cost. Table 9 indicates that the results of fundamental frequencies of the proposed model

FIGURE 11: Convergence of natural frequencies for axially exponentially FG cantilever beams with respect to spline node number N ($\alpha = -10$).

agree with those reported by Huang and Li [14] very well. The comparison of the results for Cases 1 and 2 listed in Tables 9 and 10 indicates that the gradient index α is important for the natural frequencies.

As shown in Figures 12 and 13, the natural frequencies of P-P beams and C-C beams for Case 1 with index α are identical to those for Case 2 with index $-\alpha$. However, the varying tendency of the natural frequencies for C-F and C-P beams with respect to gradient index is different because of the asymmetry of the boundary conditions, as shown in Figures 14 and 15. The fundamental natural transverse frequencies for Case 2 are larger than those for Case 1 with similar index α . However, the high-order natural frequencies, which show a complex variation tendency, do not always follow this law. For all beams, the natural frequencies for Case 2 show an ascending tendency with the increase of gradient index, whereas the natural frequencies for Case 1 show an opposite trend.

The natural transverse frequencies of the tapered beams ($c_b = c_h = c_0$) with material nonhomogeneity defined by (50) ($\alpha = -10$), which belongs to exponential law, are calculated by the proposed model to investigate the effects

of the taper ratio, and the results are tabulated in Table 11, which indicate that all natural transverse frequencies for the beams with different boundary conditions generally decrease with the taper ratio, except for the beams with C-F boundary, whose fundamental natural frequencies (as shown in italic bold fonts in Table 11) increase with the taper ratio. By contrast, the high-order frequencies show an opposite trend, and the same varying trends have been found for the tapered beams with material gradation following the polynomial law (Section 4.2) with respect to the taper ratios.

The results obtained by the numerical examples in this section indicate that the proposed model shows good adaptability to the variation of cross sections, boundary conditions, and material gradient functions in either a polynomial or an exponential gradient function. And for the free vibration of tapered beams with different type of material nonhomogeneity, the results of higher accuracy and lower computational cost are obtained for the proposed SFPM model compared with other reported models. What is more, the natural frequencies of the tapered beams of the same boundary conditions with homogenous and nonhomogeneous materials vary in a similar trend toward the variation of the taper

TABLE 11: Nondimensional natural transverse frequencies $\bar{\omega}_i$ for axially FG tapered beam with respect to taper ratio $c_b = c_h = c_0$, $\alpha = -10$.

	c_0	P-P					C-C				
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
Case 1	0	11.4560	45.6887	102.6601	182.4345	285.1358	24.0660	68.1328	135.4215	225.7493	339.2851
	0.2	10.2635	41.0849	92.2393	163.8387	256.0119	21.5207	61.0451	121.4017	202.4339	304.2973
	0.4	8.8941	36.2851	81.2897	144.1972	225.1632	18.8852	53.6029	106.6116	177.8075	267.3534
	0.6	7.2513	31.1822	69.5172	122.9219	191.6076	16.1104	45.6006	90.5999	151.0965	227.2883
	0.8	5.0932	25.5025	56.1391	98.3922	152.5382	13.0533	36.4076	71.8833	119.5672	179.6797
Case 2	0	9.9366	40.0700	90.5367	161.3146	252.4647	24.8068	66.2401	127.6871	209.1345	310.8793
	0.2	8.8494	35.9365	81.2126	144.6952	226.4368	22.3802	59.6013	114.7530	187.8207	279.0755
	0.4	7.6207	31.6402	71.4122	127.1143	198.8076	19.8231	52.5571	100.9830	165.0931	245.1303
	0.6	6.1724	27.0932	60.8831	108.0671	168.7522	17.0641	44.8967	85.9592	140.2721	208.0589
	0.8	4.3128	22.1136	49.1297	86.6404	134.9467	14.0097	36.3471	69.2653	113.0049	167.9583
	c_0	C-P					C-F				
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
Case 1	0	16.4111	54.9276	116.4430	201.0000	308.7099	3.5340	23.7075	68.1642	135.4187	225.7516
	0.2	15.3662	49.9039	105.0941	180.9523	277.5884	3.9253	22.6573	62.5140	122.8555	203.8977
	0.4	14.2204	44.6316	93.1433	159.8062	244.7549	4.4598	21.5947	56.6601	109.7005	180.9238
	0.6	12.9302	38.9644	80.2439	136.9256	209.2062	5.2495	20.5907	50.5440	95.6707	156.2332
	0.8	11.3909	32.4806	65.3274	110.2051	167.3687	6.5769	19.9787	44.1206	79.9627	127.8152
Case 2	0	17.3037	53.9917	110.4619	186.8845	283.5206	4.1801	24.4362	66.2609	127.6897	209.1407
	0.2	16.2634	49.2161	99.9160	168.4860	255.1649	4.5112	23.3314	60.8734	116.0144	189.0897
	0.4	15.1061	44.1433	88.7009	148.8877	224.9241	4.9675	22.1766	55.2115	103.6541	167.7838
	0.6	13.7727	38.6134	76.4763	127.5011	191.9048	5.6563	21.0079	49.1851	90.3173	144.6597
	0.8	12.1299	32.2583	62.5395	103.3128	154.9786	6.8715	20.0744	42.7624	75.5952	118.9682

ratios. The material properties, taper ratios, and boundary conditions play an important role in determining the natural transverse frequencies for axially FG tapered Euler-Bernoulli beams.

5. Conclusion

In this study, a new model for the analysis of the free vibration of axially FG tapered Euler-Bernoulli beams is developed with SFPM. The effects of the variation in cross-sectional and material properties along the longitudinal directions are investigated. In the proposed approach, the beam is discretized with a set of uniformly scattered spline nodes along the beam axis direction instead of meshes, thereby resulting in convenience for modeling and programming. The displacement field of the beam is approximated with cubic B-spline interpolation functions, which shows good adaptability to different boundary conditions. Given the special features of SFPM, the global structural stiffness and mass matrices for axially FG tapered beams are directly generated after spline discretization instead of element meshes, generation, and assembling. The effectiveness of the proposed method is confirmed by comparing our numerical results with those available in the literature, including the case of rectangular beams with linearly varying cross-sectional breadth or height and axially FG tapered beams with polynomial or exponential function gradient. All results show that

the proposed method can analyze the free vibration of Euler-Bernoulli beams with arbitrary axial gradient and varying cross sections in various boundary conditions. Moreover, a stable solution with high accuracy can be achieved with a small spline node number.

Appendix

A. Method of Integrations for Stiffness and Mass Matrices

Through an investigation of (34)–(38), four types of matrix integrations, $A_{x,m}$, $A_{x,\exp,p}$, $F_{x,n}$, and $F_{x,\exp,q}$, must be deduced before forming the structural stiffness and mass matrices of the proposed method. The calculation processes of matrices $A_{x,p}$ and $F_{x,q}$ are similar to $A_{x,m}$ and $F_{x,n}$, which are not repeated in this paper.

According to (10) and (20), if $t = x/h_x - i$, $k - i = c$, then $x = h_x(t + i)$, which is obtained:

$$\begin{aligned}\varphi_{3,i}^{(j)} &= \varphi_3^{(j)} \left(\frac{x}{h_x} - i \right) = h_x^{-j} \varphi_3^{(j)}(t), \\ \varphi_{3,k}^{(j)} &= \varphi_3^{(j)} \left(\frac{x}{h_x} - k \right) = h_x^{-j} \varphi_3^{(j)}(t - c) \quad (A.1) \\ (j &= 0, 1, 2, 3; i, k = -1, 0, \dots, N + 1),\end{aligned}$$

FIGURE 12: Variation of first three nondimensional natural frequencies of an axially FG beam with respect to α ; boundary condition: C-C.

FIGURE 13: Variation of first three nondimensional natural frequencies of an axially FG beam with respect to α ; boundary condition: P-P.

where subscript “ j ” for $\varphi_{3,k}^{(j)}$ or $\varphi_{3,i}^{(j)}$ means the j th differentiation with respect to x , which is defined in (10). The functions are defined as

$$\begin{aligned}
 f(t) &= \varphi_3''(t-c)(t+i)^m, \\
 \lambda(t) &= \varphi_3(t-c)(t+i)^n, \\
 \beta(t) &= \varphi_3''(t-c)(t+i)^p e^{\alpha h_x(t+i)/L}, \\
 \delta(t) &= \varphi_3(t-c)(t+i)^q e^{\alpha h_x(t+i)/L}, \\
 f(t) &= \varphi_3''(t-c)f_1(t), \quad f_1(t) = (t+i)^m, \\
 \lambda(t) &= \varphi_3(t-c)\lambda_1(t), \quad \lambda_1(t) = (t+i)^n, \\
 \beta(t) &= \varphi_3''(t-c)\beta_1(t), \quad \beta_1(t) = \beta_2(t)\beta_3(t), \\
 \beta_2(t) &= e^{\alpha h_x(t+i)/L},
 \end{aligned}$$

$$\begin{aligned}
 \beta_3(t) &= (t+i)^p, \\
 \delta(t) &= \varphi_3(t-c)\delta_1(t), \quad \delta_1(t) = \delta_2(t)\delta_3(t), \\
 \delta_2(t) &= e^{\alpha h_x(t+i)/L}, \\
 \delta_3(t) &= (t+i)^q.
 \end{aligned} \tag{A.2}$$

Equations (A.1)-(A.2) are substituted into (35) and (38), and the characteristic of $\varphi_3^{(3)}(x)$ is used, as shown in Figure 2. The matrices $A_{x,m}$, $A_{x,\exp,p}$, $F_{x,m}$, and $F_{x,\exp,q}$ can be easily deduced through partial integration methods. Each element in the matrices is calculated as follows:

$$\begin{aligned}
 A_{x,m}(i+2, k+2) \\
 = \int_0^L x^m \varphi_3''\left(\frac{x}{h_x} - i\right) \varphi_3''\left(\frac{x}{h_x} - k\right) dx
 \end{aligned}$$

FIGURE 14: Variation of first three nondimensional natural frequencies of an axially FG beam with respect to α ; boundary condition: C-F.

$$\begin{aligned}
 &= h_x^{-3+m} \int_{-i}^{N-i} (t+i)^m \varphi_3''(t) \varphi_3''(t-c) dt \\
 &= h_x^{-3+m} \sum_{l=2}^3 (-1)^{-l} \varphi_3^{(l)}(t) f(t)^{(-l+1)} \Big|_{-i}^{N-i} \\
 & \quad i, k = -1, 0, 1, \dots, N, N+1,
 \end{aligned} \tag{A.3}$$

$$F_{x,n}(i+2, k+2)$$

$$\begin{aligned}
 &= \int_0^L x^n \varphi_3 \left(\frac{x}{h_x} - i \right) \varphi_3 \left(\frac{x}{h_x} - k \right) dx \\
 &= h_x^{1+n} \int_i^{N-i} (t+i)^n \varphi_3(t) \varphi_3(t-c) dt
 \end{aligned}$$

FIGURE 15: Variation of first three nondimensional natural frequencies of an axially FG beam with respect to α ; boundary condition: C-P.

$$\begin{aligned}
 &= h_x^{1+n} \sum_{l=0}^3 (-1)^l \varphi_3^{(l)}(t) \lambda(t)^{(-l-1)} \Big|_{-i}^{N-i} \\
 & \quad i, k = -1, 0, 1, \dots, N, N+1,
 \end{aligned} \tag{A.4}$$

$$A_{x,\exp,p}(i+2, k+2)$$

$$\begin{aligned}
 &= \int_0^L x^p e^{\alpha x/L} \varphi_3'' \left(\frac{x}{h_x} - i \right) \varphi_3'' \left(\frac{x}{h_x} - k \right) dx \\
 &= h_x^{-3+p} \int_i^{N-i} (t+i)^p e^{\alpha h_x(t+i)/L} \varphi_3''(t) \varphi_3''(t-c) dt \tag{A.5} \\
 &= h_x^{-3+p} \sum_{l=2}^3 (-1)^{-l} \varphi_3^{(l)}(t) \beta(t)^{(-l+1)} \Big|_{-i}^{N-i}; \\
 & \quad i, k = -1, 0, 1, \dots, N, N+1,
 \end{aligned}$$

$$\begin{aligned}
F_{x,\exp,q}(i+2, k+2) &= \int_0^L x^q e^{\alpha x/L} \varphi_3 \left(\frac{x}{h_x} - i \right) \varphi_3 \left(\frac{x}{h_x} - k \right) dx \\
&= h_x^{1+q} \int_i^{N-i} (t+i)^p e^{\alpha h_x(t+i)/L} \varphi_3(t) \varphi_3(t-c) dt \quad (\text{A.6}) \\
&= h_x^{1+q} \sum_{l=0}^3 (-1)^{-l} \varphi_3^{(l)}(t) \delta(t)^{(-l-1)} \Big|_{-i}^{N-i}; \\
&\quad i, k = -1, 0, 1, \dots, N, N+1.
\end{aligned}$$

$f(t) = \varphi_3''(t-c)f_1(t)$ and $f_1(t) = (t+i)^m$ are substituted into (A.3). Then, the expressions of $f(t)^{(-j)}$ ($j = 1, 2$) are obtained:

$$\begin{aligned}
f(t)^{(-1)} &= \int \varphi_3''(t-c)(t+i)^m dt \\
&= \int \varphi_3''(t-c)f_1(t) dt \\
&= \varphi_3^{(2)}(t-c)f_1(t)^{(-1)} \\
&\quad - \varphi_3^{(3)}(t-c)f_1(t)^{(-2)}, \\
f(t)^{(-2)} &= \int f(t)^{(-1)} dt \\
&= \varphi_3^{(2)}(t-c)f_1(t)^{(-2)} \\
&\quad - 2\varphi_3^{(3)}(t-c)f_1(t)^{(-3)}, \\
f_1(t)^{(-k)} &= \frac{1}{\prod_{i=1}^k (m+i)} (t+i)^{m+k} \quad (k = 1, 2, 3). \quad (\text{A.7})
\end{aligned}$$

$\lambda(t) = \varphi_3(t-c)\lambda_1(t)$ and $\lambda_1(t) = (t+i)^n$ are substituted into (A.4). Then, the expressions of $\lambda(t)^{(-j)}$ ($j = 1, \dots, 4$) are obtained:

$$\begin{aligned}
\lambda(t)^{(-j)} &= \int \lambda(t)^{(-j+1)} dt \\
&= \int [\varphi_3(t-c)\lambda_1(t)]^{(-j+1)} dt \\
&= \sum_{l=0}^3 (-1)^{-l} (\eta_{j,l}) \varphi_3^{(l)}(t-c) \lambda_1(t)^{(-l-j)}, \\
\eta_{1,l} &= 1; \\
\eta_{2,l} &= \sum_{i=0}^l \eta_{1,i} = l+1; \\
\eta_{3,l} &= \sum_{i=0}^l \eta_{2,i} = \frac{(l+1)(l+2)}{2},
\end{aligned}$$

$$\begin{aligned}
\eta_{j,l} &= \sum_{i=0}^l (\eta_{j-1,i}) \quad (j \geq 2), \\
\lambda_1(t)^{(-l-j)} &= \frac{1}{\prod_{i=1}^{l+j} (m+i)} (t+i)^{m+l+j} \\
&\quad (l = 0, 1, 2, 3; j = 1, 2, 3, 4). \quad (\text{A.8})
\end{aligned}$$

$\beta(t) = \varphi_3''(t-c)(t+i)^p e^{\alpha h_x(t+i)/L} = \varphi_3''(t-c)\beta_1(t)$ is substituted into (A.5). Then, the expressions of $\beta(t)^{(-j)}$ ($j = 1, 2$) are obtained:

$$\begin{aligned}
\beta^{(-1)}(t) &= \int \varphi_3''(t-c) x^p e^{\alpha h_x(t+i)/L} dt \\
&= \int \varphi_3''(t-c) \beta_1(t) dt \\
&= \varphi_3^{(2)}(t-c) \beta_1(t)^{(-1)} \\
&\quad - \varphi_3^{(3)}(t-c) \beta_1(t)^{(-2)}, \quad (\text{A.9}) \\
\beta^{(-2)}(t) &= \int \delta^{(-1)}(t) dt \\
&= \varphi_3^{(2)}(t-c) \beta_1(t)^{(-2)} \\
&\quad - 2\varphi_3^{(1)}(t-c) \beta_1(t)^{(-3)}.
\end{aligned}$$

Then, $\beta_1(t) = \beta_2(t)\beta_3(t)$ and $\beta_2(t) = e^{\alpha h_x(t+i)/L}$ and $\beta_3(t) = (t+i)^p$ are substituted into (A.9). $\beta_1(t)^{(-j)}$ ($j = 1, 2, 3$) is obtained:

$$\begin{aligned}
\beta_1(t)^{(-j)} &= \int \beta_1(t)^{(-j+1)} dt \\
&= \int [\beta_2(t) \beta_3(t)]^{(-j+1)} dt \\
&= \sum_{k=0}^p (-1)^{-k} (\eta_{j,k}) \beta_3(t)^{(k)} \beta_2(t)^{(-k-j)},
\end{aligned}$$

$$\eta_{1,k} = 1;$$

$$\eta_{2,k} = \sum_{i=0}^k \eta_{1,i} = k+1;$$

$$\eta_{3,k} = \sum_{i=0}^k \eta_{2,i} = \frac{(k+1)(k+2)}{2},$$

$$\eta_{j,k} = \sum_{i=1}^k \eta_{j-1,i} \quad (j \geq 2),$$

$$\beta_2(t)^{(-k-j)} = \left(\frac{\alpha h_x}{L} \right)^{-k-j} e^{\alpha h_x(t+i)/L},$$

$$\beta_3(t)^{(k)} = ((t+i)^p)^{(k)} = (t+i)^{p-k} \prod_{i=0}^{k-1} (p-i)$$

$$\begin{aligned}
 & (k \geq 1), \\
 & (k = 0, \dots, p; j = 1, 2, 3). \\
 & \quad \quad \quad (A.10)
 \end{aligned}$$

$\delta(t) = \varphi_3(t-c)(t+i)^q e^{\alpha h_x(t+i)/L} = \varphi_3(t-c)\delta_1(t)$ is substituted into (A.6). $\delta(t)^{(-j)}$ ($j = 1, \dots, 4$) obtains expressions similar to (A.8):

$$\begin{aligned}
 \delta(t)^{(-j)} &= \int \delta(t)^{(-j+1)} dt \\
 &= \int [\varphi_3(t-c)\delta_1(t)]^{(-j+1)} dt \\
 &= \sum_{l=0}^3 (-1)^{-l} (\eta_{j,l}) \varphi_3^{(l)}(t) \delta_1(t)^{(-l-j)}, \\
 \eta_{1,l} &= 1; \\
 \eta_{2,l} &= \sum_{i=0}^l \eta_{1,i} = l+1, \\
 \eta_{3,l} &= \sum_{i=0}^l \eta_{2,i} = \frac{(l+1)(l+2)}{2}, \\
 \eta_{j,l} &= \sum_{i=0}^l (\eta_{j-1,i}) \quad (j \geq 2), \\
 & \quad \quad \quad (l = 0, 1, 2, 3; j = 1, 2, 3, 4).
 \end{aligned} \quad (A.11)$$

Then, $\delta_1(t) = \delta_2(t)\delta_3(t)$ and $\delta_2(t) = e^{\alpha h_x(t+i)/L}$ and $\delta_3(t) = (t+i)^q$ are substituted into (A.11). $\delta_1(t)^{(-j)}$ ($j = 1, \dots, 7$) can be calculated by

$$\begin{aligned}
 \delta_1(t)^{(-j)} &= \int \delta_1(t)^{(-j+1)} dt \\
 &= \int [\delta_2(t)\delta_3(t)]^{(-j+1)} dt \\
 &= \sum_{k=0}^q (-1)^{-k} (\eta_{j,k}) \delta_3^{(k)}(t) \delta_2(t)^{(-k-j)}, \\
 \eta_{1,k} &= 1; \\
 \eta_{2,k} &= \sum_{i=0}^k \eta_{1,i} = k+1; \\
 \eta_{3,k} &= \sum_{i=0}^k \eta_{2,i} = \frac{(k+1)(k+2)}{2}, \\
 \eta_{j,k} &= \sum_{i=1}^k \eta_{j-1,i} \quad (j \geq 2), \\
 \delta_2(t)^{(-k-j)} &= \left(\frac{\alpha h_x}{L} \right)^{-k-j} e^{\alpha h_x(t+i)/L},
 \end{aligned}$$

$$\begin{aligned}
 \delta_3(t)^{(k)} &= ((t+i)^q)^{(k)} = (t+i)^{q-k} \prod_{i=0}^{k-1} (q-i) \\
 & \quad \quad \quad (k \geq 1), \\
 & \quad \quad \quad (k = 0, \dots, q; j = 1, \dots, 7).
 \end{aligned} \quad (A.12)$$

The matrices calculated from (A.3)–(A.12) are substituted into (34) and (37). The structural stiffness and mass matrices for axially FG tapered beams can be easily calculated.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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