Research Article

Investigation for Synchronization of a Rotor-Pendulum System considering the Multi-DOF Vibration

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This work is a continuation for our published literature for vibration synchronization. A new mechanism, two rotors coupled with a pendulum rod in a multi-DOF vibration system, is proposed to implement coupling synchronization, and the dynamics equation of mechanism is derived by Lagrange equation. In addition, the coupling relationship between the vibroboddy and the pendulum rod is ascertained with the Laplace transformation method, based on the dimensionless equation of the dynamics system. The Poincare method is employed to study the synchronization state between the two unbalanced rotors, which is converted into that of existence and the stability of solutions for synchronization-balance equations. The obtained results are supported by computer simulations. It is demonstrated that the values of the spring stiffness coefficient, the length of the pendulum, and the angular installation of the pendulum are important parameters with respect to the synchronous behavior in the rotor-pendulum system.

1. Introduction

The word “synchronization” is often encountered in both science and daily life. Our surroundings are full of synchronization phenomenon, which is considered as an adjustment of rhythms of oscillating objects due to their internal weak couplings [1]. For example, violinists play in unison, insects in a population emit acoustic or light pulses with a common rate, birds in a flock flap their wings simultaneously, and the heart of a rapidly galloping horse contracts once per locomotory cycle [2]. Synchronization phenomena in large populations of interacting elements are the subjects of intense research efforts in physical, biological, chemical, and social system; however, the most representatives are synchronization of complex systems [3–5], coupled with pendula or mechanical rotors in recent years. For the synchronization of pendula, in the particular case of Huygens’ clocks system, the remarkable feature reported by Huygens in 1665 is that pendulum clocks synchronize in antiphase. Nowadays the synchronized limit behavior of Huygens’ clocks, synphase and antiphase synchronization of the pendula, is studied considering the different values of spring stiffness [6, 7]. Meanwhile, the synchronization of derivatizations of Huygens’ clocks, including two coupled double pendula [8], pendulum coupled

by an elastic force [9], and pendula connected by linear springs [10], has been attracting many scholars’ attention. For the synchronization of rotors, Blekhman [1] proposed the Poincare method for the synchronization state and stability and by now this method is widely used in engineering. Based on Blekhman’s method, many scientists have been developing the other methods to analyze the synchronization of the rotors. Wen et al. [11] developed the average method to investigate synchronization and stability of multiple rotors in after-resonance. Zhang et al. [12, 13] described the average method of modified small parameters, which immensely simplify the process for solving the problems of synchronization of the rotors. Then, Fang et al. [14] employed the average method of modified small parameters to explore the synchronization of two homodromy rotors installed on a double vibroboddy in a coupling vibration system. Sperling et al. [15] presented analytical and numerical investigation of a two-plane automatic balancing device, for equilibration of rigid-rotor unbalance. Balthazar [16, 17] examined self-synchronization of four nonideal exciters in nonlinear vibration system via numerical simulations. Djanan et al. [18] explored the condition, for which three motors working on a same plate can enter into synchronization with the phase difference depending on the physical characteristics of the motors and the plate.

The abovementioned researches are mainly synchronization of the pendula or the rotors; however, the synchronization of the rotors coupled with pendula is less reported. Recently, we have proposed synchronization of two homodromy rotors coupled with a pendulum rod in an postresonant system [19], but the influence of the multi-DOF vibration of vibroboddy on the synchronous state of the rotors is less considered. This paper is a continuation of our published literature by means of the Poincare method, building on the original work of Blekhman. Here, we consider the model of two homodromy rotors coupled with a rigid pendulum rod through a torsion spring, and the vibroboddy is connected to a fixed support by means of springs. It is demonstrated that the values of the spring stiffness coefficient, the length of the pendulum, and the angular installation of the pendulum are important parameters with respect to the synchronous behavior in the rotor-pendulum system.

This paper is organized as follows. Section 2 describes the strategy and considered model. In Section 3, we employ the Laplace transform method to calculate the value of the coupling coefficients. In Section 4, we derive the synchronization equation and the synchronization criterion of the system. In Section 5, we compare and analyze the values of the stable phase difference with theoretical computations and the computer simulations. Finally, we summarize our results in Section 6.

2. Strategy and Model

2.1. Strategy. Consider the dynamic equation of a rotation system:

$$\begin{aligned} J_s \ddot{\varphi}_s + f_s \dot{\varphi}_s &= \mu \Phi_s(\varphi_s, \ddot{x}) \quad (s = 1, \dots, k), \\ \ddot{x} + 2\omega_x \xi_x \dot{x} + \omega_x^2 x &= \sum_{j=1}^k F_j(\omega t, \alpha_1, \dots, \alpha_k) \\ &+ \mu F_{k+1}(\omega t, \alpha_1, \dots, \alpha_k), \end{aligned} \quad (1)$$

where $\mu \Phi_s = M_{es} + f(\varphi_s, \ddot{x})$, μ is a small parameter, J_s is the rotational inertia of the s th induction motors, f_s is the mechanical damping of the motors, ξ_x is the damping ratio of the system in the x -direction, and ω_x is natural frequency of the system in x -direction. ω and φ_s are mechanical velocity and phase angular of the s th unbalanced rotor, respectively.

Based on (1), the following sequence of analysis for vibration system employing synchronizing rotors can be formulated:

- (1) Steady forced vibrations with $T = 2\pi/\omega$ are determined by

$$x = x(\omega t, \alpha_1, \dots, \alpha_k) \quad (2)$$

from the supporting body or supporting system of bodies (i.e., from the second formula of (1) considering $\mu = 0$) when rotors are uniformly rotating with initial phase $\alpha_1, \dots, \alpha_k$; that is,

$$\varphi_s = (\omega t + \alpha_s). \quad (3)$$

- (2) Equation (3) mentioned above may correspond only to such values of constants $\alpha_1, \dots, \alpha_k$, which satisfy

$$P_s(\alpha_1, \dots, \alpha_k) = \mu \langle \Phi_s(\varphi_s, \ddot{x}) \rangle = 0. \quad (4)$$

Here and below the angle brackets $\langle * \rangle$ show the average within T ; that is,

$$\langle * \rangle = \frac{1}{T} \int_0^T * dt, \quad (5)$$

where symbol $*$ represents a function related to time t [1].

- (3) If a certain set of constants $\alpha_1 = \alpha_1^*, \dots, \alpha_k = \alpha_k^*$, which satisfy (4), real parts of all roots χ of the k th order algebraic equation

$$\begin{vmatrix} \frac{\partial(P_1 - P_k)}{\partial \alpha_1} - \chi & \frac{\partial(P_1 - P_k)}{\partial \alpha_2} & \dots & \frac{\partial(P_1 - P_k)}{\partial \alpha_{k-1}} \\ \frac{\partial(P_2 - P_k)}{\partial \alpha_1} & \frac{\partial(P_2 - P_k)}{\partial \alpha_2} - \chi & \dots & \frac{\partial(P_2 - P_k)}{\partial \alpha_{k-1}} \\ \dots & \dots & \dots & \dots \\ \frac{\partial(P_{k-1} - P_k)}{\partial \alpha_1} & \frac{\partial(P_{k-1} - P_k)}{\partial \alpha_2} & \dots & \frac{\partial(P_{k-1} - P_k)}{\partial \alpha_{k-1}} - \chi \end{vmatrix} = 0, \quad (6)$$

are negative, then at sufficiently small μ this set of constants is indeed correlated with the unique, analytical relative to μ , asymptotically stable periodic solution of (1). This solution changes into the fundamental solution (see (4)) at $\mu = 0$. If the real part of at least one root of (6) is positive, then the corresponding solution is unstable. With purely imaginary zero roots, an additional analysis is required in general case [1].

2.2. Model. A simplified rotor-pendulum system depicted in Figure 1(a) is considered. This model consists of a rigid vibroboddy of mass m_0 [Kg], a rigid pendulum rod, and two unbalanced rotors. The rigid vibroboddy is elastically supported via linear springs with stiffness k_x [N/m], k_y [N/m], and k_ψ [N/rad] and linear viscous dampers with damping constants f_x [Ns/m], f_y [Ns/m], and f_ψ [Nm/(rad/s)], respectively. Unbalanced rotor i is modelled by a point mass m_i [Kg] (for $i = 2, 3$) attached at the end of a massless rod of length r [m]. One of the unbalanced rotors in the system is directly mounted on the vibroboddy, and the other is fixed at the end of a pendulum rod, which is connected with the vibro-platform by a linear torque spring with stiffness k_1 [N/rad] and a linear viscous damper with damping constant f_1 [Nm/(rad/s)]. Note that the two rotors are driven separately in the same direction by two identical induction motors. The rotation angle of rotor i is denoted by $\varphi_i \in S^1$ (for $i = 2, 3$) in [rad]; the oscillating angle of the pendulum rod is denoted by $\varphi_1 \in S^1$ in [rad]; the installation angle of the pendulum rod is expressed by $\beta \in S^1$

FIGURE 1: The model of the vibration system: (a) dynamic model of the double vibrobody system with two induction motors rotating in the same direction and (b) the reference frame system.

in [rad]; M_i represents the electromagnetic torque of the induction motors in [N/m]; R_i signifies resistance torque in bearings; and x , y , and ψ are the vibration displacement of the vibrobody in [m], [m], and [rad], respectively.

As illustrated in Figure 1(b), three reference coordinates of the system can be assigned as follows: the fixed coordinate oxy ; the nonrotating moving coordinate $o'x'y'$, which undergoes the translation motion while remaining parallel to coordinate oxy , respectively; the rotating coordinate $o'x''y''$ which dedicates the rotation motion around points o' . The three reference coordinates of the vibrobody separately coincide with each other when the system is in the static equilibrium state.

In the reference frame $o'x''y''$, the center coordinate Φ_i'' ($i = 1, 2, 3$) of the rotors and pendula can be expressed as

$$\begin{aligned}\Phi_1'' &= \begin{pmatrix} l \cos(\beta + \varphi_1) \\ l \sin(\beta + \varphi_1) \end{pmatrix}, \\ \Phi_2'' &= \begin{pmatrix} l \cos(\beta + \varphi_1) + r \cos \varphi_2 \\ l \sin(\beta + \varphi_1) + r \sin \varphi_2 \end{pmatrix}, \\ \Phi_3'' &= \begin{pmatrix} -l \cos \beta + r \cos \varphi_3 \\ l \sin \beta + r \sin \varphi_3 \end{pmatrix}.\end{aligned}\quad (7)$$

In the reference frame oxy , the center coordinate of the rotors and pendula, Φ_i ($i = 1, 2, 3$), can be expressed as

$$\begin{aligned}\Phi_i &= \Phi_0 + \Gamma \Phi_i'', \\ \Gamma &= \begin{pmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{pmatrix},\end{aligned}\quad (8)$$

where Φ_0 is displacement vector of the mass center of the rigid vibrobody, $\Phi_0 = [x, y]^T$.

The kinetic energy of the system can be obtained by

$$\begin{aligned}T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J_m\dot{\psi}^2 + \frac{1}{2}\sum_{i=1}^3 J_{0i}\dot{\varphi}_i^2 \\ &+ \frac{1}{2}\sum_{i=1}^3 m_i \dot{\Phi}_i^T \dot{\Phi}_i.\end{aligned}\quad (9)$$

The potential energy of the system is approximated to

$$V = \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2 + \frac{1}{2}k_\psi \psi^2 + \frac{1}{2}k_1 \varphi_1^2. \quad (10)$$

Moreover, the viscous dissipation function D of the system is expressed by

$$\begin{aligned}D &= \frac{1}{2}f_x \dot{x}^2 + \frac{1}{2}f_y \dot{y}^2 + \frac{1}{2}f_\psi \dot{\psi}^2 + \frac{1}{2}f_1 \dot{\varphi}_1^2 + \frac{1}{2}f_2 \dot{\varphi}_2^2 \\ &+ \frac{1}{2}f_3 \dot{\varphi}_3^2.\end{aligned}\quad (11)$$

The dynamics equation is obtained by using Lagrange's equation:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial (T - V)}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i. \quad (12)$$

If $\mathbf{q} = [x, y, \psi, \varphi_1, \varphi_2, \varphi_3]^T$ is chosen as the generalized coordinates, the generalized force are $Q_x = Q_y = Q_\psi = 0$, $Q_{\varphi_1} = M_{e1} - R_{e1}$, $Q_{\varphi_2} = T_{e2} - R_{e2}$, and $Q_{\varphi_3} = 0$, respectively. As $m_i \ll m$ and $\psi, \varphi_3 \ll 1$ in the system, the inertia coupling stemming from asymmetry of the two rotors can be neglected. Substituting (9), (10), and (11) into (12), we can

yield the dynamic equation of the vibration system as the following form:

$$\begin{aligned}
M\ddot{x} + f_x\dot{x} + k_x x &= (m_1 + m_2)l [\dot{\varphi}_1^2 \cos \beta + \ddot{\varphi}_1 \sin \beta_1] \\
&+ \sum_{i=2}^3 m_i r [\ddot{\varphi}_i \sin \varphi_i + \dot{\varphi}_i^2 \cos \varphi_i], \\
M\ddot{y} + f_y\dot{y} + k_y y &= (m_1 + m_2)l [\dot{\varphi}_1^2 \sin \beta - \ddot{\varphi}_1 \cos \beta] \\
&+ \sum_{i=2}^3 m_i r [\dot{\varphi}_i^2 \sin \varphi_i - \ddot{\varphi}_i \cos \varphi_i], \\
J_{m0}\ddot{\psi} + f_\psi\dot{\psi} + k_\psi \psi &= \sum_{i=2}^3 (-1)^i \\
&\cdot m_i l r [\dot{\varphi}_i^2 \sin (\varphi_i - (-1)^i \beta) \\
&- \ddot{\varphi}_i \cos (\varphi_i - (-1)^i \beta)], \\
(m_1 l^2 + m_2 l^2) \ddot{\varphi}_1 + f_1 \dot{\varphi}_1 + k_1 \varphi_1 &= (m_1 + m_2)l [\ddot{x} \sin \beta \\
&- \ddot{y} \cos \beta] - m_2 l r [\ddot{\varphi}_2 \cos (\varphi_2 - \beta) - \dot{\varphi}_2^2 \\
&\cdot \sin (\varphi_2 - \beta)], \\
J_{20}\ddot{\varphi}_2 &= M_{e1} - R_{e1} - m_2 r [-\ddot{x} \sin \varphi_2 + \ddot{y} \cos \varphi_2] \\
&- m_2 l r [(\ddot{\varphi}_1 + \ddot{\psi}) \cos (\varphi_2 - \beta) + (\dot{\varphi}_1 + \dot{\psi})^2 \\
&\cdot \sin (\varphi_2 - \beta)], \\
J_{30}\ddot{\varphi}_3 &= M_{e2} - R_{e2} - m_3 r [-\ddot{x} \sin \varphi_3 + \ddot{y} \cos \varphi_3] \\
&+ m_3 l r [\ddot{\psi} \cos (\varphi_3 + \beta) + \dot{\psi}^2 \sin (\varphi_3 + \beta)],
\end{aligned} \tag{13}$$

where $\sum_{i=0}^3 m_i = M$, $J_{m0} \approx M l_e^2 + m_1 l^2$, $J_{20} \approx m_2 r^2 + m_3 r^2$, and $J_{30} \approx m_3 r_3^2$.

Assuming that the natural frequency of the system is unequal to or far away from the excitation frequency, thus the oscillating angle φ_1 of pendulum rod can be considered small and periodic:

$$\varphi_1 = A \sin (\omega t + \gamma), \tag{14}$$

where A is amplitude of the oscillating angle of pendulum rod (here, $A \ll 1$) and γ is the phase angular of the pendulum rod.

Having calculated the derivative of the function $\varphi_1(t)$ with respect to t , we will write the relationship under consideration as

$$\frac{\dot{\varphi}_1^2}{\varphi_1} = \frac{A^2 \omega^2 \sin^2 (\omega t + \gamma)}{A \omega^2 \sin (\omega t + \gamma)} = A \sin (\omega t + \gamma) \sim \mu. \tag{15}$$

Hence, it can be seen that the items related to symbol $\dot{\varphi}_1^2$ in (13) are nonlinear; however, the nonlinear characteristics and dampers acted on the system are weaker when the values of the parameters in the system satisfy the assumptions above. Meanwhile, considering the solution of the problem by the

Poincare method (i.e., based on the fundamental equation (1)), we will introduce the small parameter μ into (13), thus presenting it in the following form:

$$\begin{aligned}
M\ddot{x} + k_x x &= (m_1 + m_2)l \ddot{\varphi}_1 \sin \beta_1 + \sum_{i=2}^3 m_i r [\ddot{\varphi}_i \sin \varphi_i \\
&+ \dot{\varphi}_i^2 \cos \varphi_i] - \mu [f'_x \dot{x} - (m_1 + m_2)l \dot{\varphi}_1^2 \cos \beta], \\
M\ddot{y} + k_y y &= -(m_1 + m_2)l \ddot{\varphi}_1 \cos \beta \\
&+ \sum_{i=2}^3 m_i r [\dot{\varphi}_i^2 \sin \varphi_i - \ddot{\varphi}_i \cos \varphi_i] - \mu [f'_y \dot{y} - (m_1 \\
&+ m_2)l \dot{\varphi}_1^2 \sin \beta], \\
J_{m0}\ddot{\psi} + k_\psi \psi &= -m_1 l^2 \ddot{\varphi}_1 + \sum_{i=2}^3 (-1)^i \\
&\cdot m_i l r [\dot{\varphi}_i^2 \sin (\varphi_i - (-1)^i \beta) \\
&- \ddot{\varphi}_i \cos (\varphi_i - (-1)^i \beta)] - \mu f'_\psi \dot{\psi}, \\
(m_1 l^2 + m_2 l^2) \ddot{\varphi}_1 + k_1 \varphi_1 &= (m_1 + m_2)l [\ddot{x} \sin \beta - \ddot{y} \\
&\cdot \cos \beta] - m_2 l r [\ddot{\varphi}_2 \cos (\varphi_2 - \beta) - \dot{\varphi}_2^2 \sin (\varphi_2 - \beta)] \\
&- \mu f'_1 \dot{\varphi}_1, \\
J_{20}\ddot{\varphi}_2 &= \mu \Phi_1 (\varphi_1, \ddot{x}) \\
J_{30}\ddot{\varphi}_3 &= \mu \Phi_1 (\varphi_1, \ddot{x}),
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
\mu \Phi_1 &= M_{e1} - R_{e1} - m_2 r [-\ddot{x} \sin \varphi_2 + \ddot{y} \cos \varphi_2] \\
&+ m_2 l r [(\ddot{\varphi}_1 + \ddot{\psi}) \cos (\varphi_2 - \beta) \\
&+ (\dot{\varphi}_1 + \dot{\psi})^2 \sin (\varphi_2 - \beta)], \\
\mu \Phi_2 &= M_{e2} - R_{e2} - m_3 r [-\ddot{x} \sin \varphi_3 + \ddot{y} \cos \varphi_3] \\
&- m_3 l r [\ddot{\psi} \cos (\varphi_3 + \beta) + \dot{\psi}^2 \sin (\varphi_3 + \beta)].
\end{aligned} \tag{17}$$

According to [12], when the two rotors synchronously rotate, the electromagnetic torque of the inductions M_{ei} can be linearized at the vicinity of $\dot{\varphi}_i = \omega_s$ as

$$M_{ei} = n_p \frac{L_{mi}^2 U_{s0}^2}{L_{si}^2 \omega_m R_{ri}} (\omega_s - n_p \omega_m), \tag{18}$$

where L_{mi} is the mutual inductance of the i th induction motor, L_{si} is stator inductance of the i th induction motor, n_p is the number of pole pairs of the induction motor, ω_m is synchronous electric angular velocity, R_{ri} is the rotor resistance of the i th induction motor, and U_{s0} is the amplitude of the stator voltage vector.

3. Steady Response

The first four formulas in (16) are coupling dynamics equations related to DOFs x , y , ψ , and φ_1 . When the system operates in the steady state, the acceleration of the three motors changes very little (close to zero), so $\ddot{\varphi}_2$ and $\ddot{\varphi}_3$ can be neglected in these formulas. In addition, taking no account of the items related to small parameter μ and introducing the following dimensionless parameters in the mentioned formulas,

$$\begin{aligned}
 \omega_x &= \sqrt{\frac{k_x}{M}}, \\
 \omega_y &= \sqrt{\frac{k_y}{M}}, \\
 \omega_\psi &= \sqrt{\frac{k_\psi}{(Ml_e^2 + m_1 l^2)}}, \\
 \omega_1 &= \sqrt{\frac{k_1}{(m_1 l^2 + m_2 l^2)}}, \\
 r_0 &= \frac{l}{l_e}, \\
 r_l &= \frac{r}{l}, \\
 r_m &= \frac{m_3}{M}, \\
 \eta_{13} &= \frac{m_1}{m_3}, \\
 \eta_{23} &= \frac{m_2}{m_3},
 \end{aligned} \tag{19}$$

$$\begin{aligned}
 f_1(t) &= (\eta_{23} r_m r \cos \varphi_2 + r_m r \cos \varphi_3), \\
 f_2(t) &= (\eta_{23} r_m r \sin \varphi_2 + r_m r \sin \varphi_3), \\
 f_3(t) &= \frac{\eta_{23} r_m r r_o}{(1 + \eta_{13} r_m r_o^2) l_e} \sin(\varphi_2 - \beta) \\
 &\quad - \frac{r_m r r_o}{(1 + \eta_{13} r_m r_o^2) l_e} \sin(\varphi_3 + \beta), \\
 f_4(t) &= \frac{\eta_{23} r_l}{(\eta_{13} + \eta_{23})} \sin(\varphi_2 - \beta).
 \end{aligned}$$

We obtain the dimensionless dynamics equations:

$$\begin{aligned}
 \ddot{x} + \omega_x^2 x &= (\eta_{13} + \eta_{23}) r_m l \ddot{\varphi}_1 \sin \beta \\
 &\quad + \omega^2 f_1(t), \\
 \ddot{y} + \omega_y^2 y &= -(\eta_{13} + \eta_{23}) r_m l \ddot{\varphi}_1 \cos \beta \\
 &\quad + \omega^2 f_2(t),
 \end{aligned}$$

$$\begin{aligned}
 \ddot{\psi} + \omega_\psi^2 \psi &= -\frac{\eta_{13} r_m r_o^2}{(1 + \eta_{13} r_m r_o^2)} \ddot{\varphi}_1 \\
 &\quad + \omega^2 f_3(t),
 \end{aligned}$$

$$\ddot{\varphi}_1 + 2\omega_1 \xi_1 \dot{\varphi}_1 + \omega_1^2 \varphi_1 = \frac{\ddot{x} \sin \beta - \ddot{y} \cos \beta}{l} + \omega^2 f_4(t). \tag{20}$$

It can be seen that (20) are coupling dynamic equations related to DOFs x , y , ψ , and φ_1 . Applying the Laplace transform to (20), one gets

$$\begin{aligned}
 s^2 X(s) + \omega_x^2 X(s) &= (\eta_{13} + \eta_{23}) r_m l s^2 \sin \beta \Phi(s) \\
 &\quad + \omega^2 F_1(s), \\
 s^2 Y(s) + \omega_y^2 Y(s) &= -(\eta_{13} + \eta_{23}) r_m l s^2 \cos \beta \Phi(s) \\
 &\quad + \omega^2 F_2(s), \\
 s^2 \Psi(s) + \omega_\psi^2 \Psi(s) &= -\frac{\eta_{13} r_m r_o^2}{(1 + \eta_{13} r_m r_o^2)} s^2 \Phi(s) \\
 &\quad + \omega^2 F_3(s), \\
 s^2 \Phi(s) + \omega_1^2 \Phi(s) &= \frac{s^2 \sin \beta X(s) - s^2 \cos \beta Y(s)}{l} \\
 &\quad + \omega^2 F_4(s).
 \end{aligned} \tag{21}$$

Considering that the natural frequencies of the springs connected to the foundation are approximately identical (i.e., $\omega_x \approx \omega_y \approx \omega_\psi$), thus, $X(s)$, $Y(s)$, $\Psi(s)$, and $\Phi(s)$ can be expressed:

$$\begin{aligned}
 X(s) &= \left[\frac{(\eta_{13} + \eta_{23}) r_m l \omega^2 s^4 \sin^2 \beta}{G_1(s)} + \frac{\omega^2}{(s^2 + \omega_x^2)} \right] F_1(s) \\
 &\quad - \frac{(\eta_{13} + \eta_{23}) r_m l s^4 \omega^2 \sin \beta \cos \beta}{G_1(s)} F_2(s) \\
 &\quad + \frac{(\eta_{13} + \eta_{23}) r_m l s^2 \omega^2 \sin \beta}{G_2(s)} F_4(s),
 \end{aligned}$$

$$\begin{aligned}
 Y(s) &= \left[\frac{-(\eta_{13} + \eta_{23}) r_m l s^4 \omega^2 \cos \beta \sin \beta}{G_1(s)} \right] F_1(s) \\
 &\quad + \left[\frac{(\eta_{13} + \eta_{23}) r_m l s^4 \omega^2 \cos^2 \beta}{G_1(s)} + \frac{\omega^2}{(s^2 + \omega_y^2)} \right] F_2(s) \\
 &\quad - \frac{(\eta_{13} + \eta_{23}) r_m l s^2 \omega^2 \cos \beta}{G_2(s)} F_4(s),
 \end{aligned}$$

$$\begin{aligned}
\Psi(s) &= -\frac{\eta_{13}r_m r_o^2 \omega^2 s^4 \sin \beta}{G_3(s)} F_1(s) \\
&\quad + \frac{\eta_{13}r_m r_o^2 \omega^2 s^4 \cos \beta}{G_3(s)} F_2(s) + \frac{\omega^2}{(s^2 + \omega_\psi^2)} F_3(s) \\
&\quad - \frac{\eta_{13}r_m r_o^2 (\omega^2 s^4 + \omega_x^2 \omega^2 s^2)}{G_3(s)} F_4(s), \\
\Phi(s) &= \frac{s^2 \omega^2 \sin \beta F_1(s)}{G_2(s)} - \frac{s^2 \omega^2 \cos \beta F_2(s)}{G_2(s)} \\
&\quad + \frac{(s^2 + \omega_x^2) \omega^2 F_4(s)}{G_2(s)},
\end{aligned} \tag{22}$$

where $G_1(s) = (s^2 + \omega_1^2)(s^2 + \omega_x^2)^2 - (\eta_{13} + \eta_{23})r_m(s^2 + \omega_x^2)s^4$, $G_2(s) = (s^2 + \omega_1^2)(s^2 + \omega_x^2) - (\eta_{13} + \eta_{23})r_m s^4$, and $G_3(s) = (1 + \eta_{13}r_m r_o^2)(s^2 + \omega_x^2)^2(s^2 + \omega_1^2) - (1 + \eta_{13}r_m r_o^2)(s^6 + \omega_x^2 s^4)(\eta_{13} + \eta_{23})r_m$.

Applying the inverse Laplace transformation to (22), introduce frequency ratio n_x , n_y , n_ψ , and n_1 into the formulas:

$$\begin{aligned}
n_x &= \frac{\omega}{\omega_x}, \\
n_y &= \frac{\omega}{\omega_y}, \\
n_\psi &= \frac{\omega}{\omega_\psi}, \\
n_1 &= \frac{\omega}{\omega_1}.
\end{aligned} \tag{23}$$

In this case, the spring stiffness values k_x , k_y , k_ψ , and k_1 are converted into the function of frequency ratios n_x , n_y , n_ψ , and n_1 through the natural frequency, respectively. In the paper, we assume that $\omega_x \approx \omega_y \approx \omega_\psi$, and so it follows that $n_x \approx n_y \approx n_\psi$. From (22) and (23) one gets

$$\begin{bmatrix} x \\ y \\ \psi \\ \varphi_1 \end{bmatrix} = \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} \\ \mu_{21} & \mu_{22} & \mu_{23} & \mu_{24} \\ \mu_{31} & \mu_{32} & \mu_{33} & \mu_{34} \\ \mu_{41} & \mu_{42} & \mu_{43} & \mu_{44} \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{bmatrix}, \tag{24}$$

where

$$\begin{aligned}
\mu_{11} &= \frac{(\eta_{13} + \eta_{23})r_m n_1^4 \sin^2 \beta}{-(n_x^2 - 1)^2 (n_1^2 - 1) + n_x^2 n_1^2 (n_x^2 - 1) (\eta_{13} + \eta_{23}) r_m} \\
&\quad + \frac{n_x^2}{(1 - n_x^2)},
\end{aligned}$$

$$\begin{aligned}
\mu_{12} &= \frac{-(\eta_{13} + \eta_{23})r_m n_1^4 \cos \beta \sin \beta}{-(n_x^2 - 1)^2 (n_1^2 - 1) + n_x^2 n_1^2 (n_x^2 - 1) (\eta_{13} + \eta_{23}) r_m} \\
\mu_{13} &= 0, \\
\mu_{14} &= \frac{(\eta_{13} + \eta_{23})r_m (-n_x^4 n_1^2 + n_1^2 n_x^2) \sin \beta}{-(n_x^2 - 1)^2 (n_1^2 - 1) + n_x^2 n_1^2 (n_x^2 - 1) (\eta_{13} + \eta_{23}) r_m},
\end{aligned}$$

$$\begin{aligned}
\mu_{21} &= \frac{-(\eta_{13} + \eta_{23})r_m n_1^4 \sin \beta \cos \beta}{-(n_x^2 - 1)^2 (n_1^2 - 1) + n_x^2 n_1^2 (n_x^2 - 1) (\eta_{13} + \eta_{23}) r_m}, \\
\mu_{22} &= \frac{(\eta_{13} + \eta_{23})r_m n_1^4 \cos^2 \beta}{-(n_x^2 - 1)^2 (n_1^2 - 1) + n_x^2 n_1^2 (n_x^2 - 1) (\eta_{13} + \eta_{23}) r_m} \\
&\quad + \frac{n_y^2}{1 - n_y^2}, \\
\mu_{23} &= 0, \\
\mu_{24} &= \frac{(\eta_{13} + \eta_{23})r_m (n_1^2 n_x^4 - n_1^2 n_x^2) \cos \beta}{-(n_x^2 - 1)^2 (n_1^2 - 1) + n_x^2 n_1^2 (n_x^2 - 1) (\eta_{13} + \eta_{23}) r_m},
\end{aligned}$$

$$\begin{aligned}
\mu_{31} &= \frac{-\eta_{13}r_m r_o^2 n_1^4 \sin \beta}{-(n_x^2 - 1)^2 (n_1^2 - 1) + n_x^2 n_1^2 (n_x^2 - 1) (\eta_{13} + \eta_{23}) r_m}, \\
\mu_{32} &= \frac{\eta_{13}r_m r_o^2 n_1^4 \cos \beta}{-(n_x^2 - 1)^2 (n_1^2 - 1) + n_x^2 n_1^2 (n_x^2 - 1) (\eta_{13} + \eta_{23}) r_m}, \\
\mu_{33} &= \frac{n_\psi^2}{l_0 (-n_\psi^2 + 1)}, \\
\mu_{34} &= \frac{\eta_{13}r_m r_o^2 (n_1^2 n_\psi^4 - n_1^2 n_\psi^2)}{-(n_x^2 - 1)^2 (n_1^2 - 1) + n_x^2 n_1^2 (n_x^2 - 1) (\eta_{13} + \eta_{23}) r_m},
\end{aligned}$$

$$\begin{aligned}
\mu_{41} &= \frac{-n_1^2 n_x^2 \sin \beta}{(n_x^2 - 1) (n_1^2 - 1) - n_x^2 n_1^2 (\eta_{13} + \eta_{23}) r_m}, \\
\mu_{42} &= \frac{n_1^2 n_x^2 \cos \beta}{(n_x^2 - 1) (n_1^2 - 1) - n_x^2 n_1^2 (\eta_{13} + \eta_{23}) r_m},
\end{aligned}$$

TABLE 1: Parameter values.

(a) Parameter values for system equation (13)

Unbalanced rotor	Vibroboddy	Pendulum rod	Induction motor
$r = 0.05$ [m]	$l_e = 0.3$ [m]	$l = 0.3 \sim 0.9$ [m]	$m_1 = 15$ [kg]
$m_2 = 0.6 \sim 3$ [kg]	$m_0 = 100$ [kg]	$k_1 = 4980 \sim 24402510$ [N/rad]	$L_{mi} = 0.13$ [H]
$m_3 = 3$ [kg]	$k_x = 246490000 \sim 50307$ [N/m]	$f_1 = 15$ [Nm/(rad/s)]	$L_{si} = 0.1$ [H]
$\omega_m = 152 \sim 157$ [rad/s]	$f_x = 1064$ [Ns/m]	$\beta = 0, \pi/6, \pi/4, \pi/3$ [rad]	$n_p = 2$
—	$k_y = 246490000 \sim 50307$ [N/m]	—	$R_{ri} = 0.54$ [Ω]
—	$f_y = 1064$ [Ns/m]	—	$U_{s0} = 220$ [V]
—	$k_\psi = 50307 \sim 246490000$ [N/m]	—	—
—	$f_\psi = 1064$ [Nm/(rad/s)]	—	—

(b) Parameter values according to dimensionless equation (19)

$\eta_{13} = 5$
$\eta_{23} = 0.3 \sim 1$
$r_m = 0.025$
$n_x \approx n_y \approx n_\psi = 0.2 \sim 7.0$
$n_1 = 0.2 \sim 7$
$r_o = 1 \sim 3$
$r_l = 0.08, 0.17$

$$\mu_{43} = 0,$$

$$\mu_{44} = \frac{(n_1^2 n_x^2 - n_1^2)}{(n_x^2 - 1)(n_1^2 - 1) - n_x^2 n_1^2 (\eta_{13} + \eta_{23}) r_m}. \quad (25)$$

Parameter μ_{ij} ($i = 1, 2, 3, 4$, $j = 1, 2, 3, 4$) in (25) represents the mutual coupling coefficients among the rotors, the pendulum rod, and the vibroboddy through the springs. The larger the value of the coupling coefficient is, the stronger the coupling ability of the system is. In order to understand the coupling characteristics of the system, the following numerical computations have been performed with assumption that parameters η_{13} , η_{23} , l , r_o , and r_m are constants. In this case, we can confirm the value of the coupling coefficients when changing the value of parameters n_x , n_y , n_ψ , n_1 , and β within definite ranges, respectively. According to Table 1, we can determine the value of the coupling coefficients.

From Figure 2 it follows that the value of the coupling coefficients depends on the values of parameters n_x , n_y , n_ψ , n_1 , and β . In this figure, it can be seen that the peak value of the coupling coefficients is related to the frequency ratio of the system. However, frequency ratios n_x , n_y , n_ψ , and n_1 are the function of the spring stiffness values k_x , k_y , k_ψ , and k_1 , respectively. So the values of the coupling coefficients are determined by the spring stiffness. Clearly, for $0.1 < n_x, n_y, n_\psi < 0.9$ and $n_1 = 0.2$ or 0.8 , the vibration frequency of the system is less than the eigen-frequency of the system, which is denoted as before-resonance system; in this case, the absolute values of μ_{ij} are smaller except μ_{33} and μ_{44} when $n_1 = 0.8$. For $0.9 \leq n_x, n_y, n_\psi < 1.5$ and $n_1 = 1.5$, the vibration frequency of the system is approximately equal to the eigen-frequency of the system, which is denoted by resonance system; in this case, the absolute value of parameter

μ_{ij} is far larger when β is nonzero. For $n_x, n_y, n_\psi \geq 1.5$ and $n_1 = 3.0$, the vibration frequency of the system is larger than the eigen-frequency of the system, which is denoted by after-resonance system; in this case, the value of parameter μ_{ij} is no more than 2.0. Then, we can define type of the coupling system generally used in engineering according to frequency ratio, as shown in Table 2.

4. Synchronization and Stability

In this section, we analyze the synchronization and stability of the system with theoretical method. As was already mentioned in (3), the family of synchronous solutions is assumed as

$$\begin{aligned} \varphi_2 &= \omega t + \alpha_1, \\ \varphi_3 &= \omega t + \alpha_2. \end{aligned} \quad (26)$$

So in synchronous state, (24) can be rewritten as

$$\begin{aligned} x(t) &= \mu_{11} [\eta_{23} r_m r \cos(\omega t + \alpha_1) + r_m r \cos(\omega t + \alpha_2)] \\ &\quad + \mu_{12} [\eta_{23} r_m r \sin(\omega t + \alpha_1) + r_m r \sin(\omega t + \alpha_2)] \\ &\quad + \mu_{14} \frac{\eta_{23} r_l}{(\eta_{13} + \eta_{23})} \sin(\omega t + \alpha_1 - \beta), \\ y(t) &= \mu_{21} [\eta_{23} r_m r \cos(\omega t + \alpha_1) + r_m r \cos(\omega t + \alpha_2)] \\ &\quad + \mu_{22} [\eta_{23} r_m r \sin(\omega t + \alpha_1) + r_m r \sin(\omega t + \alpha_2)] \\ &\quad + \mu_{24} \frac{\eta_{23} r_l}{(\eta_{13} + \eta_{23})} \sin(\omega t + \alpha_1 - \beta), \end{aligned}$$

FIGURE 2: Continued.

FIGURE 2: The value of the coupling coefficients of the system for $\eta_{13} = 5$, $\eta_{23} = 1$, $r_m = 0.025$, $l = 0.6$, and $r_o = 2$ considering the difference of angular β .

TABLE 2: Type of the coupling system.

Abbreviation	SBCWB	SACWB	SBCWA	SACWA
Full name	System of before-resonance coupled with before-resonance	System of after-resonance coupled with before-resonance	System of before-resonance coupled with after-resonance	System of after-resonance coupled with after-resonance
Frequency ratio	$0.2 \leq n_x, n_y, n_\psi \leq 0.8$ and $0.2 \leq n_1 \leq 0.8$	$n_x, n_y, n_\psi \geq 1.5$ and $0.2 \leq n_1 \leq 0.8$	$0.2 \leq n_x, n_y, n_\psi \leq 0.8$ and $n_1 \geq 1.5$	$n_x, n_y, n_\psi \geq 1.5$ and $n_1 \geq 1.5$

$$\begin{aligned}
\psi(t) = & \mu_{31} [\eta_{23} r_m r \cos(\omega t + \alpha_1) + r_m r \cos(\omega t + \alpha_2)] \\
& + \mu_{32} [\eta_{23} r_m r \sin(\omega t + \alpha_1) + r_m r \sin(\omega t + \alpha_2)] \\
& + \mu_{33} \left[\frac{\eta_{23} r_m r r_o}{(1 + \eta_{13} r_m r_o^2) l_e} \sin(\omega t + \alpha_1 - \beta) \right. \\
& \left. - \frac{r_m r r_o}{(1 + \eta_{13} r_m r_o^2) l_e} \sin(\omega t + \alpha_2 + \beta) \right] + \mu_{34} \\
& + \left(\frac{\eta_{23} \mu_{14}}{(\eta_{13} + \eta_{23}) l} + \eta_{23} l r_m \mu_{31} \right) \cos(\alpha - \beta) \\
& + \left(\frac{\eta_{23} r_m r_o^2 \mu_{33}}{1 + \eta_{13} r_m r_o^2} + \frac{\eta_{23} \mu_{34}}{\eta_{13} + \eta_{23}} \right) \sin(\alpha - 2\beta) \\
& + \mu_{31} l r_m \cos \beta - \mu_{32} l r_m \sin \beta \Big] = 0.
\end{aligned} \tag{29}$$

$$\cdot \frac{\eta_{23} r_l}{(\eta_{13} + \eta_{23})} \sin(\omega t + \alpha_1 - \beta),$$

$$\begin{aligned}
\varphi_1(t) = & \mu_{41} [\eta_{23} r_m r \cos(\omega t + \alpha_1) \\
& + r_m r \cos(\omega t + \alpha_2)] + \mu_{42} [\eta_{23} r_m r \sin(\omega t + \alpha_1) \\
& + r_m r \sin(\omega t + \alpha_2)] + \mu_{44} \frac{\eta_{23} r_l}{(\eta_{13} + \eta_{23})} \\
& \cdot \sin(\omega t + \alpha_1 - \beta).
\end{aligned} \tag{27}$$

Specifying parameter α as the phase difference between the two rotors, we have

$$\alpha = \alpha_1 - \alpha_2. \tag{28}$$

Consequently, the basic equation (4) is expressed as

$$\begin{aligned}
P_1 = \langle \mu \Phi_1 \rangle = & M_{e1} - R_{e1} - \frac{1}{2} \\
& \cdot m_3 r^2 \omega^2 \left[\eta_{23} r_m (\mu_{11} + \mu_{22}) \sin \alpha \right. \\
& + \eta_{23} l r_m (\mu_{32} + \mu_{42}) \sin(\alpha - \beta) \\
& - \eta_{23} l r_m (\mu_{41} + \mu_{31}) \cos(\alpha - \beta) \\
& \left. - \frac{\eta_{23} \mu_{33} r_o^2 r_m}{1 + \eta_{13} r_m r_o^2} \sin(\alpha - 2\beta) \right] = 0, \\
P_2 = \langle \mu \Phi_2 \rangle = & M_{e2} - R_{e2} - \frac{1}{2} \\
& \cdot m_3 r^2 \omega^2 \left[-\eta_{23} r_m (\mu_{11} + \mu_{22}) \sin \alpha \right. \\
& \left. - \left(\frac{\eta_{23} \mu_{24}}{(\eta_{13} + \eta_{23}) l} - \eta_{23} l r_m \mu_{32} \right) \sin(\alpha - \beta) \right]
\end{aligned}$$

During the synchronous state, consider the excessive torque $Z_s(\omega)$ of the rotors to be zero [1]:

$$Z_i(\omega) = M_{ei} - R_{ei} = 0, \quad (i = 1, 2). \tag{30}$$

Therefore, according to (29) and (30), further calculations lead to the following form:

$$\begin{aligned}
P_1 - P_2 = & -(\mu_{11} + \mu_{22}) \eta_{23} r_m \sin \alpha \\
& - \frac{1}{2} \left(\eta_{23} l r_m \mu_{42} + \frac{\eta_{23} \mu_{24}}{(\eta_{13} + \eta_{23}) l} \right) \sin(\alpha - \beta) \\
& + \left[\frac{1}{2} \mu_{41} \eta_{23} l r_m + \frac{1}{2} \frac{\eta_{23} \mu_{14}}{(\eta_{13} + \eta_{23}) l} + \mu_{31} \eta_{23} l r_m \right] \\
& \cdot \cos(\alpha - \beta) + \left(\frac{\eta_{23} \mu_{33} r_o^2 r_m}{1 + \eta_{13} r_m r_o^2} + \frac{1}{2} \frac{\eta_{23} \mu_{34}}{\eta_{13} + \eta_{23}} \right) \\
& \cdot \sin(\alpha - 2\beta) + \frac{1}{2} \mu_{31} l r_m \cos \beta - \frac{1}{2} \mu_{32} l r_m \sin \beta \\
& = 0.
\end{aligned} \tag{31}$$

We can define (31) as the balance equation of synchronization state of the system, on which we can determine phase difference α with numerical computations. It should be noted that the value of the phase difference is related to parameters η_{13} , η_{23} , l , β , r_o , r_l , r_m , and μ_{ij} . However, parameter μ_{ij} is the function mainly related to frequency ratios n_x , n_y , n_ψ , n_1 , l , and β , which indicate that the spring stiffness and the structure parameters of the system are the key parameters to determine the value of phase difference α .

Now, let us consider the stability of the synchronous rotation for the rotors. Considering (31), the following criterion of synchronous stability is obtained from (6):

$$\begin{aligned}
\chi &= \frac{\partial (P_1 - P_2)}{\partial \alpha_1} = \frac{\partial (P_1 - P_2)}{\partial \alpha} \\
&= -m_3 r^2 \omega^2 \left[\eta_{23} r_m (\mu_{11} + \mu_{22}) \cos \alpha \right. \\
&\quad + \frac{1}{2} \left(\eta_{23} l r_m \mu_{42} + \frac{\eta_{23} \mu_{24}}{(\eta_{13} + \eta_{23}) l} \right) \cos (\alpha - \beta) \\
&\quad + \left(\frac{1}{2} \eta_{23} l r_m \mu_{41} + \frac{1}{2} \frac{\eta_{23} \mu_{14}}{(\eta_{13} + \eta_{23}) l} + \mu_{31} \eta_{23} l r_m \right) \\
&\quad \cdot \sin (\alpha - \beta) - \left(\frac{\eta_{23} \mu_{33} r_o^2 r_m}{1 + \eta_{13} r_m r_o^2} + \frac{1}{2} \frac{\eta_{23} \mu_{34}}{\eta_{13} + \eta_{23}} \right) \\
&\quad \cdot \cos (\alpha - 2\beta) \Big] < 0.
\end{aligned} \tag{32}$$

Rearranging (32), the stability criterion of synchronization of the system can be simplified as

$$\begin{aligned}
&(\mu_{11} + \mu_{22}) \eta_{23} r_m \cos \alpha \\
&+ \frac{1}{2} \left(\eta_{23} l r_m \mu_{42} + \frac{\eta_{23} \mu_{24}}{(\eta_{13} + \eta_{23}) l} \right) \cos (\alpha - \beta) \\
&+ \left(\frac{1}{2} \mu_{41} \eta_{23} l r_m + \frac{1}{2} \frac{\eta_{23} \mu_{14}}{(\eta_{13} + \eta_{23}) l} + \mu_{31} \eta_{23} l r_m \right) \\
&\cdot \sin (\alpha - \beta) - \left(\frac{\eta_{23} \mu_{33} r_o^2 r_m}{1 + \eta_{13} r_m r_o^2} + \frac{1}{2} \frac{\eta_{23} \mu_{34}}{\eta_{13} + \eta_{23}} \right) \\
&\cdot \cos (\alpha - 2\beta) > 0.
\end{aligned} \tag{33}$$

Only should the values of the system parameters satisfy the balance equation and the stability criterion of synchronization of the system, the synchronous state of the rotors can be determined. In this case, the phase difference between the rotors is called stable phase difference.

5. Numerical Discussions

The abovementioned sections have given some theoretical discussions in the simplified form on synchronization problem for the vibration system that the two unbalanced rotors coupled with a pendulum rod. In this section, we will quantitatively analyze the results of the stable phase difference. The parameter values corresponding to general engineering application are as given in Table 1.

5.1. Theoretical Solutions. Under the condition that the balance equation and stability criterion of synchronization between the two rotors (see (31) and (33)) are satisfied, the stable phase difference can be determined by using numerical method. It is worthy to point out that the motion type of the rigid vibrobod is dependent on the synchronous state, which is determined by the values of the stable phase difference. Based on the theoretical deductions, numerical computation will be conducted in the following sections, considering

the variations of installation angular β , frequency ratios n_x , n_y , n_ψ , and n_1 , mass ratio η_{23} , and pendulum length l .

5.1.1. Synchronous State for SBCWB and SACWB. Depending on variations of the values of the system parameters, in this subsection, we observe the synchronous states of two different systems, that is, system of before-resonance coupled with before-resonance (SBCWB), and system of after-resonance coupled with before-resonance (SACWB). We choose the systems as the comparison of the synchronous states because we want to know whether the variations of the stiffness coefficients of the springs have influence on the synchronous states, although SBCWB may be rarely applied in vibration screening engineering.

In Figure 3, we present the values of the stable phase difference for SBCWB and SACWB considering variation of frequency ratio and installation angular β . The values of the structure parameters are fixed at $\eta_{13} = 5$, $\eta_{23} = 1$, $r_o = 2$, $r_l = 0.08$, and $l = 0.6$; additionally, frequency ratio n_1 is set as 0.2, 0.4, 0.6, and 0.8, respectively. From the figure, the stable phase difference locates in the different values in the intervals of $0.2 \leq n_x, n_y, n_\psi \leq 0.8$ and $1.5 \leq n_x, n_y, n_\psi \leq 7$, respectively. It demonstrates that the stiffness coefficients of the springs determine the synchronous states in some certain extent ([1] describes that the motion, as the existence of $\alpha \in (-\pi/2, \pi/2)$, is called synphase synchronization and the motion, as the existence of $\alpha \in (\pi/2, 3\pi/2)$, is called antiphase synchronization). For SBCWB, the values of the phase difference stabilize at the region of synphase synchronization. On the contrary, for SACWB, the values of the phase difference stabilize at the region of antiphase synchronization. Further observations were as follows: the smaller the value of frequency ratio n_1 is, with adjusting installation angular β , the weaker the variation of the phase difference is. However, frequency ratios are inversely proportional to the stiffness coefficient of the springs; in this case, we can believe that extremely large value of stiffness coefficient of the torque spring leads to disadvantage of adjustment of the synchronous state.

For synchronization of asymmetric mass of the two rotors (i.e., considering $\eta_{23} \neq 1.0$), (31) is the balanced equation when the system is operated in the synchronous state, and we can find that mass ratio η_{23} of the unbalanced rotors in this equation cannot be removed. Whether it effects the synchronous state of SBCWB and SACWB depends on our following numerical computation. From Figures 4(a)–4(d), it follows the stable phase difference considering the asymmetric mass of the two rotors. In this case, we suppose that the values of the other structure parameters are fixed at $\eta_{13} = 5$, $\eta_{23} = 0.4$, $r_o = 2$, $r_l = 0.08$, and $l = 0.6$, in addition, frequency ratio n_1 is ascertained at $n_1 = 0.2$, $n_1 = 0.4$, $n_1 = 0.6$, and $n_1 = 0.8$, respectively. According to the date label in the figure, it can be seen that the values of the stable phase are less varying when changing the mass of the rotor in the pendulum rod. Therefore, the asymmetric mass of the two rotors has little influence on the synchronous state in the pendulum-rotor system.

On the other hand, whether length l of the pendulum rod affects the synchronous state of the system will be discussed here. From Figure 5 it depicts the phase difference

FIGURE 3: Stable phase difference for SBCWB and SACWB considering variation of frequency ratio n_1 with $\eta_{13} = 5$, $\eta_{23} = 1$, $r_o = 2$, $r_l = 0.08$, and $l = 0.6$: (a) $n_1 = 0.2$, (b) $n_1 = 0.4$, (c) $n_1 = 0.6$, and (d) $n_1 = 0.8$.

for SACWB considering the length of the pendulum rod. Here, the values of the structure parameters of system are identical with Figure 3 except pendulum length l ; additionally, frequency ratios n_x, n_y, n_ψ are set as $n_x, n_y, n_\psi = 3$, $n_x, n_y, n_\psi = 4$, $n_x, n_y, n_\psi = 5$, and $n_x, n_y, n_\psi = 6$, respectively. According to the data label in the figure, it can be seen that the values of the stable phase are little varying when changing the value of installation angular β of the pendulum rod, on the condition that the frequency ratio n_1 is small. In other words, if the value of stiffness coefficient of the torque spring,

between the pendulum rod and the vibroboddy, is very large in SACWB, the variation of installation angular β weakly changes the synchronous state of the system.

5.1.2. Synchronous State for SBCWA and SACWA. In the above subsection, we have computed the stable phase difference for SBCWB and SACWB considering variations of the structure parameters. Here, we will concern the phase difference for the system of before-resonance coupled with after-resonance (SBCWA, i.e., $0.2 \leq n_x, n_y, n_\psi \leq 0.8$, and

FIGURE 4: Stable phase difference for SBCWB and SACWB considering asymmetric mass of the rotors with $\eta_{13} = 5$, $n_1 = 0.4$, $r_o = 2$, $r_l = 0.08$, and $l = 0.6$: (a) $\eta_{23} = 0.2$, (b) $\eta_{23} = 0.4$, (c) $\eta_{23} = 0.6$, and (d) $\eta_{23} = 0.8$.

$n_1 \geq 2.0$), and system of after-resonance coupled with after-resonance (SACWA, i.e., $n_x, n_y, n_\psi \geq 2.0$, and $n_1 \geq 2.0$).

In Figure 6, we show the values of the stable phase difference for SBCWA and SACWA, assuming the variations of installation angular β and frequency ratios n_x, n_y, n_ψ . The values of the other structure parameters are fixed at $\eta_{13} = 5$, $\eta_{23} = 1$, $r_o = 2$, $r_l = 0.08$, and $l = 0.6$; moreover, frequency ratio n_1 is set as $n_1 = 3$, $n_1 = 4$, $n_1 = 5$, and $n_1 = 6$, respectively. According to the value of the stable phase difference, we can ascertain the synchronous characteristics of the systems.

It can be seen that, with increasing the frequency ratio n_1 , the value of the phase difference is almost absent to change. Here, antiphase and synphase synchronization appearing in SBCWA and SACWA are merely dependent on the values of frequency ratio n_x, n_y, n_ψ and installation angular β .

From Figure 7, it follows the phase difference for SACWA considering the asymmetric mass of the rotors; that is, $\eta_{23} = 0.2$, $\eta_{23} = 0.4$, $\eta_{23} = 0.6$, and $\eta_{23} = 0.8$. Meanwhile, the values of the other physics parameters are fixed at $\eta_{13} = 5$, $n_1 = 5$, $r_o = 2$, $r_l = 0.08$, and $l = 0.6$. According to the date

FIGURE 5: Stable phase difference for SACWB considering the variation of length l of pendulum rod with $\eta_{13} = 5, \eta_{23} = 1, n_x \approx n_y \approx n_\psi = 5.0, r_l = 0.08$, and $l_e = 0.3$: (a) $n_1 = 0.2$, (b) $n_1 = 0.4$, (c) $n_1 = 0.6$, and (d) $n_1 = 0.8$.

label in the figure, it can be seen that, the same as the above subsection, the values of the stable phase are varying a little when changing the mass of the rotor in the pendulum rod.

Finally, the variations of the phase difference for SACWA are compared in Figure 8, based on the variation of pendulum length l and frequency ratio n_1 of the torque spring. In the numerical computations, the values of the other structure parameters are set the same as Figure 5 except $n_1 = 3, n_1 = 4, n_1 = 5$, and $n_1 = 6$. For the synchronous state of the SACA, the phase difference from antiphase synchronization

is gradually closed to the synphase synchronization with increasing the length of the pendulum. But for the variation of frequency ratio n_1 , it can be seen that, for SACWA, the values of the stable phase are varying a little when changing the value of frequency ratio n_1 .

5.2. Sample Verifications. Further analyses have been performed by computer simulations to verify our theoretical solutions above, which can be carried out by applying the Runge-Kutta routine with adaptive stepsize control to the

FIGURE 6: Stable phase difference for SBCWA and SACWA considering $\eta_{13} = 5$, $\eta_{23} = 1$, $r_o = 2$, $r_l = 0.08$, and $l = 0.6$: (a) $n_1 = 3$, (b) $n_1 = 4$, (c) $n_1 = 5$, and (d) $n_1 = 6$.

dynamics equation (13). Here, the parameters of the two motors, which supplied the power source at the same time, are assumed to be identical.

5.2.1. Synchronization for SACWB. Simulation results for SACWB, assuming that $\eta_{13} = 5$, $\eta_{13} = 1$, $n_1 = 0.4$, $n_x \approx n_y \approx n_\psi = 5.0$, $r_o = 2.0$, $l = 0.6$, and $\beta = \pi/6$, are shown in Figure 9. When the two motors supplied the electric source at the same time, the angular accelerations of the two rotors are approximately identical as the inertia moments

of the two rotors are identical and the spring stiffness is stronger (see Figure 9(a)). During the starting process of the system, the velocity difference existing between the two rotors leads the phase difference to be unstable. However, when the angular velocities of the motors reach the operation value and the pendulum rod oscillates steadily, the synchronization phenomenon occurs. At this moment, the coupling torques of the two rotors are approximated to $0.42 \text{ [N}\cdot\text{m]}$. Meanwhile, the two motors steadily rotate in antiphase synchronization with phase difference 3.0 [rad] (see Figure 9(b)). In this

FIGURE 7: Stable phase difference for SBCWA and SACWA assuming the nonidentical mass of rotors and considering $\eta_{13} = 5$, $\eta_{23} = 1$, $\eta_1 = 5$, $r_o = 2$, $r_l = 0.08$, and $l = 0.6$: (a) $\eta_{23} = 0.2$, (b) $\eta_{23} = 0.4$, (c) $\eta_{23} = 0.6$, and (d) $\eta_{23} = 0.8$.

synchronous state, the stable trajectory of mass center of the vibroboddy in the case is a small ellipse because the centrifugal forces produced by the two rotors are counteracted. Therefore, the amplitudes of the stable responses of the vibroboddy are small (see Figure 9(e)). From the displacement responses of the vibroboddy in Figures 9(e) and 9(f), we can see that when the rotation velocity of the two rotors passes through the resonant region of the coupling system, the resonant

responses of the vibroboddy in the x -, y -, and ψ -direction appeared in the starting process except φ_1 -direction. The reason is that the natural frequency in φ_1 -direction is larger than excitation frequency of the system. Comparing the value of the phase difference with that obtained by theoretical solution in Figure 3(b), it should be emphasized here that the simulation results are in agreement with the result obtained by the theoretical solutions (i.e., the stable phase difference

FIGURE 8: Stable phase difference for SACWA assuming the nonidentical mass of rotors and considering $\eta_{13} = 5$, $\eta_{23} = 1$, $n_x \approx n_y \approx n_\psi = 5.0$, $r_l = 0.08$, and $l_e = 0.3$: (a) $n_1 = 3$, (b) $n_1 = 4$, (c) $n_1 = 5$, and (d) $n_1 = 6$.

with the theoretical calculation in Figure 3(b) is equal to 2.9 [rad]; here the stable phase difference with the computer simulation is equal to 3.0 [rad].

5.2.2. Synchronization for SACWA. Simulation results for SACWA, assuming that the values of system parameters are in accordance with Section 5.2.1 except $n_1 = 5$, are shown in Figure 10. When the two motors are supplied the electric source at the same time, the angular accelerations of the two rotors are nonidentical (see Figure 10(a)), and the

reason can be attributed to the larger oscillation amplitude of pendulum (see Figure 10(e)). Then the angular velocities of the motors reach 154 [rad/s], and the synchronization phenomenon occurs. At this moment, the coupling torques are approximated to 2.8 [N·m] (see Figure 9(c)), and the phase difference stabilized at 1.78 [rad] (as 33.18 [rad] = $(10\pi + 1.78)$ [rad]). In the case, the synchronous state for SACWA can be considered the antiphase synchronization. In addition, the two motors steadily rotate in antiphase synchronization with phase difference 3.0 [rad] (see Figure 9(b)). In this case,

FIGURE 9: Simulation results for SACWB: (a) rotational velocity, (b) phase difference of the two rotors, (c) torques of the two rotors, (d) trajectory of mass center of the vibrobod, (e) displacement responses of the vibrobod in x - and y -directions, and (f) displacement responses of the vibrobod in φ_1 - and ψ -directions.

FIGURE 10: Simulation results for SACWA: (a) rotational velocity, (b) phase difference of the two rotors, (c) torques of the two rotors, (d) trajectory of mass center of the vibrobod, (e) displacement responses of the vibrobod in x - and y -directions, and (f) displacement responses of the vibrobod in ϕ_1 - and ψ -directions.

FIGURE 11: Simulation results for asymmetry rotors system: (a) rotational velocity, (b) phase difference of the two rotors, (c) torques of the two rotors, (d) trajectory of mass center of the vibrobod, (e) displacement responses of the vibrobod in x - and y -directions, and (f) displacement responses of the vibrobod in φ_1 - and ψ -directions.

the centrifugal forces produced by the two rotors are partially offset, so the amplitudes of the responses of the vibroboddy are larger than that of Figure 9 (see Figure 10(e)). However, the stable trajectory of mass center of the vibroboddy is an ellipse. Figure 10(f) represents the oscillating responds of the vibroboddy in φ_1 - and ψ -directions. It can be seen that when the rotation velocity of the two rotors passes through the resonant region of the coupling system, the resonant responses of the vibroboddy in the x -, y -, φ_1 , and ψ -direction appeared in the starting process. Comparing simulation results with Figure 6(c), it should be stressed that the value of the stable phase difference is in accordance with the results obtained for the case of theoretical solutions (i.e., the stable phase difference with the theoretical calculation in Figure 6(c) is equal to 1.728 [rad]; here the stable phase difference with the computer simulation is 1.78 [rad]).

5.2.3. Synchronization for Asymmetry Rotors. To verify the synchronization problem of the asymmetry rotors, we take into account that the mass of the rotor, connected with pendulum rod, is nonidentical to another rotor. This simulation is performed on the assumption that $\eta_{13} = 5$, $\eta_{23} = 0.3$, $n_1 = 5$, $n_x \approx n_y \approx n_\psi = 5.0$, $r_o = 2.0$, $l = 0.6$, and $\beta = 0$. Finally, the simulation results are shown in Figure 11, from which one can see that, during the starting few seconds, angular acceleration of rotor 1 is greater than rotor 2. The reason is that the moment of inertia of rotor 1 is smaller than that of rotor 2. Meanwhile, the operation velocity of the rotors stabilizes at 156 [rad/s] in the synchronous state, and the synchronous phase difference between the rotors is 0.018 [rad] (as 33.18 [rad] = $(10\pi + 0.018)$ [rad]). In this case, the synchronous state of the system is synphase synchronization, and note that the value of the phase difference with computer simulation is in agreement with the data label in Figure 8(c). So to speak, the nonidentity of mass between the rotors is independent of the synchronous state. With regard to the displacement amplitude of the vibroboddy, the centrifugal forces produced by the two rotors are mutually added. However, the stable trajectory of mass center of the vibroboddy is still an ellipse, which is the largest amplitude, then Section 5.2.2 and the smallest Section 5.2.1; hence, the amplitude of the vibroboddy in synphase synchronization is larger than antiphase synchronization.

6. Conclusions

The occurrence of synchronized motion in the rotor-pendulum system via the multi-DOF vibration has been investigated. Firstly, the dynamics equation of the system, deduced by Lagrange equation, has been converted into the dimensionless equations. Then Laplace transformation has been employed to solve the steady responses of the system. Next, the balance equation and the stability criteria of synchronization of the system have been derived using Poincare method base on the small parameter. Only should the values of the structure parameters satisfy the balance equation and the stability criteria, the stable phase difference can be acquired. Finally, the value of the phase difference has

been calculated with numerical method; moreover, computer simulations have been performed to verify the correctness of the theoretical computations.

The analysis has revealed that the synchronous state of the rotor-pendulum system is mainly sensitive to the values of the spring stiffness coefficient, the length, and the angular installation of the pendulum rod. However, the synchronous state of the system is independent of the mass ratio between the two rotors and vibroboddy, besides the operation velocity of the motors. With regard to the displacement amplitude of the vibroboddy, the amplitude of the vibroboddy in synphase synchronization is larger than antiphase synchronization. Meanwhile, reasonable values of the structure parameter are in favour of the implementing elliptical motion with large amplitude for vibroboddy. Therefore, the vibration system proposed in this paper can be applied to design new balanced elliptical vibrating screens, when their structure parameters satisfy the balance equation and the stability criterion of synchronization. In the early stage, for the developing and understanding the internal characteristics of the system, we only consider the rotors under the assumption of the rotation in the same direction. Will these results change if the contra rotation of the rotors in the pendulum-rotor system is taken into account? We believe that finding the answer to this question is the next step in challenging task of getting a complete understanding of synchronization in such system.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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