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Research Article

On the Equivalence between Static and Dynamic Railway Track Response and on the Euler-Bernoulli and Timoshenko Beams Analogy

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The paper tries to clarify the problem of solution and interpretation of railway track dynamics equations for linear models. Set of theorems is introduced in the paper describing two types of equivalence: between static and dynamic track response under moving load and between the dynamic response of track described by both the Euler-Bernoulli and Timoshenko beams. The equivalence is clarified in terms of mathematical method of solution. It is shown that inertia element of rail equation for the Euler-Bernoulli beam and constant distributed load can be considered as a substitute axial force multiplied by second derivative of displacement. Damping properties can be treated as additional substitute load in the static case taking into account this substitute axial force. When one considers the Timoshenko beam, the substitute axial force depends additionally on shear properties of rail section, rail bending stiffness, and subgrade stiffness. It is also proved that Timoshenko beam, described by a single equation, from the point of view of solution, is an analogy of the Euler-Bernoulli beam for both constant and variable load. Certain numerical examples are presented and practical interpretation of proved theorems is shown.

1. Introduction

The problem of track dynamic response under moving load is the subject of many theoretical and experimental investigations. Under some assumptions, the beam on elastic foundation can be considered as a typical track model. It is worth mentioning the initial study of beams on the Winkler foundation subjected to a concentrated force moving with constant speed that was initiated by Timoshenko [1]. The first solution to a simple stationary case of the Euler-Bernoulli beam on elastic foundation was properly obtained by Ludwig [2]. Mathews formulated and partly solved the case of moving and oscillating force [3]. The case of varying moving force was studied, for example, by Fryba [4] and by Bogacz and Krzyzynski [5].

Many papers are devoted to study various effects of generalized models:

(1) Analysis of Timoshenko beam under moving constant and varying loads (presented, e.g., in [6–9])

(2) Analysis of a beam on elastic half-space [10, 11]

(3) Response of beam on nonlinear foundation (e.g., [8, 12–14])

(4) Dynamic response of beam on random foundation; see [15–17]

(5) Dynamic response of track as multilayered structure (see [18, 19], analytical approach; [20–22], numerical approach);

(6) Analysis of set of distributed moving forces, described by Heaviside functions (e.g., [7]), rectangular function [9, 19], cosine square formula [8, 12, 13], or Gauss function [19]

(7) Effect of axial force on dynamic response [19, 23]

(8) Analysis of set of forces varying harmonically and associated with track imperfections including the phase of sine function for particular axles (numerically [20–22]) and analytical approach [13, 19]

In all described generalizations of classical approach, the track response model was composed of rail (as the Euler-Bernoulli or Timoshenko beam) and viscoelastic or elastic foundation. The sleepers and the ballast were modelled as additional layers.

Analytical closed form solutions were obtained for infinite Euler-Bernoulli beam on elastic foundation under single concentrated load moving along the beam with constant speed using so-called matching conditions for homogeneous solution in the point related to a load position [2, 3]. Using the same approach, the damping properties of foundation or axial force in beam (rail) were also included in the closed form solutions (comp., e.g., [4, 5, 23]). For oscillating force or set of oscillating forces, the solution was obtained by using the Fourier transform and the inverse Fourier transform [5, 6, 8]. Difficulties arising from the integration of solution in the frequency domain lead to the various method of approximation of the transformed function. An interesting approach based on wavelets application is developed by Koziol and coauthors (comp. [8, 10]). Applications of Fourier series to obtain the solution of railway track response are practically limited to the case of bridges vibrations so far [4, 22]. An example of use of the Fourier series in bounded interval to study the railway track response is presented in [19]. It is shown that solutions obtained in this manner can approximate the track response with very high accuracy depending on a number of Fourier series coefficients and the assumed length of interval.

Analysis of Timoshenko beam is mainly carried out for a set of equations describing coupled beam displacements and rotations [6, 8]. The characteristics of the dynamic response of Timoshenko beam described by a single equation are rarely analysed. Hunt [24] presents numerical approach to solve the inverse Fourier transform in the case of certain simplified equation for a beam on viscoelastic foundation loaded by a single oscillating force. In [7], the problem of critical speeds for both constant and varying distributed loads is studied analytically without detailed interpretation of final solution. Single equation of dynamics of Timoshenko beam without foundation was analysed by Majkut [25].

The above-mentioned papers present many interesting results. However, one can observe a lack of simple interpretation of the dynamic railway track response under moving distributed load in terms of differences between static and dynamic solutions and also between the Euler-Bernoulli and Timoshenko beams. This paper tries to clarify the formulated problem described by linear equation of rail motion. The equivalence between static and dynamic railway track response for foundation without damping is proved. Using the Fourier series in finite interval for both distributed load and solution it is shown that damping properties of rail foundation can be interpreted as additional, substitute load. The equivalence of dynamic track solution, in terms of mathematical physics equations, between the Euler-Bernoulli and Timoshenko beams is also proved. Numerical examples are presented leading to practical interpretation of the formulated theorems in wide range of train speed and foundation damping properties. The introduced set of theorems can be treated as an important contribution to a proper arrangement and classification of knowledge that can be recognized as an essential part of fundamentals of the theory of railway track dynamics. One can observe a lack of such papers trying to formulate basics of railway track analysis in a way similar to subjects recognized more systematized research fields.

2. Track Response to a Set of Constant Forces Moving in Longitudinal Direction

2.1. Rail Modelled by the Euler-Bernoulli Beam Equation. Basic equation of motion of track modelled by the Euler-Bernoulli beam on viscoelastic foundation has the following form (comp., e.g., [4]):

$$EI\frac{\partial^4 y_r}{\partial x^4} + N\frac{\partial^2 y_r}{\partial x^2} + m\frac{\partial^2 y_r}{\partial t^2} + c\frac{\partial y_r}{\partial t} + Uy_r = q(x,t), \quad (1)$$

where E is rail Young modulus [N/m^2], I is moment of inertia of beam (rail) in vertical plane [m^4], N is axial force in rail (positive sign means compressive force) [N], m is unit mass of beam [kg/m], c is damping coefficient of foundation [N$_s$/m^2], U is foundation stiffness [N/m^2], and q is distributed load [N/m].

In the moving coordinate system ($\eta = y_r; \xi = x - vt$), if load is constant in time, (1) may be transformed to an ordinary differential equation:

$$EI\frac{d^4\eta}{d\xi^4} + \left(N + mv^2\right)\frac{d^2\eta}{d\xi^2} - cv\frac{d\eta}{d\xi} + U\eta = q(\xi). \quad (2)$$

Theorem 1. *For linear model of the track, described by (1), if load does not change in time, the steady-state solution of (2) is an equivalent static case for damping $c = 0$ by the substitution:*

$$S_{sub} = N + mv^2, \quad (3)$$

where S_{sub} is substitute axial force [N]. Effect of foundation damping, with an accuracy determined by approximation of rail displacements by Fourier series in finite interval $[0, \lambda]$, can be considered as a static case with substitution (3) and additional (substitute) load q_{sub} described as follows:

$$q_{sub} = \psi(q, P), \quad (4)$$

where q_{sub} is substitute load [N/m]; ψ is function depending on real load q and track model parameters P (i.e., EI; N; m; etc.).

Proof. The proof of the first part of the theorem (see (3)) is relatively simple. The equation of static deflection y of beam on elastic foundation (parameter U) with compressive axial force S and load q has the following form (see, e.g., [26]):

$$EI\frac{d^4 y}{dx^4} + S\frac{d^2 y}{dx^2} + Uy = q(x). \quad (5)$$

As can be seen, for assumed train speed v and unit mass of beam m, in the moving coordinate system, the sum of realistic axial force N and mv^2 in (2) for $c = 0$ expresses the substitute axial force for the beam on elastic foundation in (5), that is, $S = S_{\text{sub}} = N + mv^2$.

For proving second part of the theorem one assumes that realistic load q and the solution y_r can be described by the Fourier series in the assumed interval $[0, \lambda]$, that is,

$$q(\xi) = \frac{a_0}{2} + \sum_{i=1}^{\infty} (a_i \cdot \cos \Omega_i \xi + b_i \cdot \sin \Omega_i \xi);$$
$$y_r(\xi) = \frac{y_0}{2} + \sum_{i=1}^{\infty} (A_i \cdot \cos \Omega_i \xi + B_i \cdot \sin \Omega_i \xi); \quad (6)$$
$$\Omega_i = \frac{2\pi \cdot i}{\lambda}, \ \xi \in [0, \lambda],$$

where a_0, y_0, A_i, B_i are the coefficients of Fourier series.

Then (2) leads to the following formulas (for cosine and sine coefficients of the Fourier series):

$$A_i \left[EI\Omega_i^4 - S_{\text{sub}}\Omega_i^2 + U\right] + B_i \left[-cv\Omega_i\right] = a_i,$$
$$A_i \left[cv\Omega_i\right] + B_i \left[EI\Omega_i^4 - S_{\text{sub}}\Omega_i^2 + U\right] = b_i, \quad (7)$$

$$y_0 = \frac{a_0}{U}, \quad (8)$$

where $S_{\text{sub}} = N + mv^2$ (comp. (3)). One can derive the form of parameters A_i and B_i by solving the system of (7):

$$A_i = \frac{a_i \cdot \gamma_i + b_i \cdot \delta_i}{\gamma_i^2 + \delta_i^2};$$
$$B_i = \frac{b_i \cdot \gamma_i - a_i \cdot \delta_i}{\gamma_i^2 + \delta_i^2}, \quad (9)$$

where

$$\gamma_i = EI\Omega_i^4 - S_{\text{sub}}\Omega_i^2 + U,$$
$$\delta_i = cv\Omega_i \quad (10)$$

and the constant value y_0 is described by formula (8).

Let other coefficients C_i and D_i, together with the same value y_0 (formula (8)), describe the cosine and sine coefficients of solution of (2) in the static case with substituted axial force $S_{\text{sub}} = N + mv^2$ and the damping coefficient $c = 0$. In this case, the solution of (7) has a simple form:

$$C_i = \frac{a_i}{\gamma_i};$$
$$D_i = \frac{b_i}{\gamma_i}. \quad (11)$$

If the solution of dynamic case, for $c \neq 0$, and static solution are denoted by y_{rd} and y_{rs}, respectively, then the following formula can be written in terms of Fourier series in the assumed interval $[0, \lambda]$ (comp. (6)):

$$y_{\text{rd}}(\xi) = \frac{y_0}{2} + \sum_{i=1}^{\infty} (A_i \cdot \cos \Omega_i \xi + B_i \cdot \sin \Omega_i \xi);$$
$$y_{\text{rs}}(\xi) = \frac{y_0}{2} + \sum_{i=1}^{\infty} (C_i \cdot \cos \Omega_i \xi + D_i \cdot \sin \Omega_i \xi); \quad (12)$$
$$\xi \in [0, \lambda]; \ \Omega_i = \frac{2\pi \cdot i}{\lambda}$$

and the difference between dynamic and static solutions with S_{sub} is

$$y_{\text{rd}}(\xi) - y_{\text{rs}}(\xi) = \sum_{i=1}^{\infty} ((A_i - C_i) \cdot \cos \Omega_i \xi + (B_i - D_i) \cdot \sin \Omega_i \xi). \quad (13)$$

This difference represents function which can be considered as linked to an additional load in the static case with S_{sub}. Then one can obtain the cosine and sine parts of substitute load in the static case for any index i:

$$a_i^{\text{sub}} = (A_i - C_i) \cdot \gamma_i;$$
$$b_i^{\text{sub}} = (B_i - D_i) \cdot \gamma_i \quad (14)$$

with constant value y_0 equal to zero due to the fact that the difference between dynamic and static solution includes only cosine and sine Fourier series coefficients (the constant y_0 is the same for both dynamic and static solutions (comp. (2))). Using formulas (9), (11), and (14), one can obtain the following:

$$a_i^{\text{sub}} = \frac{-a_i \cdot \delta_i^2 + b_i \cdot \delta_i \cdot \gamma_i}{\gamma_i^2 + \delta_i^2};$$
$$b_i^{\text{sub}} = \frac{-b_i \cdot \delta_i^2 - a_i \cdot \delta_i \cdot \gamma_i}{\gamma_i^2 + \delta_i^2}. \quad (15)$$

The above formulas (15) describe the cosine and sine coefficients of substitute load:

$$q_{\text{sub}} = \sum_{i=1}^{\infty} \left(a_i^{\text{sub}} \cdot \cos \Omega_i \xi + b_i^{\text{sub}} \cdot \sin \Omega_i \xi\right). \quad (16)$$

It means that substitute load which comes from damping properties of the rail foundation is determined as a function of realistic load (coefficients a_i, b_i and speed v) and model parameters γ_i and δ_i which describe beam and foundation properties (see notation (10)). The second part of the theorem is then proved. □

For practical interpretation of Theorem 1, certain remarks should be formulated.

(1) The equivalence between static and dynamic response of railway track described by simple model, that is, the Euler-Bernoulli beam on viscoelastic foundation under moving load invariant in time, concerns only mathematical solutions.

(2) In the case when damping properties of rail foundation can be neglected, due to the load being distributed on very small spans and the distance between wheels being relatively large (practically more than about 5-6 m), dynamic factor for maximum rail displacements φ_{ds} can be described by simple formula (based on [19, 23]):

$$\varphi_{\text{ds}} = \frac{1}{\sqrt{1 - ((S_{\text{sub}} - N)/(N_{\text{cr}} - N))}}, \quad (17)$$

where S_{sub} formula (5), $N_{cr} = 2\sqrt{EIU}$ is critical value of axial force.

The sign of the substitute axial force S_{sub} depends on relation between realistic compression/tension force in rail N and the term mv^2 which has positive value. Knowing that real axial force N in rail depends mainly on the rail temperature changes in relation to the laying rail temperature ΔT,

$$N = \alpha AE\Delta T, \tag{18}$$

where α is coefficient of thermal expansion of rail steel [1/°C] and the part of substitute axial force $S_v = mv^2$ (depending on speed) can be calculated in relation to the equivalent rail temperature increase ΔT_{eq}. For typical rail, being in use in Europe, that is, 60E1, and unit mass of rail 60 kg/m (with the unit mass of half sleeper, usually equal to around $m = 327$ kg/m), one can obtain the following equivalent rail temperature increase: $\Delta T_{eq} = 0.00318v^2$ or $\Delta T_{eq} = 0.0176v^2$; see [26]. It means that

(i) $\Delta T_{eq} = 2.5$°C or $\Delta T_{eq} = 14$°C for $v = 100$ km/h (27.77 m/s);

(ii) $\Delta T_{eq} = 10$°C or $\Delta T_{eq} = 54$°C for $v = 200$ km/h (55.55 m/s);

(iii) $\Delta T_{eq} = 22.5$°C or $\Delta T_{eq} = 122$°C for $v = 300$ km/h (83.33 m/s).

For typical track structure, the critical axial force (formula (17)) is around 25–35 MN. The equivalent rail temperature increase for these values is on the level of 1500°C. Using expression (17), it follows that, for realistic rail temperature increase (maximum 50–60°C) and speed up to 300 km/h, the dynamic factor for track without imperfections and damping is lower than 1%, in the case of typical track structure.

(3) The dynamic factor associated with damping properties of rail foundation in relation to the static response with substitute axial force cannot be described by simple formula. Also, the substitute load linked to the foundation damping (formulas (15)) cannot be simply explained. Currently, these problems can be numerically solved. The following track and load parameters are assumed in further study [19]:

(i) Rail: 60E1, Young modulus, $E = 2.1 * 10^{11}$ N/m^2; cross-sectional area, $A = 7687 * 10^{-6}$ m^2; thermal expansion coefficient, $\alpha = 1.15 * 10^{-5}$ 1/°C; unit mass, $m = 60$ kg/m (or $m = 327$ kg/m if rail and a half of concrete sleeper PS-94 type are taken into account); rail temperature increase, $\Delta T = 50$°C

(ii) Rail foundation: substitute foundation stiffness due to fasteners and sleeper foundation elastic properties $U = 3.8 * 10^7$ N/m^2, damping properties – fasteners $c_{ro} = 2370/0.6$ Ns/m^2 (reference value obtained during laboratory tests and also $10c_{ro}$ and $100c_{ro}$), sleepers foundation $c_{po} = 49000$ Ns/m^2 (reference value and $10c_{po}$)

(iii) Load: 4 wheels of EMU-250 Pendolino train with configuration: 2.7 m, 4.5 m, and 2.7 m; see [27], wheel load 80 kN; load distribution represented by the Gauss function with the parameter $\sigma = 0.005$ m, number of Fourier series coefficients equal to 3000

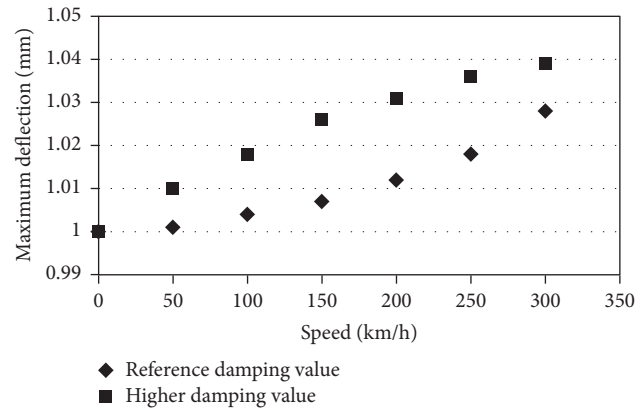

Figure 1: Dynamic factor for the Euler-Bernoulli beam and various damping properties of rail foundation.

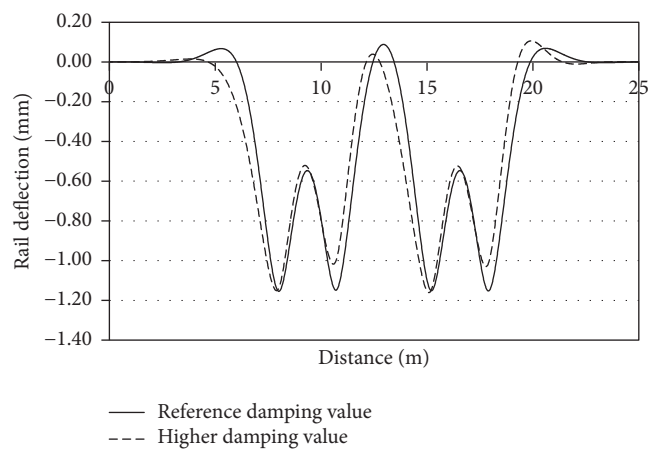

Figure 2: Deflection line of rail modelled by the Euler-Bernoulli beam and various damping properties of rail foundation.

Figure 1 presents the dynamic factor due to rail foundation damping for two options: higher values describe dynamic factor for viscous properties of fasteners $c_r = 100c_{ro}$ and sleeper foundation properties $c_p = 10c_{po}$; lower values describe dynamic factor for the reference values c_{ro} and c_{po}. As can be observed, these relations have various curvatures and show sensitivity related to the effect of damping on dynamic factor. In the considered range of operating train speeds and viscous coefficients, the maximum value of dynamic factor can reach 4%.

Figure 2 presents rail deflection lines for two rail foundation damping properties (as shown in Figure 1) and speed of 300 km/h. Figures 3 and 4 show the substitute load associated with similar options of rail foundation damping properties. It can be seen that the substitute load due to damping properties of rail foundation strongly depends on train speed and damping coefficients.

2.2. Rail Modelled by the Timoshenko Beam and Comparative Study of the Obtained Results. It is known that various authors use the term Timoshenko beam for different equations or systems of equations. This problem is more discussed

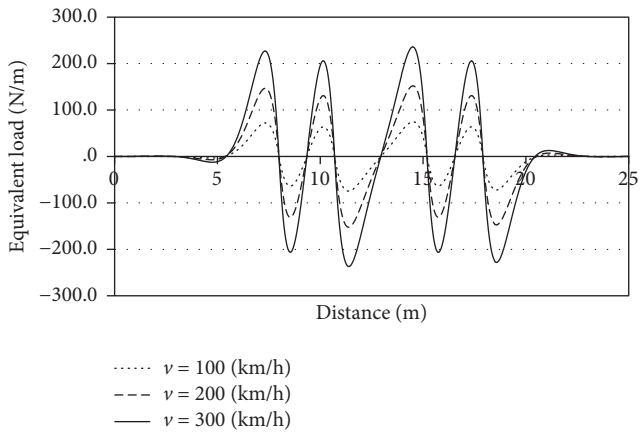

Figure 3: Equivalent load related to damping properties of rail foundation (reference values of viscous coefficients of fasteners and sleeper foundation).

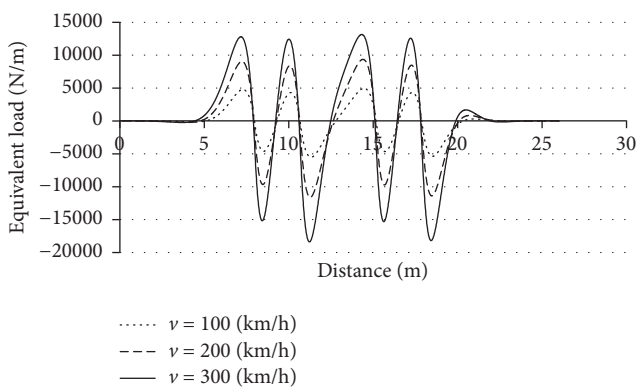

Figure 4: Equivalent load related to damping properties of rail foundation (higher values of viscous coefficients of fasteners and sleeper foundation).

in Appendix, at the end of this paper. On the basis of [7, 25] the following single differential equation of rail vertical motion described by the Timoshenko beam theory can be written:

$$EI\frac{\partial^4 y_r}{\partial x^4} + N\frac{\partial^2 y_r}{\partial x^2} + m\frac{\partial^2 y_r}{\partial t^2} - \left(\frac{EI\rho}{\kappa G} + \rho I\right)\frac{\partial^4 y_r}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{\kappa G}\frac{\partial^4 y_r}{\partial t^4} + R_p(x,t) = q_T(x,t), \tag{19}$$

where the used notation is similar to ones used in (1) and, additionally, ρ is rail mass density [kg/m^3], κ is rail cross-sectional shear coefficient, G is steel shear modulus [N/m^2], R_p is rail foundation viscoelastic reaction [N/m], and q_T is substitute load [N/m].

The foundation reaction R_p and the substitute load q_T are characterized by the following expressions:

$$R_p(x,t) = c\frac{\partial y_r}{\partial t} + c\frac{\rho I}{\kappa AG}\frac{\partial^3 y_r}{\partial t^3} - c\frac{EI}{\kappa AG}\frac{\partial^3 y_r}{\partial x^2 \partial t} + Uy_r + U\frac{\rho I}{\kappa AG}\frac{\partial^2 y_r}{\partial t^2} - U\frac{EI}{\kappa AG}\frac{\partial^2 y_r}{\partial x^2};$$
$$q_T(x,t) = q(x,t) + \frac{\rho I}{\kappa AG}\frac{\partial^2 q}{\partial t^2} - \frac{EI}{\kappa AG}\frac{\partial^2 q}{\partial x^2}, \tag{20}$$

where A [m^2] is a rail cross-sectional area.

In the moving coordinate system ($\eta = y_r$; $\xi = x - vt$), for the load constant in time, (19) and expression (20) lead to the ordinary differential equation:

$$B_{\text{sub}}\frac{d^4\eta}{d\xi^4} - cvw_T\frac{d^3\eta}{d\xi^3} + S_{\text{sub}}\frac{d^2\eta}{d\xi^2} - cv\frac{d\eta}{d\xi} + U\eta = q(\xi) + w_T\frac{d^2 q}{d\xi^2}, \tag{21}$$

where

$$w_T = \frac{\rho I v^2 - EI}{\kappa AG};$$
$$S_{\text{sub}} = N + mv^2 + Uw_T; \tag{22}$$
$$B_{\text{sub}} = EI + mv^2\left(w_T - \hat{\eta}\right); \quad \hat{\eta} = \frac{I}{A}.$$

The load q and rail displacements function y_r are described by Fourier series in the assumed interval $[0, \lambda]$ (see formulas (6)). Following the methodology used in previous subsection, one can obtain the closed form solution in the case of the infinite Fourier series:

$$A_{Ti} = \frac{a_{Ti}\cdot\gamma_{Ti} + b_{Ti}\cdot C_{\text{sub},i}}{\gamma_{Ti}^2 + C_{\text{sub},i}^2};$$
$$B_{Ti} = \frac{b_{Ti}\cdot\gamma_{Ti} - a_{Ti}\cdot C_{\text{sub},i}}{\gamma_{Ti}^2 + C_{\text{sub},i}^2}, \tag{23}$$
$$Y_{r0} = \frac{a_{T0}}{U},$$

where

$$a_{Ti} = a_i\left(1 - w_T\Omega_i^2\right);$$
$$b_{Ti} = b_i\left(1 - w_T\Omega_i^2\right);$$
$$a_{T0} = a_0; \tag{24}$$
$$\gamma_{Ti} = B_{\text{sub}}\Omega_i^4 - S_{\text{sub}}\Omega_i^2 + U;$$
$$C_{\text{sub},i} = cv\Omega_i\left(1 - w_T\Omega_i^2\right)$$

and a_i, b_i, a_0 denote Fourier coefficients of distributed load function q (comp. formulas (6)). One can observe that formulas (23), with a use of symbols (24), have the same mathematical form as formulas (9) with notations (10). Therefore, using procedure like the one used in the case of the Euler-Bernoulli beam, one can prove relatively easy the following theorem.

Theorem 2. *For linear model of the track, modelled by the Timoshenko beam on viscoelastic foundation (see (19)), if load does not change in time, the steady-state solution of (21) is an equivalent static case* ($c = 0$) *with substitution (22). Effect of foundation damping* ($c > 0$), *with accuracy determined by approximation of rail displacements using Fourier series in finite interval* $[0, \lambda]$, *can be considered as a static case with substitution (22) and additional (substitute) load* q_{sub} *described as follows:*

$$q_{sub} = \sum_{i=1}^{\infty} \left(a_i^{sub} \cdot \cos \Omega_i \xi + b_i^{sub} \cdot \sin \Omega_i \xi \right), \tag{25}$$

where

$$\begin{aligned} a_i^{sub} &= \frac{-a_{Ti} \cdot C_{sub,i}^2 + b_{Ti} \cdot C_{sub,i} \cdot \gamma_{Ti}}{\gamma_{Ti}^2 + C_{sub,i}^2}; \\ b_i^{sub} &= \frac{-b_{Ti} \cdot C_{sub,i}^2 - a_{Ti} \cdot C_{sub,i} \cdot \gamma_{Ti}}{\gamma_{Ti}^2 + C_{sub,i}^2}; \\ \Omega_i &= \frac{2\pi i}{\lambda}; \\ \xi &\in [0, \lambda]. \end{aligned} \tag{26}$$

Proof. For a sake of reading simplicity the proof of Theorem 2 is left to readers as simple symbols manipulation using remarks formulated above. □

It is also worth formulating the following theorem being a consequence of the studies carried out.

Theorem 3. *For the linear model of the track, if load does not change in time, the steady-state solution of the Timoshenko beam equation (expression (21)) is equivalent to the steady-state solution of the Euler-Bernoulli beam (2), for any speed* v *and foundation damping* $c = 0$, *with the following substitution:*

$$\begin{aligned} EI &:= EI + mv^2 \left(w_T - \widehat{\eta} \right); \\ N + mv^2 &:= N + mv^2 + U w_T; \\ q(\xi) &:= q(\xi) + w_T \frac{d^2 q}{d\xi^2}, \end{aligned} \tag{27}$$

where

$$\begin{aligned} w_T &= \frac{\rho I v^2 - EI}{\kappa A G}, \\ \widehat{\eta} &= \frac{I}{A}. \end{aligned} \tag{28}$$

Effect of foundation damping for the Timoshenko beam, with an accuracy determined by approximation of rail displacements using the Fourier series in finite interval $[0, \lambda]$, *is equivalent to the Euler-Bernoulli beam in terms of algebraic equations for cosine and sine part of solution (comp. expressions (7)) with substitution*

$$c v \Omega_i := c v \Omega_i \left(1 - w_T \Omega_i^2 \right); \quad \Omega_i = \frac{2\pi i}{\lambda}. \tag{29}$$

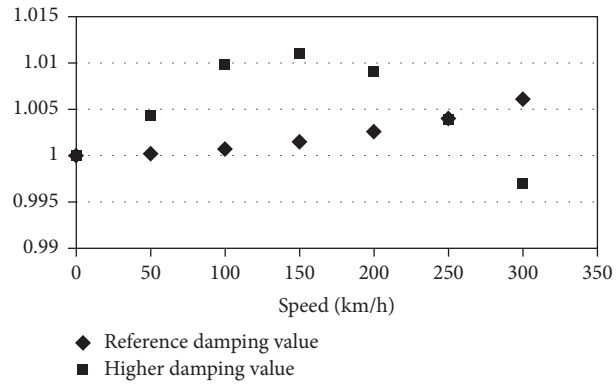

Figure 5: Dynamic factor due to damping in the case of the Timoshenko beam for various properties of rail foundation damping.

Proof. The theorem can be relatively easy proved based on the previous studies (see Theorem 1) and, here, similar to Theorem 2, its proof is omitted in order to avoid repetitions. Instead, the practical interpretation is presented as more important for understanding of the theorems formulation in terms of scientific arrangement of the subject. □

For practical interpretation of Theorems 2 and 3, certain remarks and conclusions are presented on the basis of numerical examples using the track and load parameters assumed in Section 2.2 for the Euler-Bernoulli beam. Values of additional parameters are taken from [6]: mass density of rail steel $\rho = 7850$ kg/m; shear coefficient of rail cross section $\kappa = 0.4$; shear modulus of rail steel $G = 0.77 * 10^{11}$ N/m^2:

(1) The dynamic factor for maximum rail displacements φ_{ds} for the Timoshenko beam without damping, in the speed range [0; 300] km/h, is less than 1%, which is similar to the Euler-Bernoulli beam case.

(2) Rail foundation damping for the Timoshenko beam has significantly smaller effect in relation to the Euler-Bernoulli beam case. This is shown in Figure 5 in the case of typical damping and high values of damping coefficient (the same values are used in Figure 1). The dynamic factor for maximum rail displacements due to damping reaches a level of a few percent (3-4%) in the case of the Euler-Bernoulli beam and, at the same time, it reaches only 1% in the case of the Timoshenko beam.

The peak corresponding to this one observed in Figure 5 for the Timoshenko beam case and higher damping value appears also in the case of the Euler-Bernoulli beam. However, this peak is reached for higher speed, that is, around 400 km/h (150 km/h in the case of the Timoshenko beam). In both cases, the dynamic factor decreases above these speed values but it remains positive. It reaches the level of 0.6 for both the Timoshenko and the Euler-Bernoulli beams around 2000 km/h. It also becomes slightly higher in the case of the Timoshenko beam compared to the Euler-Bernoulli model. For higher speeds, reaching

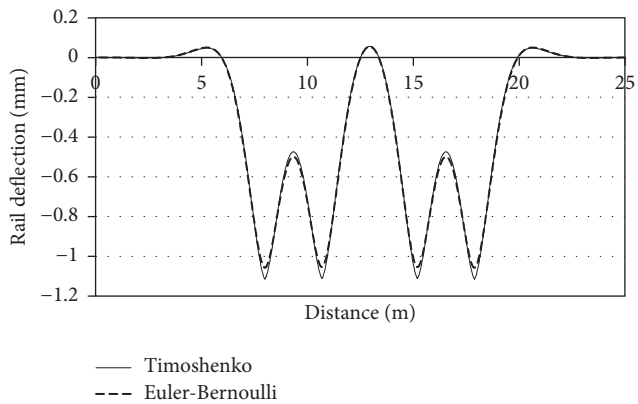

Figure 6: Rail deflection line for $v = 0$ km/h and for two types of models: the Euler-Bernoulli and the Timoshenko beams.

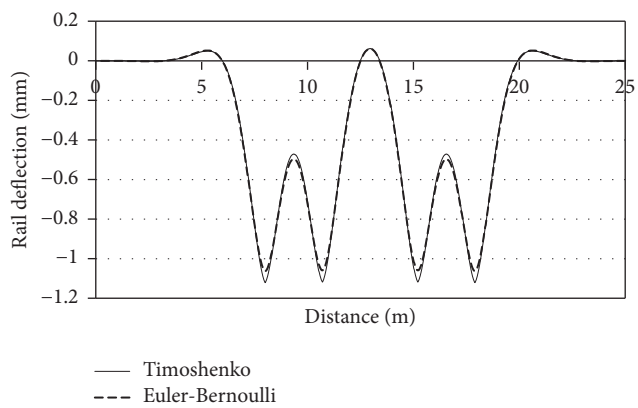

Figure 7: Rail deflection line for $v = 300$ km/h and for two types of models: the Euler-Bernoulli and the Timoshenko beams.

3000 km/h, the dynamic factor can be estimated around 0.5 in both cases.

One should note that, in the case of high damping value, unsymmetrical response of the track in relation to the centre of the load system can be observed. This effect is minimal when one deals with low speeds. For these parameters, the maximal displacements increase along with the speed increase up to 150 km/h (for the Timoshenko beam) and 400 km/h (for the Euler-Bernoulli beam). Above these values, maximal deflections become smaller.

In the case of the reference damping value, the dynamic factor increases without any extremal values. One can show that it reaches the level of 1.2 near 700 km/h.

(3) The coefficient describing the ratio between maximum rail displacements for the Timoshenko and Euler-Bernoulli beams is on the level of 5.4% for the analysed speed range and it is practically independent of speed. Hence one can conclude that the static effect associated with the shear force can be treated as the main factor characteristic for the analysed model with moving load invariant in time.

This observation can be additionally justified by the fact that the dynamic factor for the Timoshenko beam is relatively small in both cases, that is, with and without damping (comp. Figures 6 and 7).

3. Track Response for a Set of Oscillating Forces Moving along the Track with Constant Speed

3.1. Rail Modelled by the Euler-Bernoulli Beam Equation. In the case of forces oscillating with circular frequency ω, (2) obtains the following form in the moving coordinate system ($\eta = y_r, \xi = x - vt$):

$$EI\frac{\partial^4 y_r}{\partial\xi^4} + N\frac{\partial^2 y_r}{\partial\xi^2} + m\left(\frac{\partial^2 y_r}{\partial t^2} - 2v\frac{\partial^2 y_r}{\partial t\partial\xi} + v^2\frac{\partial^2 y_r}{\partial\xi^2}\right) + c\left(\frac{\partial y_r}{\partial t} - v\frac{\partial y_r}{\partial\xi}\right) + Uy_r = q(\xi, t). \tag{30}$$

The assumed form of solution is as follows (comp. [5, 19]):

$$y_r(\xi, t) = Y_c(\xi)\cdot\cos\omega t + Y_s(\xi)\cdot\sin\omega t. \tag{31}$$

One should mention that, in general, a set of forces can be also described as follows:

$$q(\xi, t) = q_c(\xi)\cdot\cos\omega t + q_s(\xi)\cdot\sin\omega t. \tag{32}$$

By differentiating (31) and substituting (32), (30) becomes a set of ordinary equations associated with cosine ($\cos\omega t$) and sine parts ($\sin\omega t$):

$$\begin{aligned} &EI\frac{d^4Y_c}{d\xi^4} - m\omega^2 Y_c - 2mv\omega\frac{dY_s}{d\xi} + \left(N + mv^2\right)\frac{d^2Y_c}{d\xi^2} \\ &\quad + c\omega Y_s - cv\frac{dY_c}{d\xi} - UY_c = q_c, \\ &EI\frac{d^4Y_s}{d\xi^4} - m\omega^2 Y_s + 2mv\omega\frac{dY_c}{d\xi} + \left(N + mv^2\right)\frac{d^2Y_s}{d\xi^2} \\ &\quad - c\omega Y_c - cv\frac{dY_s}{d\xi} + UY_s = q_s. \end{aligned} \tag{33}$$

The functions Y_c, Y_s, q_c and q_s are expressed in terms of Fourier series in the assumed interval $[0, \lambda]$:

$$\begin{aligned} Y_c(\xi) &= \frac{Y_{c0}}{2} + \sum_{i=1}^{\infty}\left(A_i\cdot\cos\Omega_i\xi + B_i\cdot\sin\Omega_i\xi\right); \\ Y_s(\xi) &= \frac{Y_{s0}}{2} + \sum_{i=1}^{\infty}\left(C_i\cdot\cos\Omega_i\xi + D_i\cdot\sin\Omega_i\xi\right); \\ q_c(\xi) &= \frac{a_0}{2} + \sum_{i=1}^{\infty}\left(a_i\cdot\cos\Omega_i\xi + b_i\cdot\sin\Omega_i\xi\right); \\ q_s(\xi) &= \frac{c_0}{2} + \sum_{i=1}^{\infty}\left(c_i\cdot\cos\Omega_i\xi + d_i\cdot\sin\Omega_i\xi\right), \\ &\xi\in[0,\lambda];\ \Omega_i = \frac{2\pi\cdot i}{\lambda}. \end{aligned} \tag{34}$$

After differentiation and rearranging one obtains the following set of algebraic equations determining unknown parameters A_i, B_i, C_i, D_i and Y_{c0}, Y_{s0}:

$$\begin{aligned} &A_i\left[P_{1E}\right]+B_i\left[P_{2E}\right]+C_i\left[P_{3E}\right]+D_i\left[P_{4E}\right]=a_i,\\ &A_i\left[-P_{2E}\right]+B_i\left[P_{1E}\right]+C_i\left[-P_{4E}\right]+D_i\left[P_{3E}\right]=b_i,\\ &A_i\left[-P_{3E}\right]+B_i\left[-P_{4E}\right]+C_i\left[P_{1E}\right]+D_i\left[P_{2E}\right]=c_i,\\ &A_i\left[P_{4E}\right]+B_i\left[-P_{3E}\right]+C_i\left[-P_{2E}\right]+D_i\left[P_{1E}\right]=d_i, \end{aligned} \tag{35}$$

$$\begin{aligned} Y_{c0}&=\frac{a_0\left[P_{5E}\right]-c_0\left[P_{6E}\right]}{\left(P_{5E}\right)^2+\left(P_{6E}\right)^2},\\ Y_{s0}&=\frac{c_0\left[P_{5E}\right]+a_0\left[P_{6E}\right]}{\left(P_{5E}\right)^2+\left(P_{6E}\right)^2}, \end{aligned} \tag{36}$$

where

$$\begin{aligned} P_{1E}&=EI\Omega_i^4-\left(N+mv^2\right)\Omega_i^2+U-m\omega^2,\\ P_{2E}&=-cv\Omega_i,\\ P_{3E}&=c\omega,\\ P_{4E}&=-2mv\omega\Omega_i,\\ P_{5E}&=U-m\omega^2,\\ P_{6E}&=c\omega. \end{aligned} \tag{37}$$

3.2. Rail Modelled by the Timoshenko Beam Equation. In the case of forces oscillating with circular frequency ω, (19) and (20) obtain the following form in the moving coordinate system ($\eta=y_r$, $\xi=x-vt$):

$$\begin{aligned} &EI\frac{\partial^4 y_r}{\partial\xi^4}+N\frac{\partial^2 y_r}{\partial\xi^2}+m\left(\frac{\partial^2 y_r}{\partial t^2}-2v\frac{\partial^2 y_r}{\partial t\partial\xi}+v^2\frac{\partial^2 y_r}{\partial\xi^2}\right)\\ &+\frac{\rho^2 I}{\kappa G}\left(\frac{\partial^4 y_r}{\partial t^4}-4v\frac{\partial^4 y_r}{\partial\xi\partial t^3}+6v^2\frac{\partial^4 y_r}{\partial\xi^2\partial t^2}\right.\\ &\left.-4v^3\frac{\partial^4 y_r}{\partial\xi^3\partial t}+v^4\frac{\partial^4 y_r}{\partial\xi^4}\right)-\left(\rho I+\frac{\rho EI}{\kappa G}\right)\cdot\left(\frac{\partial^4 y_r}{\partial\xi^2\partial t^2}\right.\\ &\left.-2v\frac{\partial^4 y_r}{\partial\xi^3\partial t}+v^2\frac{\partial^4 y_r}{\partial\xi^4}\right)+R_p\left(\xi,t\right)=q_T\left(\xi,t\right), \end{aligned} \tag{38a}$$

where the rail foundation reaction R_p and the load q_T are described by the following expression:

$$\begin{aligned} R_p\left(\xi,t\right)&=c\left(\frac{\partial y_r}{\partial t}-v\frac{\partial y_r}{\partial\xi}\right)-\frac{EIc}{\kappa AG}\left(\frac{\partial^3 y_r}{\partial\xi^2\partial t}\right.\\ &\left.-v\frac{\partial^3 y_r}{\partial\xi^3}\right)+\frac{\rho^2 I}{\kappa G}\left(\frac{\partial^3 y_r}{\partial t^3}-3v\frac{\partial^3 y_r}{\partial\xi\partial t^2}+3v^2\frac{\partial^3 y_r}{\partial\xi^2\partial t}\right.\\ &\left.-3v^3\frac{\partial^3 y_r}{\partial\xi^3}\right)+Uy_r-\frac{EIU}{\kappa AG}\cdot\frac{\partial^2 y_r}{\partial\xi^2}+\frac{\rho IU}{\kappa AG}\left(\frac{\partial^2 y_r}{\partial t^2}\right.\\ &\left.-2v\frac{\partial^2 y_r}{\partial\xi\partial t}+v^2\frac{\partial^2 y_r}{\partial\xi^2}\right);\\ q_T\left(\xi,t\right)&=q\left(\xi,t\right)+\frac{\rho I}{\kappa AG}\cdot\left(\frac{\partial^2 q}{\partial t^2}-2v\frac{\partial^2 q}{\partial\xi\partial t}\right.\\ &\left.+v^2\frac{\partial^2 q}{\partial\xi^2}\right)-\frac{EI}{\kappa AG}\cdot\frac{\partial^2 q}{\partial\xi^2}. \end{aligned} \tag{38b}$$

Under assumption that the load q_T and the unknown function of rail displacements y_r are represented by cosine and sine parts,

$$\begin{aligned} q_T\left(\xi,t\right)&=q_{Tc}\left(\xi\right)\cdot\cos\omega t+q_{Ts}\left(\xi\right)\cdot\sin\omega t,\\ y_r\left(\xi,t\right)&=Y_c\left(\xi\right)\cdot\cos\omega t+Y_s\left(\xi\right)\cdot\sin\omega t, \end{aligned} \tag{39}$$

differentiating functions (39) and applying the following notation:

$$\begin{aligned} w_T&=\frac{\rho Iv^2-EI}{\kappa AG};\\ S_{\text{sub}}&=N+mv^2+Uw_T+\left(\rho I+\frac{\rho EI}{\kappa G}\right)\cdot\omega^2\\ &\quad-\frac{6\rho^2 Iv^2\omega^2}{\kappa G};\\ B_{\text{sub}}&=EI+mv^2\left(w_T-\hat{\eta}\right);\quad \hat{\eta}=\frac{I}{A};\\ K_{\text{sub}}&=U-m\omega^2+\frac{\rho^2 I\omega^4}{\kappa G}-U\frac{\rho I\omega^2}{\kappa AG};\\ H_{c1}&=-cvw_T;\\ H_{c2}&=-cv\left(1-\frac{3\rho I\omega^2}{\kappa AG}\right);\\ H_{c3}&=c\omega\cdot\frac{3\rho Iv^2-EI}{\kappa AG};\\ H_{c4}&=c\omega\left(1-\frac{\rho I\omega^2}{\kappa AG}\right);\\ H_{\omega1}&=-\frac{4\rho^2 Iv^3\omega}{\kappa G}+2v\omega\left(\rho I+\frac{\rho EI}{\kappa G}\right);\\ H_{\omega2}&=\frac{4\rho^2 Iv\omega^3}{\kappa G}-2mv\omega-\frac{2\rho Iv\omega U}{\kappa AG}, \end{aligned} \tag{40}$$

lead to a possibility of transformation of formulas (38a) into the system of two ordinary differential equations related to cosine and sine parts:

$$\begin{aligned} &B_{\text{sub}}\frac{d^4Y_c}{d\xi^4}+S_{\text{sub}}\frac{d^2Y_c}{d\xi^2}+K_{\text{sub}}Y_c+H_{c1}\frac{d^3Y_c}{d\xi^3}+H_{c2}\frac{dY_c}{d\xi}\\ &\quad+H_{\omega1}\frac{d^3Y_s}{d\xi^3}+H_{c3}\frac{d^2Y_s}{d\xi^2}+H_{\omega2}\frac{dY_s}{d\xi}\\ &\quad+H_{c4}Y_s=q_{Tc}(\xi);\\ &B_{\text{sub}}\frac{d^4Y_s}{d\xi^4}+S_{\text{sub}}\frac{d^2Y_s}{d\xi^2}+K_{\text{sub}}Y_s+H_{c1}\frac{d^3Y_s}{d\xi^3}+H_{c2}\frac{dY_s}{d\xi}\\ &\quad+H_{\omega1}\frac{d^3Y_c}{d\xi^3}+H_{c3}\frac{d^2Y_c}{d\xi^2}+H_{\omega2}\frac{dY_c}{d\xi}\\ &\quad+H_{c4}Y_c=q_{Ts}(\xi). \end{aligned} \tag{41}$$

The right-hand sides of these two equations, representing the load, obtain the following form:

$$\begin{aligned} q_{Tc}(\xi)&=q_c(\xi)\cdot Q_{\omega1}+w_T\cdot\frac{d^2q_c}{d\xi^2}-Q_{\omega2}\cdot\frac{dq_s}{d\xi};\\ q_{Ts}(\xi)&=q_s(\xi)\cdot Q_{\omega1}+w_T\cdot\frac{d^2q_s}{d\xi^2}+Q_{\omega2}\cdot\frac{dq_c}{d\xi};\\ Q_{\omega1}&=1-\frac{\rho I\omega^2}{\kappa AG};\ Q_{\omega2}=\frac{2\nu\omega\rho I}{\kappa AG}. \end{aligned} \tag{42}$$

The quantities q_c and q_s represent cosine and sine part of realistic load. w_T is defined above (see formulas (40)). The functions Y_c, Y_s, q_c, and q_s are expressed in terms of Fourier series in the assumed interval $[0,\lambda]$ (see (34)). By substituting the functions Y_c, Y_s, q_c, q_s and their derivatives to (41) and (42), one can obtain, after rearranging, the following system of algebraic equations for unknown values A_i, B_i, C_i, D_i:

$$\begin{aligned} &A_i[P_{1T}]+B_i[P_{2T}]+C_i[P_{3T}]+D_i[P_{4T}]=a_{Ti},\\ &A_i[-P_{2T}]+B_i[P_{1T}]+C_i[-P_{4T}]+D_i[P_{3T}]=b_{Ti},\\ &A_i[-P_{3T}]+B_i[-P_{4T}]+C_i[P_{1T}]+D_i[P_{2T}]=c_{Ti},\\ &A_i[P_{4T}]+B_i[-P_{3T}]+C_i[-P_{2T}]+D_i[P_{1T}]=d_{Ti}, \end{aligned} \tag{43}$$

$$\begin{aligned} Y_{c0}&=\frac{a_{T0}[P_{5T}]-c_{T0}[P_{6T}]}{(P_{5E})^2+(P_{6E})^2},\\ Y_{s0}&=\frac{c_{T0}[P_{5T}]+a_{T0}[P_{6T}]}{(P_{5T})^2+(P_{6T})^2}, \end{aligned} \tag{44}$$

where

$$\begin{aligned} P_{1T}&=B_{\text{sub}}\Omega_i^4-S_{\text{sub}}\Omega_i^2+K_{\text{sub}},\\ P_{2E}&=-H_{c1}\Omega_i^3+H_{c2}\Omega_i,\\ P_{3E}&=-H_{c3}\Omega_i^2+H_{c4},\\ P_{4E}&=-H_{\omega1}\Omega_i^3+H_{\omega2}\Omega_i,\\ P_{5E}&=K_{\text{sub}},\\ P_{6E}&=H_{c4}. \end{aligned} \tag{45}$$

The quantities for the right-hand side of (45) are defined by (40), whereas load coefficients (right-hand side of (43) and coefficients in expression (44)) are described as follows:

$$\begin{aligned} a_{T0}&=a_0\cdot Q_{\omega1};\\ c_{T0}&=c_0\cdot Q_{\omega1},\\ a_{Ti}&=a_i\cdot Q_{\omega1}-a_i\cdot w_T\cdot\Omega_i^2-d_i\cdot Q_{\omega2}\cdot\Omega_i,\\ b_{Ti}&=b_i\cdot Q_{\omega1}-b_i\cdot w_T\cdot\Omega_i^2+c_i\cdot Q_{\omega2}\cdot\Omega_i,\\ c_{Ti}&=c_i\cdot Q_{\omega1}-c_i\cdot w_T\cdot\Omega_i^2+b_i\cdot Q_{\omega2}\cdot\Omega_i,\\ d_{Ti}&=d_i\cdot Q_{\omega1}-d_i\cdot w_T\cdot\Omega_i^2-a_i\cdot Q_{\omega2}\cdot\Omega_i. \end{aligned} \tag{46}$$

3.2.1. Equivalence between the E-B and the Timoshenko Beams in the Case of Varying Load. It is easy to observe that solution for rail displacements determined in the case of the Timoshenko beam ((43) and (44) with notation (45) and (46)) has the same mathematical form as solution for rail displacements determined in the case of the Euler-Bernoulli beam (formulas (35) and (36) with notation (37)). Therefore, the following theorem can be formulated.

Theorem 4. *For the linear model of the track, if load is varying in time, the steady-state solution for the Timoshenko beam (see (38a)) is equivalent to the steady-state solution for the Euler-Bernoulli beam (see (30)), with accuracy determined by approximation of rail displacements by using Fourier series in finite interval* $[0,\lambda]$, *in terms of the following substitution in matrix (35) and (36):*

$$\begin{aligned} P_{jE}&\coloneqq P_{jT};\quad j=1,\dots,6,\\ a_0&\coloneqq a_{T0};\\ c_0&\coloneqq c_{T0};\\ \alpha_i&\coloneqq\alpha_{Ti};\quad \alpha=a,b,c,d, \end{aligned} \tag{47}$$

where P_{jT} $(j=1,\dots,6)$ *defines formulas (45) and* a_{T0}, c_{T0}, *and* α_{Ti} $(\alpha=a,b,c,d)$ *are described by (46).*

Proof. The proof is omitted as a direct observation arising from results and analyses presented in Sections 3.1 and 3.2. □

Certain remarks can be formulated for practical interpretation of the above theorem:

(1) The Timoshenko and the Euler-Bernoulli beams analogy concern only mathematical form of solution of the steady-state response.

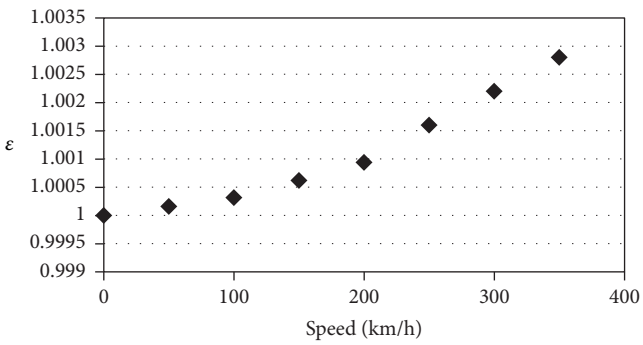

Figure 8: Effect of train speed on relative rail bending stiffness for the Timoshenko and Euler-Bernoulli beams ($\varepsilon = EI/B_{\text{sub}}$).

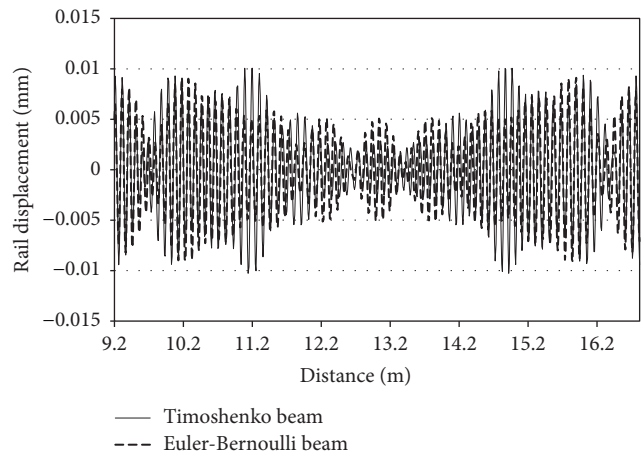

Figure 9: Steady-state response of rail modelled by Timoshenko and Euler-Bernoulli beams ($\Delta P = 10$ kN, $\omega = 5236$ rad/s, $v = 300$ km/h).

(2) For comparative analysis of the steady-state response of Timoshenko and Euler-Bernoulli beams under moving load oscillating with circular frequency ω and having an amplitude ΔP, the same track parameters as used in Sections 2.1 and 2.2 are considered. Firstly, the substitute bending stiffness of Timoshenko beam is analysed:

$$B_{\text{sub}} = EI + mv^2 \left(\frac{\rho I v^2 - EI}{\kappa AG} - \frac{I}{A} \right) \tag{48}$$

This quantity expresses the constant associated with the fourth derivative (comp. (41)). The comparison concerns the bending stiffness EI for the Euler-Bernoulli beam and formula (48). Figure 8 shows the effect of train speed on the relation $\varepsilon = EI/B_{\text{sub}}$. As can be seen, in the case of 60E1 rail typically used in Europe for main railway lines decrease of rail bending stiffness B_{sub} for Timoshenko beam in relation to the bending stiffness EI for Euler beam is on the level less than 0.3% for speed up to 350 km/h. It means that, in practical calculations, simple rail bending stiffness EI can be used, also if Timoshenko beam is taken into account.

(3) Figure 9 presents the steady-state response of Timoshenko and Euler-Bernoulli beams to a set of distributed forces with amplitudes $\Delta P = 10$ kN circular frequencies $\omega = 236$ rad/s (833.33 Hz) and train speed $v = 300$ km/h. One can see that the difference between displacements of rail modelled by Timoshenko and Euler-Bernoulli beams is quite significant: the maximum value exceeds 10%.

4. Conclusions

In this paper, analytical solutions of railway track response to moving forces are studied in the case of linear models. The obtained results lead to the following conclusions:

(1) Dynamic equation of vertical track motion in the case of the Euler-Bernoulli beam resting on elastic foundation and subjected to a constant moving load is equivalent to a static beam equation with substitute axial force for fixed unit mass of beam and train speed. In the case of the Timoshenko beam, the substitute axial load depends additionally on shear properties of rail section, rail bending stiffness, and subgrade stiffness. This conclusion is valid for each method applicable to solution in the moving coordinate system in the case of constant speed, due to its formal justification based on mathematical formulation of the problem taking into account equations of mathematical physics only. Depending on analysed features of the modelled railway system, an appropriate approach based on the Euler-Bernoulli or the Timoshenko beam formulation can be applied. The obtained results allow suggesting that the Euler-Bernoulli beam should be preferred in the scope discussed in this paper, due to its simpler formulation leading to relatively easier computations.

(2) Damping properties of track (beam) foundation can be treated as an additional substitute load taking into account a substitute axial force in the static case. The equivalence between dynamic and static responses of beam resting on elastic foundation with damping is clarified in terms of mathematical method of solution for fixed unit mass and train speed. This equivalence is shown by using the Fourier series representation for both the distributed vertical load and the vertical rail displacement. The equivalence remains correct for both track models: the Euler-Bernoulli and the Timoshenko beams.

(3) The Timoshenko beam described by a single equation is an analogy of the Euler-Bernoulli beam for both constant and variable moving distributed loads, in terms of form of solution in the moving coordinate system.

(4) The set of theorems formulated in the paper can be treated as an important part of fundamentals of the theory of railway track dynamics, being at the same time a trial approach to the systematization of the subject principles.

Appendix

Original approach presented by Timoshenko [28] provides the beam equation considering rotary inertia and shear effect in the following form:

$$EI\frac{\partial^4 y}{\partial x^4}+\frac{\rho}{A}\frac{\partial^2 y}{\partial t^2}-\rho I\left(1+\frac{E}{\kappa G}\right)\frac{\partial^4 y}{\partial x^2\partial t^2}+\frac{\rho^2 I}{\kappa G}\frac{\partial^4 y}{\partial t^4}=0, \tag{A.1}$$

where y is transverse displacement [m], E is Young's modulus of beam [N/m^2], I is moment of inertia of cross-sectional area [m^4], beam mass density [kg/m^3], A is cross-sectional area [m^2], G is shear modulus [N/m^2],and κ is shear coefficient of beam cross section.

Equation (A.1) does not include an external load. Forced vibrations of Timoshenko beam are presented in, for example, [25]. The beam equation is formulated in this case as follows

$$EI\frac{\partial^4 y}{\partial x^4}+\frac{\rho}{A}\frac{\partial^2 y}{\partial t^2}-\rho I\left(1+\frac{E}{\kappa G}\right)\frac{\partial^4 y}{\partial x^2\partial t^2}+\frac{\rho^2 I}{\kappa G}\frac{\partial^4 y}{\partial t^4}=q(x,t)-\frac{EI}{\kappa AG}\frac{\partial^2 q}{\partial x^2}+\frac{\rho I}{\kappa AG}\frac{\partial^2 q}{\partial t^2}. \tag{A.2}$$

The left-hand side of (A.2) is similar to this one in (A.1), whereas q [N/m] is an external load distribution.

If viscoelastic continuous foundation of beam is taken into account (railway track case), then one usually uses two equations describing the Timoshenko beam (e.g., [26, 29]):

$$\begin{gathered}\frac{\rho}{A}\frac{\partial^2 y}{\partial t^2}+c\frac{\partial y}{\partial t}+Uy+\kappa AG\left(\frac{\partial\psi}{\partial x}-\frac{\partial^2 y}{\partial x^2}\right)=q(x,t),\\ EI\frac{\partial^2\psi}{\partial x^2}+\kappa AG\left(\frac{\partial y}{\partial x}-\psi\right)-\rho I\frac{\partial^2\psi}{\partial t^2}=0,\end{gathered} \tag{A.3}$$

where Ψ is beam slope due to bending, U is elastic coefficient of beam foundation [N/m^2], and c is damping coefficient of foundation [Ns/m^2].

After rearrangement of the two above equations, one can obtain the following equation for Timoshenko beam deflection:

$$\begin{aligned}&EI\frac{\partial^4 y}{\partial x^4}+\frac{\rho}{A}\frac{\partial^2 y}{\partial t^2}-\rho I\left(1+\frac{E}{\kappa G}\right)\frac{\partial^4 y}{\partial x^2\partial t^2}+\frac{\rho^2 I}{\kappa G}\frac{\partial^4 y}{\partial t^4}\\ &\quad+c\frac{\partial y}{\partial t}+\frac{c\rho I}{\kappa AG}\frac{\partial^3 q}{\partial t^3}-\frac{cEI}{\kappa AG}\frac{\partial^3 y}{\partial^2\partial t}+Uy\\ &\quad+U\frac{\rho I}{\kappa AG}\frac{\partial^2 y}{\partial t^2}-U\frac{EI}{\kappa AG}\frac{\partial^2 y}{\partial x^2}\\ &=q(x,t)+\frac{\rho I}{\kappa AG}\frac{\partial^2 q}{\partial t^2}-\frac{EI}{\kappa AG}\frac{\partial^2 q}{\partial x^2}.\end{aligned} \tag{A.4}$$

One should note that various authors use the term Timoshenko beam for different equations. For example, [30] describes the Timoshenko beam by using the following expression:

$$\begin{aligned}&EI\frac{\partial^4 y}{\partial x^4}+\frac{\rho}{A}\frac{\partial^2 y}{\partial t^2}-\rho I\left(1+\frac{E}{\kappa G}\right)\frac{\partial^4 y}{\partial x^2\partial t^2}+\frac{\rho^2 I}{\kappa G}\frac{\partial^4 y}{\partial t^4}\\ &\quad+c\frac{\partial y}{\partial t}+Uy=q(x,t).\end{aligned} \tag{A.5}$$

It can be seen that four terms related to damping and elastic properties of beam foundation, as well as derivatives of the load, are neglected in the above (A.5) compared to (A.4). Further doubts arise from consideration of the axial force included in Timoshenko beam equation. In [31], equation of Timoshenko beam without foundation is considered:

$$EI\frac{\partial^4 y}{\partial x^4}-\frac{\rho EI}{\kappa G}\frac{\partial^4 y}{\partial x^2\partial t^2}+\frac{\rho}{A}\frac{\partial^2 y}{\partial t^2}+N\frac{\partial^2 y}{\partial x^2}=0. \tag{A.6}$$

Here in (A.6), in relation to (A.1)-(A.2) and (A.4)-(A.5), the fourth derivative with respect to time is neglected and a simplified expression of fourth-order mixed derivative is proposed. It can be shown that these changes lead to the error of order around 13% in the case of typical track parameters. In the contrast, in [32–34], two equations are considered:

$$\begin{gathered}\kappa AG\left(\frac{\partial\psi}{\partial x}-\frac{\partial^2 y}{\partial x^2}\right)+N\frac{\partial^2 y}{\partial x^2}-\rho A\frac{\partial^2 y}{\partial t^2}=q(x,t),\\ EI\frac{\partial^2\psi}{\partial x^2}+\kappa AG\left(\frac{\partial y}{\partial x}-\psi\right)-\rho I\frac{\partial^2\psi}{\partial t^2}=0.\end{gathered} \tag{A.7}$$

After rearrangement of these two equations, one can obtain the part of equation for the beam deflection depending on axial force:

$$\begin{aligned}f(N)&=EI\left(1-\frac{N}{\kappa AG}\right)\frac{\partial^4 y}{\partial x^4}\\ &\quad+\left(\frac{\rho IN}{\kappa AG}-\rho I-\frac{\rho EI}{\kappa G}\right)\frac{\partial^4 y}{\partial x^2\partial t^2}+N\frac{\partial^2 y}{\partial x^2}.\end{aligned} \tag{A.8}$$

Assuming that the axial force comes from the rail temperature changes ΔT (in relation to neutral temperature), one can write the following [26]:

$$\frac{N}{\kappa AG}=\frac{\alpha EA\Delta T}{\kappa AG}=\frac{\alpha\Delta TE}{\kappa G}. \tag{A.9}$$

This shows that the above term is independent of a cross-sectional area of the beam (rail). For typical rail steel parameters: $E = 210 * 10^3$ MN/m^2; $G = 77 * 10^3$ MPa $= 77 * 10^3$ MN/m^2; $\kappa = 0.4$; [1/°C]; maximal rail temperature increase/decrease $\Delta T = \pm 50$°C; and the thermal expansion coefficient $\alpha = 1.15 * 10^{-5}$ 1/°C, one obtains

$$\left|\frac{\alpha EA\Delta T}{\kappa AG}\right|<0.004=4‰. \tag{A.10}$$

Thus, the considered terms can be neglected in terms of railway tracks applications. Knowing that steel materials (G; E; α) and geometrical parameters (especially κ) may vary

for various types of rails, expression (A.8) can be written as follows:

$$f(N) \cong N\frac{\partial^2 y}{\partial x^2} + EI\frac{\partial^4 y}{\partial x^4} - \rho I\left(1 + \frac{E}{\kappa G}\right)\frac{\partial^4 y}{\partial x^2 \partial t^2}. \quad \text{(A.11)}$$

One should underline that, in the case of the Euler-Bernoulli beam, the problem of the axial force introduction is clearly interpretable, whereas in the case of the Timoshenko beam the axial force can be introduced in various ways, by inclusion of appropriate term in either equation for vertical deflection or torsional vibrations. Hence it is worth investigating how important is the difference between these two approaches, depending on the application analysed. In present paper, the presented analysis deals with typical parameters of railway track only. One can see that also other effects, for example, those associated with fourth derivative with respect to the time variable or derivatives of external load, are taken into account or omitted depending on the actual Timoshenko beam formulation.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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