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Research Article

Fractional Critical Damping Theory and Its Application in Active Suspension Control

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In this paper, the existence condition of critical damping in 1 DOF systems with fractional damping is presented, and the relationship between critical damping coefficient and the order of the fractional derivative is derived. It shows only when the order of fractional damping and its coefficient meet certain conditions, the system is in the critical damping case. Then the vibration characteristics of the systems with different orders located in the critical damping set are discussed. Based on the results, the classical skyhook damping control strategy is extended to the fractional one, where a switching control law is designed to obtain a more ideal control effect. Based on the principle of modal coordinate transformation, a new design method of fractional skyhook damping control for full-car suspension is given. The simulation results show that the proposed control method has a good control effect, even in some special cases, such as roads bumps.

1. Introduction

The vibrations of linear 1 DOF systems with ordinary damping can be classified as underdamped, critically damped, and overdamped according to the magnitude of the damping coefficient. Critical damping is defined as the threshold between overdamping and underdamping. In the case of critical damping, the oscillator returns to the equilibrium position as quickly as possible, without oscillating, and passes it once at most [1]. Considering the particularity of critical damping, it is frequently studied in other systems. The criterion for critical damping of viscously damped multi-degree-of-freedom systems is provided by Bulatovic [2]. The existence conditions for the critical damping in second-order pendulum-like systems are established by Li et al. [3]. A general method that determines the "critical damping surfaces" of a certain linear continuous dynamic system is proposed by Beskos and Boley [4]. However, so far, there are only a few researches on the critical damping in fractionally damped systems. In 1984, Torvik and Bagley [5] proposed a mechanical model with fractional derivatives in the study of the motion of a rigid plate immersed in a Newton fluid, and the study results in [6, 7] make the fractional calculus attractive for many engineers and technicians [8].

Vehicle suspension is an important component for improving the driving comfort and the handling performance [9], the research on its control strategy is a hot spot. In these control approaches, the skyhook control strategy proposed by Karnopp et al. [10, 11] is widely applied because of its simple algorithm and good control performance. The classical skyhook control principle is based on a SDOF vibration system, which is suitable for the vertical vibration control of two DOFs quarter-car models. In recent years, many scholars have studied the application of skyhook algorithm in full-car suspension model. The mainstream skyhook control strategies for full-car suspension systems are based on physical thinking; these strategies are the application extensions of the classical skyhook method that is widely used to control the quarter-vehicle suspension systems. The full-car suspension model is regarded as a simple combination of four 1/4 subsuspension models, and it is assumed that there is a "skyhook" connected with each 1/4 car body by four skyhook dampers to control the vibration of the car body, whereas the full-vehicle suspension has requirements of multiobjective suspension performances involving the vertical, pitch, and roll motions [12]. Therefore, the problem of how to coordinate the forces of the four independent controllers to keep a good body posture [13] needs to be solved and the typical solution is adding a decision-making system.

Although mainstream algorithms can achieve a good control performance, it is inconsistent with the original skyhook control principle. From the perspective of mathematical principles, the classic skyhook control principle is used to control a SDOF system with one skyhook damper. Whereas the vehicle suspension is a system with multi-DOFs (the existing models have seven or more DOFs), thus the same number of controllers is required. However, in reality, there are only four controllers. How to tackle this problem?

This work is divided into two parts. In the first part, the critical damping in fractional order system is studied. The existence conditions of the critical damping are given, and the relationship order is derived. Then the vibration attenuation characteristics of fractional critical damping systems with different order are discussed. In the second part, the fractional critical damping is applied to the control strategy of the vehicle suspension system. The method of modal decoupling is used to solve the problem that the number of required controllers is not consistent with that of the actual ones. In the modal space, the classical skyhook control strategy is used for depressing the decoupled single mode vibration. Here, the fractional critical damping coefficients are chosen as the skyhook damping coefficients. In this way, the number of designed controllers is consistent with that of DOFs of the system, then these modes are recoupled and the actual controllers are used to control the suspension. A four-wheel-correlated random road time domain model is built to test the effect of fractional derivative skyhook control strategy; a road bump is especially designed to demonstrate the advantages of the fractional derivative critical damping.

The organization of the paper is as follow. In Section 2, the conditions of fractional damped systems being in critical damping case are given first. Then the properties of the vibration with critical damping are studied. In Section 3, a new fractional skyhook control algorithm for full-car suspension systems is proposed. In Section 4, the simulation results are discussed. Conclusions are given in Section 5.

2. Critical Damping of the System with Fractional Derivative Damping

2.1. Formula Derivation. The free vibration differential equation of a SDOF system with fractional derivative damping has the form

$$m\ddot{x}(t) + c_0 D^{\alpha} x(t) + kx(t) = 0, \tag{1}$$

where $x(t)$ is the displacement, ${}_0D^{\alpha}x(t)$ is the fractional time derivative of $x(t)$, and m, c, and k are the mass, damping, and stiffness coefficient, respectively.

There are many definitions for fractional derivatives [14], among which Riemann-Liouville definition and Caputo definitions are most widely used [15]. The former is frequently used for problem description because of its demand moderately for the continuity of the function. The latter has the same Laplace transform as the integer order one, so it is widely used in control theory. In this paper, the fractional derivative damping force is regarded as a control force to study the properties of free damped vibration of the system, so the Caputo definition is used here.

By the Laplace transform method, the characteristic equation of the system takes the form

$$ms^2 + cs^{\alpha} + k = 0, \tag{2}$$

where s is the complex variable. By substituting its polar form $s = re^{i\theta}$ into (2), we have

$$mr^2 e^{i(2\theta)} + cr^{\alpha} e^{i(\alpha\theta)} + k = 0, \tag{3}$$

Considering the Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$, (3) takes the form

$$mr^2 (\cos 2\theta + i\sin 2\theta) + cr^{\alpha}(\cos\alpha\theta + i\sin\alpha\theta) + k = 0, \tag{4}$$

The establishment-condition of (4) is that both the real and imaginary parts are equal to zero, so we obtain

$$\begin{aligned} mr^2\cos 2\theta + cr^{\alpha}\cos\alpha\theta + k &= 0 \\ mr^2\sin 2\theta + cr^{\alpha}\sin\alpha\theta &= 0. \end{aligned} \tag{5}$$

It is known that when the imaginary part of the roots of (2) is non-zero, the damped free motion of the system is always oscillating. To avoid the oscillation, the characteristic roots must lie in the negative real axis. Assume that $\theta = (2k_1 + 1)\pi$ where k_1 is an integer, so $\cos 2\theta = 1$ and $\sin 2\theta = 0$ are obtained, then (5) can be simplified as

$$mr^2 + cr^{\alpha}\cos\alpha\theta + k = 0, \tag{6}$$

$$cr^{\alpha}\sin\alpha\theta = 0. \tag{7}$$

The establishing condition for (7) is $\sin\alpha\theta = 0$, which means that $\cos\alpha\theta = \pm 1$. Therefore, it can be obtained that

$$\alpha\theta = \alpha(2k_1 + 1)\pi = k_2\pi, \tag{8}$$

where k_2 is an integer. As a result, we have

$$\alpha = \frac{k_2}{2k_1 + 1}. \tag{9}$$

We find that the set of α is dense, but the probability density of any α locating in this domain is small, so the existence condition of critical damping is strict.

From (6), a negative damping coefficient c is obtained when $\cos\alpha\theta = 1$, which represents an energy input to the system. In this case, the system oscillation is strengthened, and there is no critical damping, while it is the opposite when $\cos\alpha\theta = -1$; that is, k_2 is odd, so substituting $\cos\alpha\theta = -1$ into (6) and then (10) is obtained. In summary, in (9) k_1 is an integer, k_2 is odd, and $\alpha \in (0, 2)$. The existence conditions of critical damping in the vibration systems with fractional derivative damping and its calculation formula are presented.

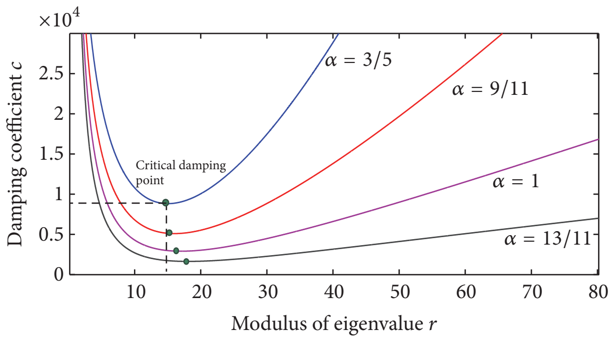


FIGURE 1: The relation between the three parameters in (12).

For linear 1 DOF fractionally damped systems, only when (9) is satisfied by the order of fractional operator, there is a critical value of damping coefficients. To make the solutions of (1) be without oscillation, the relation between the damping coefficient and the order is

$$c = \frac{mr^2 + k}{r^\alpha}, \tag{10}$$

where $r \in R^+$. In (10), when $dc/dr = 0$, i.e. $r = \sqrt{k\alpha/m(2-\alpha)}$, we have the minimum value of damping coefficient c which represents the critical damping coefficient c_c.

The curves that represent the relation between the variables in (10) are plotted in Figure 1. Take $\alpha = 3/5$, for example, the lowest point of the curve represents the critical damping point and its corresponding damping coefficient is the critical value of damping coefficient. It is worth noting that many previous researches on 1 DOF fractionally damped systems focus on the solutions of the characteristic equations. From this perspective, we find when $c < c_c$, the characteristic equations only have complex or conjugate roots and they have negative real roots when $c \geqq c_c$. Therefore, when $c > c_c$, it represents the overdamping coefficient, and when $c < c_c$, it is the underdamping coefficient. In the case of critical damping, the characteristic equation has the root $s = -r$, which represents the convergence rate. When α increases from 0 to 2, the critical damping point is shifted to the lower right in the figure, which indicates that with larger α, it turns out a smaller c_c and larger r; that is, with a smaller eigenvalue, the system is a faster convergent.

It should be noted also that Sakakibara [16] studied the properties of vibration with fractional derivative damping of order 1/2. By the analysis of the solutions of (1), it is concluded that there is no critical value of damping coefficient, which is not against the conclusions of this paper because $\alpha = 1/2$ is not located in the set represented by (9). In fact, it is easy to understand that by reduction to absurdity, that is, when the roots s are negative real, they do not hold by substituting $\alpha = 1/2$ into (2). This means that when $\alpha = 1/2$, the eigenvalues cannot be negative real and always contain an imaginary part. Furthermore, we find that when $\alpha = 1$, the critical damping coefficients $c_c = 2\sqrt{mk}$, $r = \sqrt{k/m}$, and $s = -\sqrt{k/m} = -\omega_n$ are obtained, which are consistent with the critical damping in an integer order system. Because it is not our main objective to solve the equation and the critical damping coefficients can be obtained without analyzing the solutions, we will not return to these questions here and refer the interested reader to [17, 18]. As is shown in Figure 2, when $\alpha = 9/11$, the critical damping coefficient is figured out according to the above analysis.

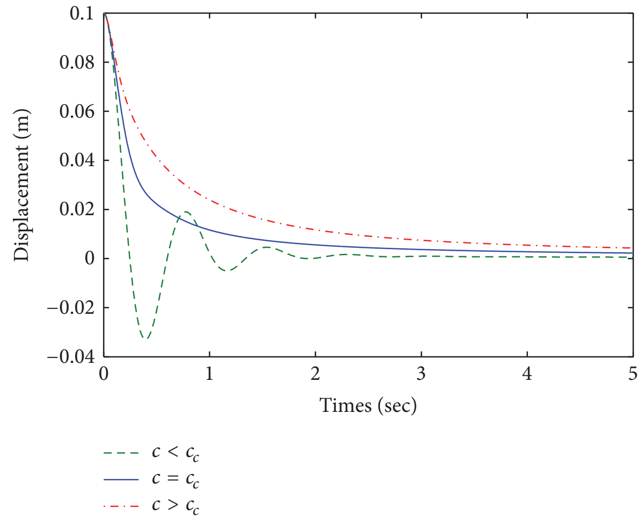


FIGURE 2: When $\alpha = 9/11$, the three cases of free decay vibration.

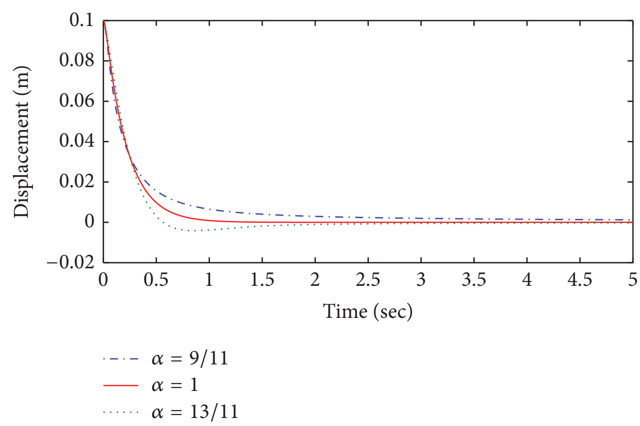


FIGURE 3: The curves of free damped motion of critical damping system with different orders.

2.2. Properties of Vibration with Fractional Derivative Critical Damping. When $\alpha \in (0, 1)$, the fractional damping plays not only the role of a conventional damping, but also the role of a supplementary spring [19]. If $\alpha \to 0$ or $\alpha \to 2$, the damping effect of the system will be weakened, and there is a typical behavior of the oscillation. Furthermore, the fractional order systems are easily affected by the initial state. Therefore, in practice, α should lie within the range of engineering interest.

Figure 3 shows the curves of decaying free motions of critical damping systems with different orders under the initial state $x_0 = 0.1$, $\dot{x}_0 = 0$. It shows that in the case of the same other parameters, the systems with a large α return back to equilibrium position faster. When $\alpha \in (0, 1)$, the systems are relatively slow as they go back to balance position

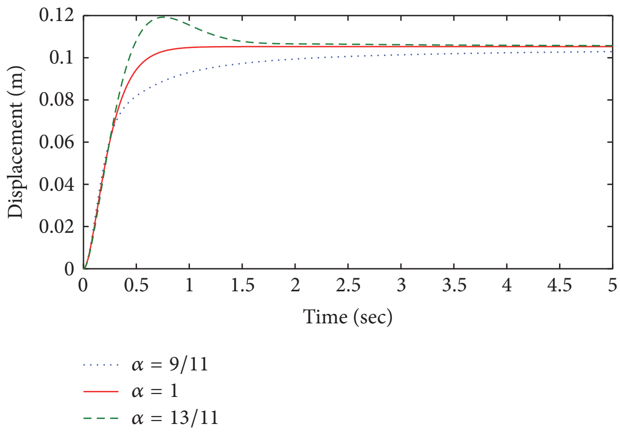


Figure 4: Step response curves of critical damping systems with different orders.

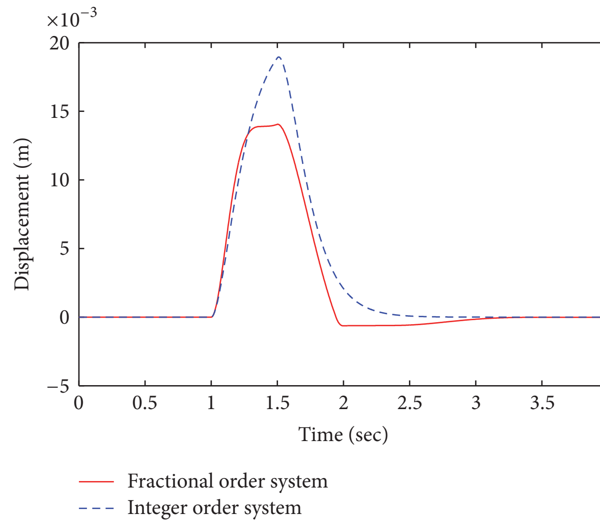


Figure 5: Impulse response curves of integer order and fractional orders switched systems.

and do not cross it. Otherwise when $\alpha \in (1,2)$, the systems are relatively fast and cross through the static equilibrium position once (overshoot occurs), which is different from the ordinary critical damping. Although the systems with a large α return back to the equilibrium position at a faster speed, it is easily to be aroused by external excitation such as step input; the response curves are shown in Figure 4.

It is expected that under the premise of nonoscillatory, the system is not easy to be aroused by external excitation and can return back to the equilibrium position as quickly as possible when there is no external force. A switch control law is designed to make the displacement as small as possible when the system is away from the equilibrium position and to limit the time it takes to reach the asymptotically stable position when there is no external force. The designed control law is

$$u = \begin{cases} c_1 D^{\alpha_1} x, & \alpha_1 < 1, x\dot{x} > 0, \\ c_2 D^{\alpha_2} x, & \alpha_2 \geq 1, x\dot{x} \leq 0, \end{cases} \tag{11}$$

where u is the control force, α_1 and α_2 are the orders of fractional derivative, and c_1 and c_2 are the corresponding fractional derivative critical damping coefficients, x is the displacement. The effectiveness of the proposed control strategy is tested by a pulse excitation. Figure 5 shows that, under impulse input, the switching control law makes the vibration performance of the fractional order system better than that of the integer order one.

3. Vehicle Skyhook Control Strategy

According to vehicle dynamics theory, the dynamic model of the vehicle with seven DOFs is established. The seven DOFs Z_b, θ, Ψ, Z_{wA}, Z_{wB}, Z_{wC}, and Z_{wD} are the heave, pitch, roll displacement of the body, and the four wheels displacement, respectively. This model is similar to those used by [20, 21], here the matrix differential equation of the model can be described as

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{K}_t\mathbf{Z}_g + \mathbf{R}\mathbf{U}, \tag{12}$$

where $\mathbf{X} \in \mathbf{R}^{(7\times1)}$ is a vector consisting of Z_b, θ, Ψ, Z_{wA}, Z_{wB}, Z_{wC}, and Z_{wD}. $\mathbf{M} \in \mathbf{R}^{(7\times7)}$, $\mathbf{C} \in \mathbf{R}^{(7\times7)}$ and $\mathbf{K} \in \mathbf{R}^{(7\times7)}$ are the mass, damping, and stiffness matrix, respectively. $\mathbf{K}_t \in \mathbf{R}^{(7\times4)}$ is the input matrix and $\mathbf{Z}_{g(4\times1)}$ is a vector that stands for the four-wheel-related road excitation. $\mathbf{R} \in \mathbf{R}^{(7\times4)}$ is the control vector and $\mathbf{U}_{(4\times1)}$ is the vector of active control force. Equation (12) represents a passive suspension when $\mathbf{U}_{(4\times1)}$ is a zero vector.

According to the linear vibration theory, the decoupled suspension system turns into isolated linear subsystems that can be controlled independently [22]. Therefore, a systematic modal decoupling method [23] is considered, with which the mass and stiffness matrix can be completely decoupled; however, the damping matrix cannot be completely decoupled generally. Here only the diagonal elements of the damping matrix are controlled to verify the effectiveness of the control strategy. The matrix differential equation of the fully decoupled system is considered

$$\overline{\mathbf{M}}\ddot{\boldsymbol{\eta}} + \overline{\mathbf{C}}_{\mathrm{Di}}\dot{\boldsymbol{\eta}} + \overline{\mathbf{K}}\boldsymbol{\eta} = \overline{\mathbf{F}} + \overline{\mathbf{U}}, \tag{13}$$

where

$$\begin{aligned} \overline{\mathbf{M}} &= \mathbf{Q}^T\mathbf{M}\mathbf{Q}, \\ \overline{\mathbf{C}} &= \mathbf{Q}^T\mathbf{C}\mathbf{Q}, \\ \overline{\mathbf{K}} &= \mathbf{Q}^T\mathbf{K}\mathbf{Q}, \\ \overline{\mathbf{F}} &= \mathbf{Q}^T\mathbf{K}_t\mathbf{Z}_g, \\ \overline{\mathbf{U}} &= \mathbf{Q}^T\mathbf{R}\mathbf{U}, \end{aligned} \tag{14}$$

$\boldsymbol{\eta} \in \mathbf{R}^{(7\times1)}$ is the vector of principal coordinates, $\boldsymbol{\eta}=\mathbf{Q}^{-1}\mathbf{X}$, $\mathbf{Q}$ is the feature matrix, and $\overline{\mathbf{C}}_{\mathrm{Di}}$ is the diagonal matrix whose diagonal elements are equal to those in vector $\overline{\mathbf{C}}$. In (13), $\overline{\mathbf{M}}$, $\overline{\mathbf{K}}$, and $\overline{\mathbf{C}}_{\mathrm{Di}}$ are seven-order diagonal matrices and, assuming that

$\overline{\mathbf{U}}$ is also a seven-order diagonal matrices, seven differential equations of independent scalar function η_i are obtained; fractional skyhook control is used here to depress each independent modal vibration. Let $u_{isky} = -c_{isky}\eta_i^{(\alpha)}$ $(i = 1 \sim 7)$, so the seven independent differential equations have the form

$$m_i\ddot{\eta} + c_i\dot{\eta} + k_i\eta + c_{isky}\eta^{(\alpha)} = F_i, \quad (i = 1 \sim 7), \tag{15}$$

where F_i is the external excitation for modal vibration systems, $c_{isky}\eta^{(\alpha)}$ represents the fractional skyhook damping force.

The free vibration equations of the modal systems are considered, namely,

$$m_i\ddot{\eta} + c_i\dot{\eta} + k_i\eta + c_{ishk}\eta^{(\alpha)} = 0, \tag{16}$$

where the control force $c_{isky}\eta^{(\alpha)}$ is used to keep the system in the case of critical damping. According to the method in Section 2, the relation between the damping coefficient and the order is obtained

$$c_{ishk} = \frac{m_i r_i^{\,2} - c_i r_i + k_i}{r_i^{\alpha_i}}. \tag{17}$$

When $r_i = (c_i(1-\alpha) + \sqrt{(c_i(1-\alpha))^2 + 4\alpha(2-\alpha)m_ik_i})/2m_i(2-\alpha)$, the fractional derivative skyhook damping coefficient c_{ishk} is equal to the fractional derivative critical damping coefficient. In the same way, it is hoped that with the fractional damping force, the modal system is not easily aroused by external force and returns to equilibrium position as fast as possible without oscillating when there is no force. Here, a switching control law is given as follows:

$$u_{ishk} = \begin{cases} c_1 D^{\alpha_1}\eta_i, & \alpha_1 < 1, \eta_i\dot{\eta}_i > 0, \\ c_2 D^{\alpha_2}\eta_i, & \alpha_2 \geq 1, \eta_i\dot{\eta}_i \leq 0. \end{cases} \tag{18}$$

In practice, with a larger or smaller α, these problems, such as the limitation of actuator force and the work efficiency of the actuator, arise. In order to achieve a relatively good control effect, only the limitation of actuator force is considered. Seven skyhook damping coefficients c_{ishk} of the system are obtained. By coordinate reduction, the final control force vector is

$$\mathbf{U} = \left(\mathbf{Q}^T\mathbf{R}\right)^{-} \times \mathbf{Q}^{-1} \times \mathbf{C}_{shk} \times \mathbf{X}^{(\alpha)}, \tag{19}$$

where $\mathbf{C}_{shk} = \mathrm{diag}([c_{ishk}])$ $(i = 1 \sim 7)$. Equation (19) represents the force of integer order skyhook damping control strategy when $\alpha = 1$. The generalized inverse matrix of $\mathbf{Q}^T\mathbf{R}$ is used here because it is not a square matrix.

4. Simulation Results and Discussions

A four-wheels-correlated random road time domain model [24] is used here and the road profile is C grade. To verify the characteristics of fractional critical damping, a work condition is designed as follows: when the simulation goes to 5s, on the left side of the vehicle, the front and rear wheels have been raised successively by road bump shaped like a sine wave with a height of 0.1 m. Vehicle suspension parameters are shown in Notations. For validating the superiority of the fractional derivative critical damping, meanwhile avoiding the following negative effects with a large or small α, in the switching control law, the orders are chosen as $\alpha_1 = 9/11$ and $\alpha_2 = 13/11$.

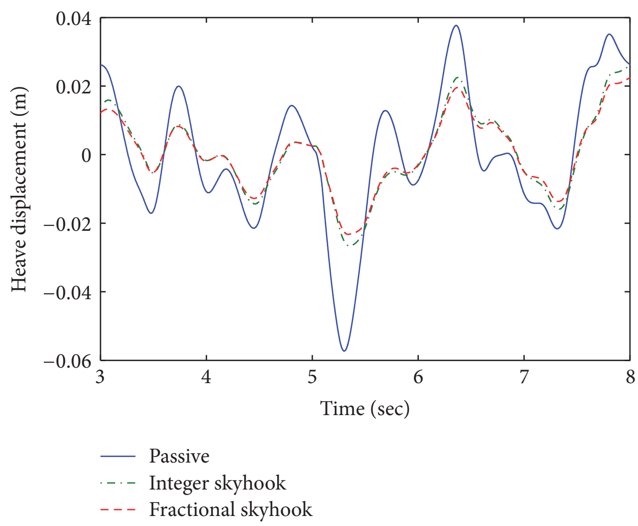
Figures 6 and 7 show that the proposed vehicle skyhook control strategy can effectively suppress the vibration of the body; both vibration amplitude and acceleration are decreased significantly; the performance especially is good after crossing the road bump. Figure 6 shows that the vibration with fractional derivative critical damping has a better performance on amplitude responses than that with integer one. And Figure 7 shows that fractional order skyhook damping control strategy has no significant deterioration in acceleration response. But for a large or small α, the acceleration responses become worse than those in integer order control strategy, and that is why the order should locate within a reasonable domain in engineering application.

Compared with many other full-car suspension control strategies, there are two main advantages for the method in this paper. Firstly, the proposed method is much more simple than most of the control methods. For example, these methods presented in [25] are also tested by a road bump and can improve the vibration performance of the vehicle, but they are too complicated. Actually, the skyhook control strategy is one of several simple and practical methods which is widely applied. Among the full-car skyhook control algorithms, a skyhook-based asynchronous semiactive controller proposed by Zhang et al. can control each subsystem independently; the results show that the peak amplitudes of body accelerations increase more than those in the passive suspension when they are tested by a pulse excitation. Therefore, it is not easy to keep a good body posture particularly when a car crosses through a road bump. The existing solution is to introduce new controls such as modularize parallel fuzzy control in [26] and human-like intelligent control in [27] and this makes the strategies complex and difficult to apply. Secondly, there are many active suspension control strategies which are designed based on a more comprehensive usage of road preview information facilitated by utilising on-board cameras and global positioning systems [28]. For example, it is provided that the road preview is available in the control method in [29]. However, our control method does not need such facilities.

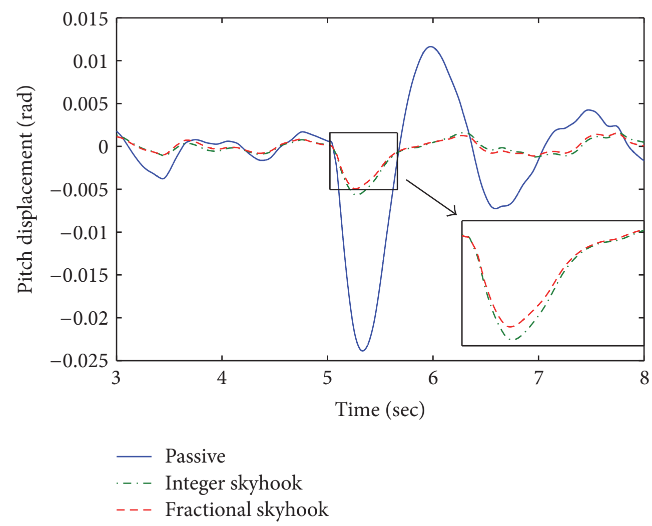
In a word, the proposed skyhook control has a simple algorithm and is consistent with the original skyhook damping scheme in principle. The strategy with integer order critical damping coefficients has a good effect, and the fractional one is seen as a supplement, which provides more parameter selection and has a better performance on amplitude responses.

5. Conclusions

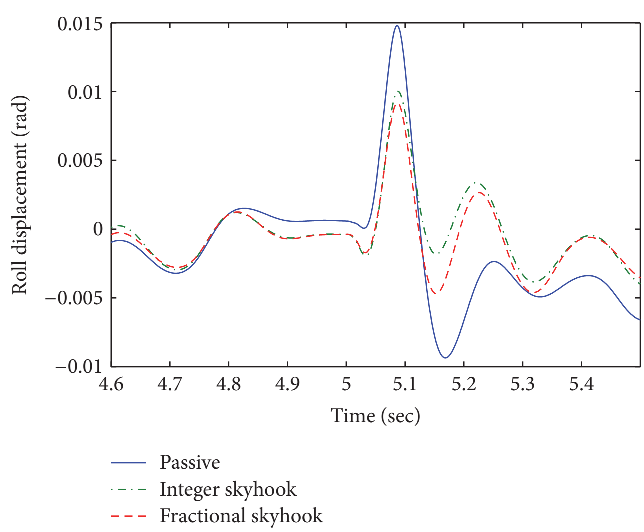
(1) The free damped motion of SDOF systems with fractional derivative damping is firstly studied. Conditions of existing critical damping are given and the relation between the



(a) Heave displacement



(b) Pitch displacement



(c) Roll displacement

Figure 6: Body motion amplitude response of three kinds of suspension.

critical damping coefficient and the order fractional derivative is derived. It is also found that when the order increases from 0 to 2, the critical damping coefficient is getting small, but it is faster to return back to equilibrium position.

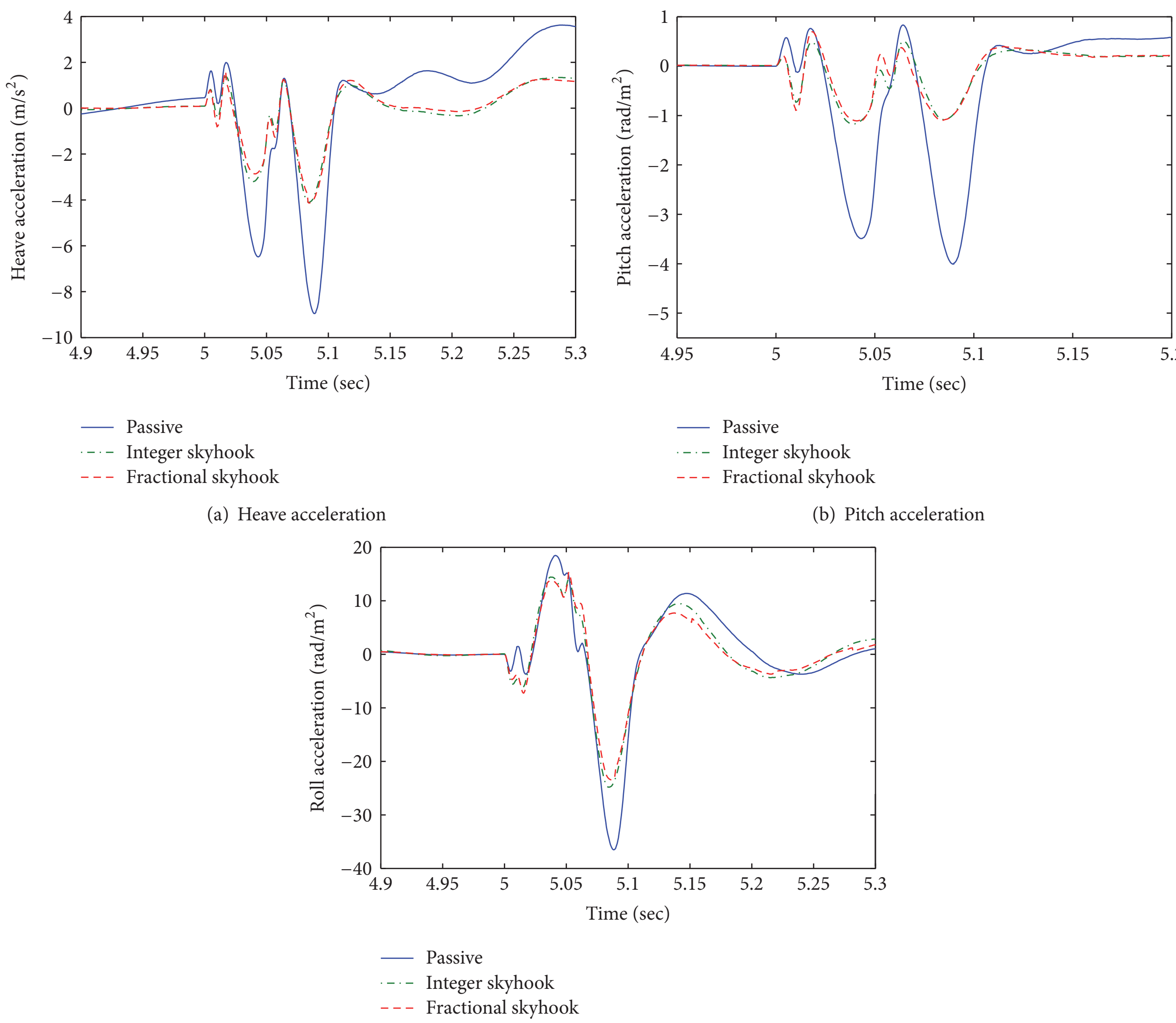
(2) Based on the mathematical thinking, a new full-car skyhook damping control strategy is proposed, which is different from the logical thinking of most scholars. The mainstream algorithm can also achieve a good performance; here, it is not the purpose to deny its effectiveness, but to give a new perspective for scholars to re-examine the intrinsic mathematical logic of classic skyhook damping principle. The fractional order critical damping coefficient is selected as the skyhook damping coefficient to clarify the superiority of proposed fractional order critical damping in practical application.

(3) Simulation results show that compared with the passive suspension, the skyhook controlled active suspension has a better performance on vibration suppression. Furthermore, the fractional skyhook controlled suspension has better responses of the body vibrating, especially when the vehicle passes the road bump. The results not only confirm the superiority of fractional critical damping, but also validate the effectiveness of this control strategy.

Abbreviations

Vehicle Parameters

M_b: Sprung mass, 810 kg
I_p: Inertia moment of vehicle pitch, 300 kg·m^2
I_r: Inertia moment of vehicle roll, 1058 kg·m^2
A: Distance from axle to 1.14 m
B: Center of gravity, 1.22 m
$K_{sf1/2}$: Front suspension stiffness, 20600 N/m
$K_{sr1/2}$: Rear suspension stiffness, 15200 N/m
$C_{f1/2}$: Front suspension damping, 1570 N/m



(c) Roll acceleration

FIGURE 7: Body motion acceleration response of three kinds of suspension.

$C_{r1/2}$: Rear suspension damping, 1760 N/m
K_w: Tire stiffness, 138000 N/m
M_{wf}: Front tire mass, 26.5 kg
M_{wr}: Rear tire mass, 24.4 kg
B_1: Distance between two tires, 1.3 m
V: Vehicle speed, 50 km/h.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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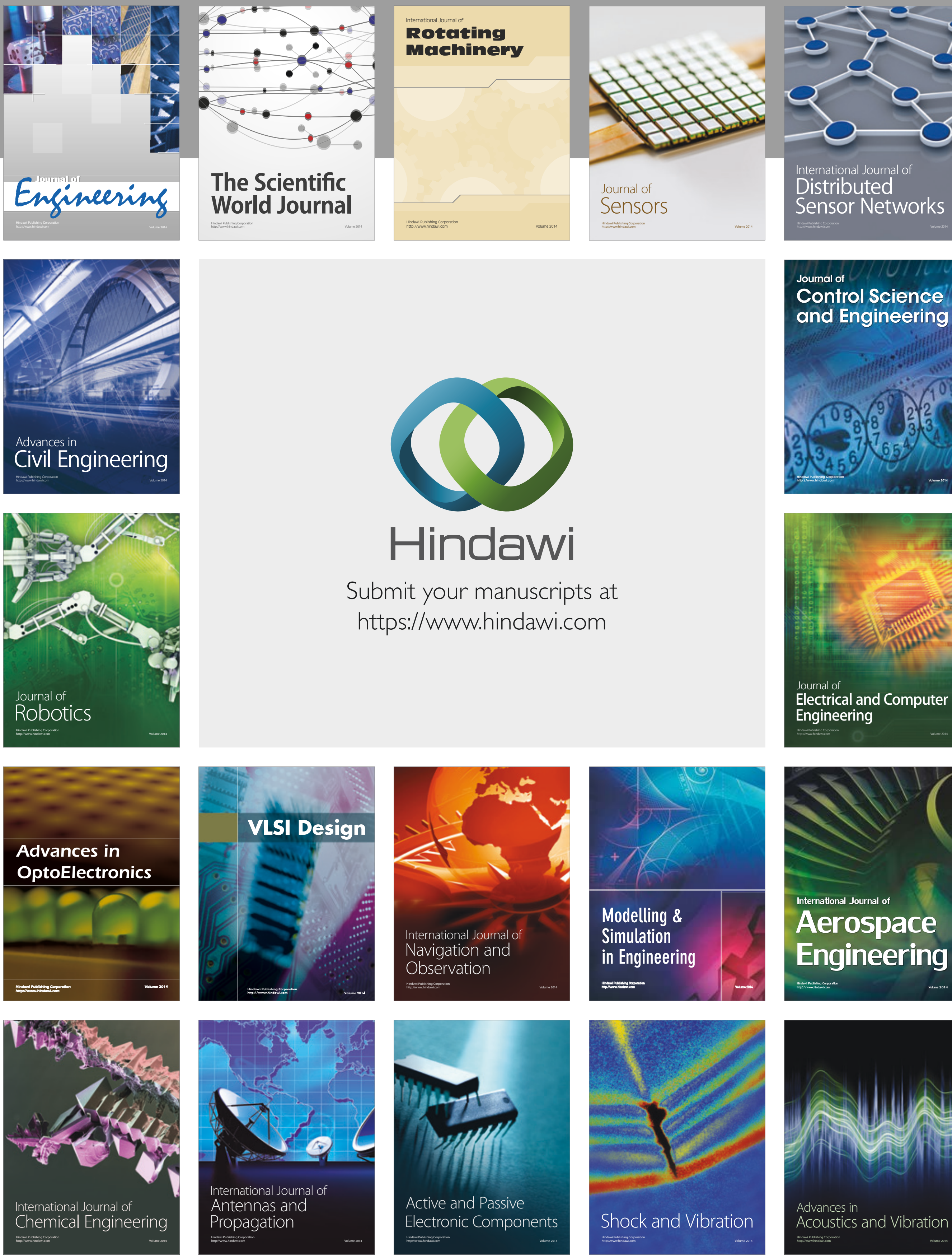
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