Impedance Synthesis Based Vibration Analysis of Geared Transmission System

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The severity of gear noise response depends on the sensitivity of geared rotor system dynamics to the transmission error. As gearbox design trending towards lighter weight and lower noise, the influence of housing compliance on system dynamic characteristics cannot be ignored. In this study, a gear-shaft-bearing-housing coupled impedance model is proposed to account for the effect of housing compliance on the vibration of geared transmission system. This proposed dynamic model offers convenient modeling, efficient computing, and ability to combine computed parameters with experimental ones. The numerical simulations on system dynamic characteristics are performed for both a rigid housing configuration and a flexible one. Natural frequencies, dynamic mesh forces, and dynamic bearing reaction loads are computed, and the housing compliance contribution on system dynamic characteristics is analyzed. Results show that increasing housing compliance will decrease the system natural frequencies and will affect the dynamic bearing reaction loads significantly but have very little influence on the dynamic mesh force. Also, the analysis shows that bearing stiffness has significant influence on the degree of housing contribution on system dynamic characteristics.

1. Introduction

Due to the time-varying mesh stiffness and manufacture error of gear teeth, geared transmission systems will inevitably induce unwanted vibration and noise. The key to predict system dynamic characteristics reasonably is to take into account influences of all main mechanical components. As an important component of a gearbox, the housing not only plays an important role in the vibration propagation and noise radiation but also affects system dynamic characteristics significantly, especially in geared transmissions of helicopters, marine vessels, and automobiles.

A simple way to analyze geared transmission system vibration and noise responses is to apply a lumped parameter model to calculate the bearing reaction loads [1–3]. A more sophisticated approach is to construct a finite element (FE)/boundary element (BE) model of gearbox housing and apply the bearing reaction loads to the housing as forced excitation to simulate structure-borne and air-borne noise responses [4, 5]. However, in most research efforts, the housing dynamics is frequently ignored and bearings are considered to be directly connected to a rigid foundation when calculating the bearing reaction loads. As these models neglect the housing compliance, they may differ from the practical case. With the gearbox design trending towards lighter weight and lower noise, the influence of housing on gearbox vibration and noise becomes more pronounced.

Dynamic modeling methods that are able to take into account the housing compliance include finite element method (FEM) [6–8], static substructure synthesis method (often referred to as simply substructure synthesis method) [9, 10], multibody dynamic modeling method [11], substructural modal synthesis method (often referred to as dynamic substructural synthesis method) [12], and substructural mobility synthesis method [13, 14]. Lim and Singh [15] reviewed the overall gearbox dynamics and discussed the major modeling challenges. Rigaud and Sabot [16] reported that natural frequency and modal shape both changed when geared transmission system is coupled with housing. Hambric et al. [6]
built a finite element model to investigate the influence of housing on the modal shape of shaft. Choy et al. [17] concluded that the influence of housing on system vibration is more pronounced in a stiffer rotor. Liu et al. [11] reported that housing compliance will deteriorate bearing vibration, but He et al. [9] showed that housing compliance will decrease vibration of both gear and bearing. Parker et al. [8, 18] established both a finite element model and a lumped parameter model and concluded that housing has limited influence on the gear dynamic transmission error but can play an important role on the bearing reaction load fluctuation and noise radiation. Abbes et al. [12] investigated vibration response of the housing and showed that housing vibration will be reduced when a stretcher is applied. Hajžman and Zeman [19] and Zhu et al. [7] also studied the housing vibration and noise in a coupled geared transmission-housing model. Jauregui et al. [20] estimated the influence of housing stiffness on the synchronization of nonlinear elements in a wind turbine gearbox. Rook and Singh [13] suggested that increasing housing stiffness, gear damping, shaft damping, or bearing damping will reduce vibration and noise. Leung and Pinnington [14] pointed out that the vibratory energy transmitted to the housing by gear tangential excitation force is much bigger than that excited by gear normal excitation force. Zhang et al. [21] evaluated the coupling loss factor between plate and gear shaft and concluded that symmetric layout reduces housing vibration. Gao et al. [22] demonstrated that high frequency vibration energy induced by torque fluctuation attenuates substantially after being transmitted through the gear-shaft-bearing path.

When the housing geometry is too complex to construct from a solid model, or too complex for finite element analysis, experimental setup has to be applied to extract the relevant parameters. For example, the geometrical design of a marine gearbox with a flexible housing, isolators, and a foundation is often very complex. Most of the previous studies applying the methods described above did not make use of the measured data directly. The impedance synthesis approach is an effective way to combine theoretical parameter with experimental ones and has been widely used in the past [23, 24]. However, this method is seldom applied in the dynamic analysis of the coupled gearbox housing and internal geared transmission. This study attempts to address these gaps.

In this study, a coupled gear-shaft-bearing-housing model is proposed based on the impedance synthesis approach. This proposed approach only needs independent modeling of each subsystem and combines their impedances at their interfacial connections instead of directly representing the entire system in a single model. The gear, shaft, and bearing impedances can be derived from a lumped parameter model, and the housing impedance can be obtained from either a finite element (FE) model or an appropriate experimental setup. The proposed modeling is convenient to implement and computationally efficient. System dynamic characteristics are computed for both a rigid housing configuration and a flexible one using this method, and the influence of housing impedance on dynamic characteristics of geared transmission system is studied.

### 2. Subsystem Impedance

Composition of a typical gearbox structure is shown in Figure 1. Gears are mounted on shafts, and gears-shafts are coupled to the housing through bearings. Conventional lumped parameter models assume bearings to be directly mounted on a rigid foundation and hence ignored the effect of gearbox housing on the system dynamic characteristics. In this study, housing compliance will be taken into consideration by coupling the entire gearbox into a single system. To absorb vibrations, the gear housing is sometimes mounted on isolators. Though isolators are ignored here, they can be modeled in the same way like the housing.

Geared transmission error excitation is applied on the gear system, while other force and moment fluctuations are ignored. Housing deformation or shaft assembly error may lead to nonparallel gearing condition and thus influences the mesh stiffness [25]. However, in order to simplify the problem, this factor is ignored and will be studied in the future.

The studied gearbox with a transmission system and housing is shown in Figure 2. The transmission system contains a helical gear pair, two shafts, and four bearings. Rotational speed of the input shaft is 1000 rev/min, and driven torque of the output shaft is 1200 N-m. Basic gear parameters are tabulated in Table 1. Parabolic profile modification is

![Figure 1: Schematic of a typical gearbox structure.](image-url)
Table 2: Shaft segments dimensions (mm).

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Figure 2: Gearbox layout design.

Power in Bearing 1 Bearing 2 Pinion Bearing 3 Wheel Bearing 4 Power out

2.1. Gear Pair Impedance. The gear pair lumped model is extensively studied by many researchers [1–3, 25–27]. One with time-varying stiffness is adopted here, and an impedance model is transformed directly from this lumped parameter representation. The spring-mass dynamic model of the gear pair is shown in Figure 3. Time-varying mesh stiffness excitation is applied, but nonlinear terms are ignored. Let the mesh element node displacement vector be \( \mathbf{x} = \{x_p, y_p, z_p, \theta_{xp}, \theta_{yp}, \theta_{zp}, x_g, y_g, z_g, \theta_{xg}, \theta_{yg}, \theta_{zg}\}^T \). Gear pair relative deflection \( \delta \) along the line of action can be yielded from

\[
\delta = \mathbf{V}_p^T \mathbf{x} - e_m, \tag{1}
\]

where \( e_m \) is the composite mesh error [26, 27], and gear profile modification is treated as profile error here. \( \mathbf{V}_p \) is the project vector transferring displacement to the line of action, which can be expressed as

\[
\mathbf{V}_p = \begin{bmatrix}
\cos \beta_b \sin \varphi, \pm \cos \beta_b \cos \varphi, \sin \beta_b, \\
\mp r_p \sin \beta_b \sin \varphi, -r_p \sin \beta_b \cos \varphi, \mp r_p \cos \beta_b, \\
-\cos \beta_b \sin \varphi, \mp \cos \beta_b \cos \varphi, -\sin \theta_{zb}, \\
\mp r_g \sin \beta_b \sin \varphi, -r_g \sin \beta_b \cos \varphi, \mp r_g \cos \beta_b
\end{bmatrix}^T,
\]

where \( r_p \) is the base circle radius of pinion, \( r_g \) is the base circle radius of gear, \( \beta_b \) is the base helix angle, \( \varphi = \alpha \mp \phi \) is the angle between the transverse line of action and \( y \)-axis, \( \alpha \) is the mesh angle, \( \phi \) is the fix angle, the top side of \( \pm \) or \( \mp \) is for anticlockwise rotation of pinion, and the bottom side is for clockwise rotation of pinion.

Equations of motion of mesh element are established according to the Newton's second law and are given by

\[
\mathbf{M} \ddot{\mathbf{x}}(t) + \mathbf{C} \dot{\mathbf{x}}(t) + \mathbf{K}(t)(\mathbf{x}(t) - \mathbf{e}(t)) = \mathbf{f}^{GP}(t), \tag{3}
\]

where \( \mathbf{M} = \text{diag}(m_p, m_p, m_p, I_{xp}, I_{yp}, I_{zp}, m_g, m_g, m_g, I_{xg}, I_{yg}, I_{zg}) \), \( \mathbf{K} = k_m \mathbf{V}_p \mathbf{V}_p^T \), \( \mathbf{C} = c_m \mathbf{V}_p \mathbf{V}_p^T \), \( \mathbf{e} \) is the equivalent...
displacement vector of composite mesh error, $\mathbf{f}^{\text{GP}}$ is the external force vector applied on the gear pair, that is, dynamic reaction force applied by the shafts here, $m_i$ ($i = p, g$) is the mass of pinion or gear, $I_{x_i}, I_{y_i}, I_{z_i}$ ($i = p, g$) are the moment of inertia about $x$, $y$, and $z$, respectively, $k_m$ is the mesh stiffness, and $c_m$ is the mesh damping that can be expressed as

$$
c_m = 2\zeta \sqrt{\frac{k_m}{(1/m_{eq,p} + 1/m_{eq,g})}}, (4)
$$

where $\zeta$ is the damping ratio that is generally equal to 0.03–0.17 [11, 28, 29], $k_m$ is the mean mesh stiffness, and $m_{eq,i} = I_{i}/r_i^2$ is the equivalent mass of gear $i$ ($i = p, g$).

Time-varying mesh stiffness can be decomposed into constant part representing the average value and a fluctuating component,

$$
K(t) = K_0 + \Delta K(t), \quad (5)
$$

where $K(t)$ is the time-varying mesh stiffness [27], $K_0$ is the average stiffness, and $\Delta K$ is the fluctuating component of stiffness. A combination of (3) and (5) yields

$$
\mathbf{M} \ddot{x}(t) + \mathbf{C} \dot{x}(t) + \mathbf{K}_0 x(t) = \mathbf{f}^{\text{GP}} - \Delta \mathbf{K}(t) \cdot \mathbf{x}(t) + \mathbf{K}(t) \cdot \mathbf{e}(t). \quad (6)
$$

As $\mathbf{x}(t)$ is unknown, let $\mathbf{x}(t) = \mathbf{x}_i(t)$ in the right side of the equation for approximation, and (6) then can be simplified as

$$
\mathbf{M} \ddot{x}_i(t) + \mathbf{C} \dot{x}_i(t) + \mathbf{K}_0 \mathbf{x}_i(t) = \mathbf{f}^{\text{GP}} + \mathbf{f}^{\text{mesh}}, \quad (7)
$$

where $\mathbf{f}^{\text{mesh}} = -\Delta \mathbf{K}(t) \cdot \mathbf{x}_i(t) + \mathbf{K}(t) \cdot \mathbf{e}(t)$, $\mathbf{x}_i(t) = \mathbf{x}_i(t) \mathbf{V}_p$, and $\mathbf{x}_i(t)$ is the static transmission error. $k_m(t), x_i(t)$, and $e(t)$ can be solved through method mentioned in [26].

Since displacement and force are periodic functions, they can be described in the Fourier series with a base frequency of gear mesh frequency. In the frequency domain, (7) can be transformed into the following term:

$$
\mathbf{M} \mathbf{X}^\omega(\omega) + \mathbf{C} \mathbf{X}^\omega(\omega) + \mathbf{K}_0 \mathbf{X}^\omega(\omega) = \mathbf{f}^{\text{GP}}(\omega) + \mathbf{f}^{\text{mesh}}(\omega). \quad (8)
$$

Let $Z^{\text{GP}}(\omega) = k_0 \mathbf{M} + \mathbf{C} \mathbf{X}^\omega(\omega)$ and $\mathbf{V}^{\text{GP}}(\omega) = \mathbf{X}^\omega(\omega)$; then the impedance equation of gear pair is given by

$$
Z^{\text{GP}} \mathbf{V}^{\text{GP}} = \mathbf{F}^{\text{GP}} + \mathbf{F}^{\text{mesh}}, \quad (9)
$$

where $Z^{\text{GP}}$ is the gear pair impedance matrix, $\mathbf{V}^{\text{GP}}$ is the gear pair velocity vector, $\mathbf{F}^{\text{GP}}$ is the applied external force vector, and $\mathbf{F}^{\text{mesh}}$ is the static transmission error excitation force vector.

2.2. Shaft Impedance. The shafts are divided into numerous segments, and each segment is modeled as a Timoshenko beam with 2 nodes and 6 degrees of freedom at each node as shown in Figure 4.

![Figure 4: Model of the shaft structure.](image)

Let the general displacement vector of shaft segment beam element be $\mathbf{x} = \{x_1, y_1, z_1, \theta_1, \phi_1, x_2, y_2, z_2, \theta_2, \phi_2\}^T, x_i, y_i, z_i (i = 1, 2)$ are displacement of node $i$ along direction of local coordinate, respectively; and $\theta_{x1}, \theta_{y1}, \theta_{z1}, \phi_{x1}$ are angle of node $i$ rotated about the corresponding axis.

The transverse shear parameter is

$$
\Phi = \frac{12EI}{\kappa AG^2}, \quad (10)
$$

where $E$ is Young’s modulus, $I$ is the area of inertia, $\kappa$ is the shape factor, $\kappa = 6(1 + \nu)/(7 + 6\nu)$ for a solid shaft, $\nu$ is the Poisson’s Ratio, $A$ is the shaft cross-sectional area, $G$ is the shear modulus, and $l$ is the element length.

The radius of gyration is defined as

$$
r_g = \sqrt{\frac{I}{A}}, \quad (11)
$$

The shaft element mass matrix is given by [30]

$$
M^* = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & -c & 0 & g & c & 0 \\
0 & 0 & -c & 0 & g & c \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

sym

$$
= \rho Al \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

\begin{bmatrix}
\frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{6} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6}
\end{bmatrix}
$$

\begin{bmatrix}
\frac{1}{3A} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3A} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3A} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3A} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{3A} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3A}
\end{bmatrix}
$$
The shaft element stiffness matrix is given by [30]

\[
K' = \begin{bmatrix}
a & 0 & \frac{AE}{I} & 0 & 0 & 0 \\
0 & a & -b & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{GJ}{I} \\
0 & 0 & -b & -b & 0 & 0
\end{bmatrix}
\]

The shaft mass matrix \( M \) and the stiffness matrix \( K \) can be yielded through a matrix assembly according to element nodes.

The shaft damping matrix can be described as the Rayleigh damping.

Motion equation of shafts can be yielded by

\[
M \ddot{x} + C \dot{x} + Kx = f.
\] (14)

Equation (14) can be transformed into (15) in the frequency domain

\[
M \omega^2 x + C \omega \dot{x} + Kx = F.
\] (15)

Impedance equation of shaft is expressed as

\[
Z' V' = F'.
\] (16)

2.3. Bearing Impedance. Bearing is modeled as a spring and a damper acting in parallel and connected through two coincident nodes. The 1st node is connected to the shafts and the 2nd node is connected to the housing. Bearing stiffness matrix [31] is defined in (17). Nonlinear time-varying bearing stiffness [20] is ignored to simplify the problem, and only an average value is used here. Angular contact ball bearings are used in this study, and the detailed numerical scheme for \( K_m \) can be found in [31].

\[
[K]_m = \begin{bmatrix}
\frac{\partial F_{jm}}{\partial \delta_{jm}} & \frac{\partial F_{jm}}{\partial \beta_{jm}} \\
\frac{\partial M_{jm}}{\partial \delta_{jm}} & \frac{\partial M_{jm}}{\partial \beta_{jm}}\end{bmatrix}_{i,j} \quad (i, j = x, y, z).
\] (17)

The final bearing stiffness matrix with six DOF per node is shown in (18).

\[
K = \begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix}
\]

\[
K_{11} = K_{22} = -K_{12} = -K_{21}
\]

\[
K = \begin{bmatrix}
k_{xx} & k_{xy} & k_{xz} & k_{x\delta x} & k_{x\theta y} & 0 \\
k_{yx} & k_{yy} & k_{yz} & k_{y\delta x} & k_{y\theta y} & 0 \\
k_{zx} & k_{zy} & k_{zz} & k_{z\delta x} & k_{z\theta y} & 0 \\
k_{\theta xx} & k_{\theta xy} & k_{\theta xz} & k_{\theta x\delta x} & k_{\theta x\theta y} & 0 \\
k_{\theta yy} & k_{\theta yy} & k_{\theta yz} & k_{\theta y\delta x} & k_{\theta y\theta y} & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

When bearing damping is taken into consideration, the bearing impedance matrix will be yielded,

\[
Z' Br(\omega) = C + \frac{K}{(j\omega)}. \] (20)

The impedance matrix can be obtained as follows:

\[
Z' Br V' = F' Br
\] (21)

where \( Z' Br \) is the impedance matrix, \( V' Br \) is the velocity vector, and \( F' Br \) is the excitation force vector.

2.4. Gearbox Housing Impedance. The mobility matrix of gearbox housing can be measured from experiment directly, and housing impedance can be obtained through a matrix inversion of the mobility function. In this study, harmonic analysis is employed to simulate the experiment.

A finite element model of gearbox housing is built, as shown in Figure 5. The material is aluminum alloy, density is 2700 kg/m³, Young’s modulus is 7.1 \times 10¹⁰ Pa, Poisson’s ratio is 0.33, and viscous damping ratio is 2%. The housing is meshed with 4 nodes’ tetrahedron element and with over 75,000 nodes and more than 310,000 elements. The bearing holes are coupled at external nodes which are connected with bearings, and the housing is fixed to the foundation through the bottom area.

Harmonic analysis within a frequency range of 1 Hz to 20 kHz with a step of 1 Hz is adopted in this study, and the modal superposition method with 500 modes is used to solve this problem.

The equation of motion of housing is given by

\[
M \ddot{x} + C \dot{x} + Kx = f,
\] (22)

where \( M \) is the mass matrix, \( C \) is the damping matrix, \( K \) is the stiffness matrix, \( x \) is the nodal displacement vector, \( \dot{x} \) is the velocity vector, \( \ddot{x} \) is the acceleration vector, and \( f \) is the external force vector.

Equation (22) can be converted to a modal form

\[
\ddot{y}_i + 2\omega_i \xi_i \dot{y}_i + \omega_i^2 y_i = f_i,
\] (23)
where $y_i$ is the modal coordinate, $\omega_i$ is the natural circular frequency of mode $i$, $\xi_i$ is the fraction of critical damping for mode $i$, and $f_i$ is the force in modal coordinate.

For a steady-state sinusoidal vibration, $f$ and $f_i$ have the form

$$f = f_c e^{j\Omega t},$$

$$f_i = f_{ic} e^{j\Omega t},$$

where $f_c$ is the complex amplitude, $f_{ic}$ is the complex amplitude of the modal coordinate for mode $i$, $j = \sqrt{-1}$, and $\Omega$ is the imposed circular frequency. To keep (23) true at all times, $y_i$ must have the following form:

$$y_i = y_{ic} e^{j\Omega t},$$

where $y_{ic}$ is the complex amplitude of the modal coordinate for mode $i$.

Differentiating (25), substituting (24) and (25) into (23) yields

$$(-\Omega^2 + j2\omega_i\Omega \xi_i + \omega_i^2)y_{ic} = f_{ic}.$$  

(26)

The complex displacement can be solved from

$$\mathbf{x}_c = \sum_{i=1}^{n} \phi_i y_{ic},$$

(27)

where $\phi_i$ is the modal shape for mode $i$.

Then the complex velocity can be obtained

$$\mathbf{v}_c = j\Omega \mathbf{x}_c.$$  

(28)

By applying unit force on term $j$ ($j = 1, 2, \ldots, n$), one will yield velocity at term $i$ ($i = 1, 2, \ldots, n$). Mobility matrix element $Y_{ij}$ can be obtained from

$$Y_{ij} = \frac{v_i}{F_j} = \frac{v_{ic}}{F_{jc}}.$$  

(29)

Even though $Y_{ij}$ is solved by finite element method here, it can also be measured directly from experiment. A typical simulated mobility element is shown in Figure 6. Peak frequencies in the frequency-amplitude curve denote the natural frequencies of the housing.

In this study, only four external nodes with six degrees of freedom (DOF) per node are needed for further study, so the total DOF needed for housing impedances is $n = 24$. When each mobility element $Y_{ij}$ ($i = 1, 2, \ldots, n; j = 1, 2, \ldots, n$) has been solved, mobility matrix can be expressed as

$$\mathbf{Y} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1n} \\ Y_{21} & Y_{22} & \cdots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{nn} & \cdots & \cdots & Y_{nn} \end{bmatrix}_{24 \times 24}.$$  

(30)

Impedance matrix can be obtained through a matrix inversion of $\mathbf{Y}$

$$\mathbf{Z}^{GB} = \mathbf{Y}^{-1}.$$  

(31)

And the impedance equation yields

$$\mathbf{Z}^{GB} \mathbf{V}^{GB} = \mathbf{F}^{GB}.$$  

(32)

3. Gear-Shaft-Bearing-Housing Impedance

3.1. Impedance Synthesis. When all subsystem impedance models have been established, the impedances of gear pairs, shafts, bearings, and housing can be coupled through the impedance synthesis approach. For illustration facility, the degrees of freedom (DOF) of subsystems of gear pairs, shafts, bearings, and housing are rearranged according to the coupling relationship, and the corresponding impedance equations are presented below.

Gear pair element impedance equation is shown below as

$$[\mathbf{Z}^{GP}_b] \{\mathbf{V}^{GP}_b\} = \{\mathbf{F}^{GP}_b\} + \{\mathbf{F}^{mnh}\}.$$  

(33)
Shaft element impedance equation is expressed as

\[
\begin{bmatrix}
Z^S_{tt} & Z^S_{tm} & Z^S_{tb} \\
Z^S_{mt} & Z^S_{mm} & Z^S_{mb} \\
Z^S_{et} & Z^S_{em} & Z^S_{eb}
\end{bmatrix}
\begin{bmatrix}
V^S_t \\
V^S_m \\
V^S_b
\end{bmatrix}
= \begin{bmatrix}
F^S_t \\
F^S_m \\
F^S_b
\end{bmatrix}
\] \tag{34}

Bearing element impedance equation is given by

\[
\begin{bmatrix}
Z^{Br}_{tt} & Z^{Br}_{tb} \\
Z^{Br}_{bt} & Z^{Br}_{bb}
\end{bmatrix}
\begin{bmatrix}
V^{Br}_t \\
V^{Br}_b
\end{bmatrix}
= \begin{bmatrix}
F^{Br}_t \\
F^{Br}_b
\end{bmatrix}
\] \tag{35}

Housing element impedance equation is formulated as

\[
\begin{bmatrix}
Z^{GB}
\end{bmatrix}
\begin{bmatrix}
V^{GB}
\end{bmatrix}
= \begin{bmatrix}
F^{GB}
\end{bmatrix}
\] \tag{36}

where \(Z\) is the impedance matrix, \(V\) is the velocity vector, \(F\) is the excitation force vector, superscript GP means gear pair, \(S\) means shaft, \(Br\) means bearing, \(GB\) means gearbox or housing, subscript \(b\) means bottom interface of subsystem in Figure 1 or right interface of subsystem in Figure 7, \(t\) means top or left part, and \(m\) means the middle part.

Coupling relationship is shown in Figure 7. The whole coupled model is a nonlinear system including nonlinear mesh stiffness and nonlinear bearing stiffness [20]; however, to simplify the problem, nonlinear time-varying mesh stiffness term is approximated as an average stiffness term and an excitation force term in (7), and nonlinear time-varying bearing stiffness is ignored here. So the whole model is simplified as a linear system which is suitable for the impedance method. Gear pairs and shafts are coupled through interface 1, shafts and bearings are coupled through interface 2, and bearings and gearbox housing are coupled through interface 3. Assume no external force acting at these interfaces, and only transmission error excitation is applied on the gear pair. The continuity equations and force equilibrium equations are also shown in Figure 7. Since static force and displacement have no direct influence on vibration and noise, they will be ignored here.

System impedance equation can be obtained through a combination of components impedance equations, continuity equations, and force equilibrium equations, or directly by assembling impedance matrix element according to node number

\[
\begin{bmatrix}
Z^G_{tt} & Z^G_{tm} & Z^G_{tb} \\
Z^G_{mt} & Z^G_{mm} & Z^G_{mb} \\
Z^G_{et} & Z^G_{em} & Z^G_{eb}
\end{bmatrix}
\begin{bmatrix}
V^G_t \\
V^G_m \\
V^G_b
\end{bmatrix}
= \begin{bmatrix}
F^G_t \\
F^G_m \\
F^G_b
\end{bmatrix}
\] \tag{37}

Impedance \(Z\) and force \(F\) in (37) are known, so velocity which is unknown can be solved from the equation below

\[
V(\omega) = Z(\omega)^{-1} F(\omega).
\] \tag{38}

Displacement vector can then be obtained as

\[
X(\omega) = \frac{V(\omega)}{j \cdot \omega},
\] \tag{39}

where \(X\) is the displacement vector, \(V\) is the velocity vector, \(j = \sqrt{-1}\), and \(\omega\) is the circular frequency.

Displacement in time domain can be formulated as

\[
x(t) = x_0 + \sum_{i_{GP}=1}^{n_{GP}} \sum_{k=1}^{K} X(k \cdot \omega_{i_{GP}}) \cdot e^{j k \omega_{i_{GP}} t},
\] \tag{40}

where \(x_0\) is the static displacement, \(i_{GP}\) is the index of gear pairs, \(n_{GP}\) is the number of gear pairs, \(K\) is the number of harmonic component, \(K\) is the maximum harmonic order, \(j = \sqrt{-1}\), and \(\omega_{i_{GP}}\) is the mesh frequency of gear pair \(i_{GP}\).

3.2. Model Validation. Even though conventional lumped parameter model can incorporate an equivalent mass matrix and stiffness matrix of housing into the whole system [9, 10, 18, 20], they failed to take into consideration the housing impedance directly when the housing model is not given and only impedance is known. In order to compare
the impedance model with the lumped model, the geared transmission system (without housing) is studied by both impedance synthesis method and lumped method. In this case, the impedance model can be transformed directly from the lumped model [27], and the only difference occurs between the solving processes. Natural frequencies are computed for both impedance model and lumped model, and they yield the same results. The root mean square (RMS) of mesh force and bearing reaction loads for rigid housing configuration at different rotational speeds are computed by both Fourier series method [27] and impedance method. Here, the system total DOF is 150, and it takes about 14 seconds to compute system vibration at 600 different speeds for impedance method while taking 77 seconds for Fourier series method using a relatively modern computer. A good coincidence between the two methods is shown in Figure 8.

For a gear-shaft-bearing-housing coupled case, a FE model is built, as shown in Figure 9. The gear pair is modeled as two masses connected by a mesh spring. Bearings are modeled as springs. Shafts and the housing are represented with 4-node tetrahedron elements with a maximum element size of 5 mm. The shaft material is steel and the housing material is aluminum alloy. There are more than 120,000 nodes and nearly 600,000 elements in total. The block Lanczos method is adopted to extract the modal frequency, while, in the impedance model, system DOF is 174 and frequency is swept from 0.1 Hz to 5000 Hz with a step of 0.1 Hz to solve the natural frequency.

Natural frequencies are obtained for the impedance model and the FEM, respectively, and the results are listed in Table 3. The maximum relative error is 3.2%.

4. Effect of Housing Impedance

4.1. Effect of Housing on System Natural Frequency. Damping terms are ignored in this section. Each mobility element is a function of frequency, and natural frequencies are the peak frequencies of these frequency-mobility curves. It is challenging to recognize all modes on a single frequency-mobility curve; however it is possible to find all modes when all mobility matrix elements are taken into consideration. Summation function denoted as SUM is the sum of all
mobility matrix elements. The seeking process of identifying
the peak at each natural frequency is adopted here to find the
system natural frequency. This analysis is given by

\[
\text{SUM} = \sum_{i=1}^{n} \sum_{j=1}^{n} \left| Y_{ij} \right|. 
\]  
(41)

Frequency is swept from 0.1 Hz to 5000 Hz with a step of
0.1 Hz to find peak frequencies of the summation function,
and the first 5 system natural frequencies are listed in Table 4.
When the gearbox housing is coupled to the transmission
system, system natural frequency decreases significantly.

It is difficult to show the mode shapes of the whole system
using the impedance model, because only four external nodes
are used to represent the structure of the housing. In order
to study the mode shapes, a FE model shown in Section 3.2 is
used instead of the impedance model.

The first four mode shapes of the whole coupled system
are shown in Figure 10. The 1st mode is a composite motion
of the whole system, and the housing is strongly coupled with
the shafts. The 2nd mode is mainly the swing motion of the
output shafts, with the input shaft and the housing slightly
coupled. The 3rd mode is mainly the flexural deformation
of the output shaft. The 4th mode is mainly the flexural
deformation of the input shaft, with the output shaft and the
housing slightly coupled.

4.2. Effect of Housing on Dynamic Mesh Force. When the
system dynamic displacement is solved from (42), the DTE
(dynamic transmission error) can be obtained as

\[
\text{DTE} = V_{p}^{T}X_{m}. 
\]  
(42)

**Figure 10:** Mode shapes of the whole coupled system: (a) 1st mode; (b) 2nd mode; (c) 3rd mode; (d) 4th mode.
In order to investigate influence of different housing configurations on gear vibration, housing impedance is computed and compared with the geared transmission system. The peak values of mesh force at some operation speeds but decreases it at some other speeds, which has a good agreement with results reported by Parker et al. [8].

The RMS function of bearing reaction load for modified housing impedances is shown in Figure 12(b). When the housing has been coupled with the geared transmission system, the first main peak frequency shifts −15.3%, −20.8%, and −29.2%, respectively, for the stiffened housing configuration, the original case, and the softened one. Vibration amplitude decreases 49.5%, 33.7%, and 56.3%, respectively, for the stiffened housing, the original housing, and the softened one. The housing with lower impedance has more potential influence on bearing reaction load. More peaks appear when the housing is softened.

4.3. Effect of Housing on Bearing Reaction Load. Bearing reaction load can be computed from (35) by solving for the system velocity. The RMS function of the 1st bearing reaction load at different rotational speed is shown in Figure 12.

Housing flexibility is noted to contribute greatly to the RMS of bearing reaction load. Also, the following specific observations are seen. (1) The two main peak frequencies for the rigid housing configuration are corresponding to the 10th natural frequency and the 15th natural frequency. For the flexible housing, these frequencies are changed to the 10th natural frequency and the 47th natural frequency. (2) When flexible housing has been taken into consideration, the first main critical speed decreases 20.8%. (3) About 33.7% peak amplitude reductions are seen for the first main resonance peak. (4) New peaks such as those corresponding to the 28th and 33rd natural frequency emerge when housing impedance is considered. (5) Housing increases bearing reaction load fluctuation at some operation speeds but decreases it at some other speeds, which has a good agreement with results reported by Parker et al. [8].

The RMS function of bearing reaction load for modified housing impedances is shown in Figure 12(b). When the housing has been coupled with the geared transmission system, the first main peak frequency shifts −15.3%, −20.8%, and −29.2%, respectively, for the stiffened housing configuration, the original case, and the softened one. Vibration amplitude decreases 49.5%, 33.7%, and 56.3%, respectively, for the stiffened housing, the original housing, and the softened one. The housing with lower impedance has more potential influence on bearing reaction load. More peaks appear when the housing is softened.
5. Effect of Bearing on Housing Contribution

A bearing structure will propagate structure-borne noise from the shaft to the housing, and it also plays an important role on the housing contribution on geared transmission system vibration. However, bearing stiffness and damping are difficult to calculate exactly, and currently any bearing vibration model will be a simplified approximation [32]. In order to evaluate the housing contribution on system dynamics, a parametric study of bearing characteristics is applied. Since average bearing stiffness [31] and damping are used in this model, only stiffness and damping influences are studied here.

The RMS function of mesh force for different bearing stiffness models with both rigid housing and flexible one is computed, and results are shown in Figure 13(a). Damping is kept constant in order to study the influence of bearing stiffness. As bearing displays a dashpot behavior in the propagation of structure-borne noise, it only allows relative motion at low frequencies. When bearing stiffness increases, gear vibration increases too. Comparison between rigid housing models and flexible ones with different bearing stiffness shows that housing reduces the amplitude by 3.5%, 9.0%, and 11.9% for the softened bearing model, the original one, and the stiffened one, respectively.

In order to study the influence of bearing damping on the housing contribution on mesh force, different bearing damping is used while bearing stiffness remains constant. Results are shown in Figure 13(b). Since damping will reduce vibration amplitude, peak value decreases as bearing damping increases. When the housing impedance is considered, the peak amplitude reduces about 4.6%, 9.0%, and 9.6% for 10 times’ bearing damping model, the original damping one, and 0.1 times’ damping one, respectively.

The RMS function of bearing reaction load for different bearing stiffness with a rigid housing and a flexible one is shown in Figure 13(c). Bearing stiffness not only influences system dynamic characteristics and bearing vibration directly, but also has significant influence on the housing contribution on bearing reaction load. As bearing stiffness increases, −11.3%, −20.8%, and −25.3% peak frequency shifts are yielded, and 32.9%, −33.7%, and −69.5% peak value changes are obtained, respectively, for the first main peaks. New peaks emerge as bearing stiffness increases.

Influence of bearing damping on housing contribution on bearing reaction load is shown in Figure 13(d). As bearing damping decreases, −23.7%, −20.8%, and −20.8% peak frequency shifts are obtained, and −34.0%, −33.7%, and −45.2% peak value changes can be observed for the first main peaks.

Comparison between Figures 13(a), 13(b), 13(c) and 13(d) shows that housing compliance has limited influence on gear vibration, but it will affect the bearing reaction load significantly. Bearing stiffness has prominent influence on the contribution of housing compliance on system vibration. When bearing stiffness increases, the housing influence on system dynamic characteristics increases.

6. Conclusions

The main objective of this paper is to incorporate housing impedance with the gear transmission system when the housing model is not given and only the impedance is known. A coupled gear-shaft-bearing-housing dynamic model is developed applying the impedance synthesis approach. Though the whole system is nonlinear, an approximate linear model is used to simplify the problem. The static transmission error excitation is applied on the model while the nonlinear time-varying bearing stiffness excitation is ignored for simplification. The proposed method is convenient to model and efficient to compute. Impedances of gear pairs, shafts, and bearings are transformed directly from lumped parameter representations, and the housing impedance is obtained through harmonic analysis instead of using an experimental
Figure 13: Influence of bearing properties on housing contribution on system vibration: (a) bearing stiffness, mesh force; (b) bearing damping, mesh force; (c) bearing stiffness, bearing reaction load; (d) bearing damping, bearing reaction load.

setup. Influence of housing impedance on system dynamic is analyzed. Specific conclusions are listed below.

(1) The housing compliance will decrease the system natural frequency and induces new natural frequencies as compared to rigid foundation assumption.

(2) Housing impedance has limited influence on gear vibration but will affect bearing reaction load significantly. The specific influence of the housing on the RMS function of bearing reaction load depends on the operating speed.

(3) Bearing stiffness has significant influence on housing contribution to the overall system vibration. For a stiffer bearing configuration, the housing influence will be more prominent.

(4) The impedance synthesis model proposed in this study is able to combine theoretical parameters with experimental data, which provides an effective way to couple complex housing and foundation with the transmission system. Also, the impedance method is computationally more efficient than the Fourier series method when performing the dynamic response calculation.

(5) It takes more computational time for the impedance model than the lumped parameter model in the modal analysis. Also, the impedance method cannot be adopted directly in a nonlinear analysis.

Further study is needed to take into consideration the non-parallel gearing condition caused by housing deformation and the time-varying bearing stiffness excitation.

Nomenclature

\( A \): Section area
\( c_m \): Mesh damping
\( C \): Damping matrix
DTE: Dynamic transmission error
E: Material Young’s modulus
\( e_m \): Composite mesh error
f: External force vector
\( f_{\text{mesh}} \): Excitation force vector
F: External force vector in frequency domain
\( F_j \): Force/moment applied at point j
\( f_{\text{mesh}}^j \): Excitation force in frequency domain
G: Material shear elastic modulus
I: Area moment of inertia
\( I_i \): Moment of inertia
j: j = √-T
\( j \): Polar moment of inertia
\( k_{\text{m}} \): Mesh stiffness
K: Stiffness matrix
\( \Delta K \): Fluctuant component of stiffness matrix
\( K_0 \): Average stiffness matrix
L: Length
\( m_{\text{eq},i} \): Equivalent mass of gear i \((i = p, g)\)
\( m_i \): Mass of pinion or gear \((i = p, g)\)
M: Mass matrix
\( n_B \): Number of bearings
\( n_G \): Number of gears
\( n_{\text{GF}} \): Number of gear pairs
OD: Outer diameter
\( r_i \): Base circle radius of pinion or gear \((i = p, g)\)
SUM: Summation function
\( v_i \): Translation/rotation velocity measured at point i
V: Velocity vector
\( V_p \): Project vector
X: Displacement vector
\( X_i \): Displacement vector in frequency domain
\( Y_i \): Modal coordinate
Y: Mobility matrix
\( Y_{ij} \): Mobility matrix element
Z: Impedance matrix
\( \alpha \): Mesh angle
\( \alpha_0 \): Mass scale factor
\( \alpha_1 \): Stiffness scale factor
\( \beta_b \): Base helix angle
\( \delta \): Gear pair relative deflection along the line of action
\( \phi \): Angle between transverse line of action and y-axis
\( \phi_i \): Fix angle
\( \Phi_i \): Modal shape for mode i
\( \Phi \): Transverse shear parameter
\( \kappa \): Shape factor
\( \rho \): Material density
\( \theta_i \): Rotation angle of node i
\( \omega \): Circular frequency
\( \Omega \): Angular speed
\( \zeta \): Damping ratio.

**Superscript**

Br: Bearing subsystem
GB: Gearbox or housing subsystem

**Subscript**

b: Bottom interface part (in Figure 1)
g: Gear
m: Middle or internal part (in Figure 1)
p: Pinion
t: Top interface part (in Figure 1).

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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