Research Article

A New Approach on Vibrating Horns Design

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An optimization method of the vibrating horns is presented considering the smallest action principle and the attached cutting tool mass. The model is based on Webster's wave propagation equation and as an objective function the minimization of the volume in structural equilibrium conditions was considered. The considered input parameters were working frequency, maximum cross-sectional area, magnification coefficient, and the attached mass. At the end of the study, a new shape function of the horn's cross section is obtained. The particularity of the new obtained shape is given by the nodal point position that is the same with the position of the maximum cross-sectional area. The obtained horn was analyzed from the modal point of view using theoretical and experimental methods. As theoretical methods, both the state-space method and the finite element method were used. An experimental setup for frequency response function determination was developed using a random input signal. The verification of the magnitude value was done considering a harmonic steady-state signal. The recorded values were compared with the predicted values. The numerical simulations and tests support the validity of the assumptions used in the horns optimization design.

1. Introduction

Ultrasonic horns are axisymmetric devices that provide, with high efficiency, acoustic energy developed by a transducer that is working based on the magnetostriction effect, to a tool. These devices can be themselves tools or can be used with a tool attached at the small end, and thus they are tied with high-power applications of ultrasound, like cutting [1–3], drilling [4–6], turning [7], welding [8, 9], and so forth, or for machining different materials as carbon fiber reinforced composites [10–12].

From a physical point of view, the work of such a system consists of an initial signal, emitted by a transducer at a given frequency, with an amplitude u_0 , which is mechanically amplified by the horn, with an imposed magnitude coefficient q . The whole system is working in a resonance regime at an imposed frequency. The mechanical magnification of the input signal amplitude is generated by the horn varying cross section from an initial value S_0 , in the origin of the reference system, to an end one value S_L (Figure 1).

Generally, the cross section variation is given by different mathematical functions (exponential [13], linear [14], tapered

(or stepped) [15], catenoidal [14], Bézier [16], Gaussian, etc.). Studies concerning the wave propagation and basic concepts of horns design are presented in different papers [17–21].

A problem that is associated with the horns design refers to the shape optimization. The optimization procedure is connected with design parameters like frequency working values [21], amplitude of the signal [22, 23], load transferred in the manufacturing area [24], the objective function [25, 26], the impact of different design variables on the objective function [27], combined signals transmitted in the horn [28], and so forth.

In the present paper, we present an approach based on the principle of the least action that takes into consideration as an objective function the volume minimization. The considered input parameters are the working frequency, the maximum cross-sectional area, the magnification coefficient, and the attached mass.

2. Mathematical Model

In case of ultrasonic horns, a planar stress waves state is assumed. The general equation that describes one-dimensional

FIGURE 1: The solid horn geometry.

propagation, in the longitudinal direction, of ultrasonic waves along a rigid horn with a variable cross-sectional area is Webster's horn equation [29]:

$$\frac{\partial^2 u(x, t)}{\partial x^2} + \frac{\partial u(x, t)}{\partial x} \frac{\partial}{\partial x} (\ln S_x) = \frac{1}{c^2} \frac{\partial^2 u(x, t)}{\partial t^2}, \quad (1)$$

where $u(x, t)$ is the signal, $S_x = S(x)$ represents the cross-sectional area of the considered horn at the distance x from the reference (point 0), and c is the wave velocity in the horn's material.

One of the widely used techniques to find the explicit analytical solutions of the linear partial differential equations is the method of variables separation [30]. The method assumes that the solution of (1) can be written as a product of two functions, one in space domain $u(x) = u_x$ and the other one in time domain $u(t) = u_t$:

$$u(x, t) = u(x) u(t) = u_x u_t. \quad (2)$$

Introducing solution (2) in Webster's equation (1), the new shape is obtained:

$$\frac{u_x''}{u_x} + \frac{d}{dx} (S_x) \frac{u_x'}{u_x} = \frac{1}{c^2} \frac{\ddot{u}_t}{u_t}, \quad (3)$$

where the following notations were done: $u_x'' = d^2 u_x / dx^2$, $u_x' = du_x / dx$, and $\ddot{u}_t = d^2 u_t / dt^2$.

This way, in (3), there were separate variables on each side of the equation.

The congruency of both sides, left and right, of (3) is achieved only if the two functions are equal with the same separation constant β . It is considered as a constant value $\beta = -\omega^2$, where ω is the angular wave frequency (rad/s).

Thus, from (3), we obtained the following ordinary differential equations:

$$\begin{aligned} \ddot{u}_t + \omega^2 u_t &= 0; \\ u_x'' + \frac{d}{dx} (S_x) u_x' + k^2 u_x &= 0, \end{aligned} \quad (4)$$

where $k = \omega/c$ is the wave number.

The general solution of the time domain equation from (4) is given by

$$u_t = D_1 \cos \omega t + D_2 \sin \omega t \quad (5)$$

with the initial conditions

$$\begin{aligned} u_t|_{t=0} &= u_0; \\ \frac{du_t}{dt}|_{t=0} &= 0, \end{aligned} \quad (6)$$

where u_0 is the initial amplitude of the ultrasonic wave.

Thus, solution (5) becomes

$$u_t = u_0 \cos \omega t, \quad (7)$$

and the total solution (2) becomes

$$u(x, t) = u(x) u(t) = u_x u_t = u_x u_0 \cos \omega t. \quad (8)$$

In case of a steady-state mode, considering the assumption of an infinitesimal motion in a beam made of nondissipative medium and with a variable cross-sectional area, the second equation from (4), which represents the plane wave motion, can be rewritten as

$$\frac{d}{dx} \left(S_x \frac{du_x}{dx} \right) + k^2 S_x u_x = 0. \quad (9)$$

It is considered that the maximum diameter of the horn should not exceed one-quarter of the wavelength λ and the maximum cross section is denoted as S_m . Thus, we can write the condition that, on the whole length of the horn, the cross section value has to be smaller than the maximum cross section:

$$S(x) \leq S_m. \quad (10)$$

Considering relation (10), one can write the following mathematical relation:

$$S_m - S(x) - a^2(x) = 0, \quad (11)$$

where $a(x)$ is an unspecified function of the variable x . In the following, the notations $a(x) = a_x$ and $S(x) = S_x$ will be used.

An optimization problem of the horn's shape refers to minimizing the horn's volume for a given initial set of design data. Based on the principle of the least action, one can write the action relation as

$$A = \int_{t_1}^{t_2} (T - V) dt, \quad (12)$$

where T is the kinetic energy and V represents the potential energy.

Considering the wave shape as

$$u = u(x, t) = u(x) \cos(\omega t) = u_x \cos(\omega t) \quad (13)$$

for a beam of length L with attached mass at one end, the case of an ultrasonic horn with the attached tool, the total energies of the system horn tool, mentioned in (12), are given by

$$\begin{aligned} T &= \frac{1}{2} \int_0^L (m_x \dot{u}^2) dx + \frac{M_t \dot{u}^2(L)}{2}, \\ V &= \frac{1}{2} \int_0^L \left(\frac{E_h m_x}{\rho_h} \right) (u')^2 dx + \frac{\sigma^2}{2E_t} V_t, \end{aligned} \quad (14)$$

where m_x is the specific mass of the horn, M_t is the attached mass (toll mass), E_h is Young's modulus of the horn material, E_t is Young's modulus of the tool material, V_t is the volume of the attached tool, ρ_h represents the horn material density, $\dot{u} = du/dt$, and $u' = du/dx$.

Considering a period, action (12) becomes

$$A|_{t_1}^{t_2+2\pi/\omega} = \bar{A} = \pi (\bar{T} - \bar{V}) \quad (15)$$

with

$$\begin{aligned} \bar{T} &= \frac{\omega^2}{2} \left[\int_0^L \rho \cdot S \cdot u^2 dx + M_t u^2(L) \right], \\ \bar{V} &= \frac{1}{2} \left[\int_0^L E_h \cdot (u')^2 \cdot S dx + \sigma \cdot u'(L) \cdot V_t \right]. \end{aligned} \quad (16)$$

Based on [31, 32], the following functional is defined:

$$\mathbf{I} = \bar{T} - \bar{V} + \alpha_1 V + \int_0^L \alpha_x (S_m - S_x - a_x^2) dx, \quad (17)$$

where α_1 and $\alpha_x = \alpha(x)$ are Lagrange multipliers and V is the volume of the horn, given by

$$V = \int_0^L S_x dx. \quad (18)$$

The optimization problem has a solution for the imposed limit values if the functional $\mathbf{I}$ given by (17) has to be stationary related to the variations of u_x and to the design parameter S_x . Thus, based on (18), it is necessary for Euler's equations [33] to be satisfied:

$$\begin{aligned} \frac{\partial \mathbf{I}}{\partial u_x} - \frac{d}{dx} \frac{\partial \mathbf{I}}{\partial u'_x} &= 0, \\ \frac{\partial \mathbf{I}}{\partial S_x} - \frac{d}{dx} \frac{\partial \mathbf{I}}{\partial S'_x} &= 0, \\ \frac{\partial \mathbf{I}}{\partial a_x} - \frac{d}{dx} \frac{\partial \mathbf{I}}{\partial a'_x} &= 0, \end{aligned} \quad (19)$$

where $u'_x = du_x/dx$, $S'_x = dS_x/dx$, and $a'_x = da_x/dx$.

Considering the functional given by (17) and (19), one can obtain the following differential equations:

(a) From the first equation in (19) results

$$\begin{aligned} \frac{\omega^2}{2} 2u_x \rho S_x - \frac{d}{dx} \left[-\frac{E_h}{2} 2u'_x S_x \right] \\ = \omega^2 u_x \rho S_x + E_h (S_x u_x)' = 0 \end{aligned} \quad (20)$$

which when divided by E_h becomes

$$(S_x u'_x)' + \frac{\rho_h \omega^2}{E_h} S_x u_x = 0. \quad (21)$$

Considering the velocity of the longitudinal wave in the horn as $c_l = \sqrt{E_h/\rho_h}$ and the wave number $k = \omega/c_l$, (21) can be rewritten as

$$(S_x u'_x)' + k^2 S_x u_x = 0. \quad (22)$$

(b) From the second equation, one obtains

$$\frac{\omega}{2} \rho_h u_x^2 - E_h (u'_x)^2 + 2\alpha_1 - 2\alpha_x = 0, \quad (23)$$

which when divided by E_h becomes

$$k u_x^2 - (u'_x)^2 + \frac{2}{E_h} \alpha_1 - \frac{2}{E_h} \alpha_x = 0. \quad (24)$$

(c) From the third equation results, it is found that

$$2\alpha_x a_x = 0. \quad (25)$$

Analyzing the previous equations, one can obtain the following remarks about the optimal design of the ultrasonic horn:

(a) Equation (21) is identical to the longitudinal wave propagation equation along a horn and it is available along the whole horn.

(b) Equation (25) leads to two other conditions:

$$\begin{aligned} \alpha_x &= 0 \\ \text{or } a_x &= 0. \end{aligned} \quad (26)$$

Conditions (26) involve the following aspect: the function a_x is different from zero on the whole length of the horn except for one point, where $S_x = S_m$.

Consequently, for (25) to be satisfied for any point in the range $0 \leq x \leq L$, it is necessary to have the first condition from (26) $\alpha_x = 0$ in the abovementioned range. Moreover, all Lagrange's coefficients have to be constants.

(a) From (24) and (26), the following is obtained:

$$\alpha_x = \frac{E_h}{2} \left[k^2 u_x^2 - (u'_x)^2 + \frac{2\alpha_1}{E_h} \right] = 0 \quad (27)$$

or

$$(u'_x)^2 - k^2 u_x^2 - \frac{2\alpha_1}{E_h} = 0. \quad (28)$$

3. The Solution of the Differential Equation

The solution of (28) can be found considering the initial functional conditions of the ultrasonic horn. The following are considered as initial conditions:

- (a) The length of the horn has to be equal to a wavelength.
- (b) To achieve the assumption that the propagated waves are only longitudinal ones, a maximum value is considered for the magnification coefficient $q = 5$ [17].
- (c) The initial value of the signal amplitude, at the connection point between the transducer and the horn ($x = 0$), is u_0 equal to the emitted signal by the transducer:

$$u(x) = u(0) = u_0. \quad (29)$$

- (d) Considering the initial signal and the magnification coefficient, the signal at the end of the horn (for $x = L$) is given by

$$u(x) = u(L) = -qu_0. \quad (30)$$

- (e) In (30), the negative value ($-$) is considered in order to obtain a nodal point in a position along the horn length.
- (f) Between the two ends of the horn, there is no other extreme of the signal except for the maximum developed at the end of the horn, in the connection point with the tool.
- (g) The following condition is considered [32]:

$$mu'_{x=L} - k^2 Mu_{x=L} = 0, \quad (31)$$

where m is the mass on the unit length, at the end of the horn, $u'_{x=L}$ is the strain at the end of the horn, M is the attached mass, and $u_{x=L}$ is the signal at the end of the horn, given by (30).

Denoting

$$C = \frac{2}{E} \alpha_1, \quad (32)$$

equation (28) can be rewritten as

$$(u')^2 - k^2 u^2 - C = 0, \quad (33)$$

and integrating it with the derivative method, one can obtain the following equation [34]:

$$2u'u'' - k^2 2uu' = 0. \quad (34)$$

Dividing (34) by $u' \neq 0$ results in the differential equation

$$u'' - k^2 u = 0, \quad (35)$$

with the solution

$$u(x) = C_1 e^{kx} + C_2 e^{-kx}. \quad (36)$$

Introducing solution (36) in (33), after some calculations, the following is obtained:

$$-4k^2 C_1 C_2 = C, \quad (37)$$

$$C_1 = -\frac{C}{4k^2 C_2}. \quad (38)$$

Considering (38) in solution (36) results in

$$u(x) = -\frac{C}{4k^2 C_2} e^{kx} + C_2 e^{-kx}. \quad (39)$$

The constants C and C_2 can be found considering the initial conditions mentioned above, for $x = 0$. Combining (38) and (30), the following relation is obtained:

$$u_0 = -\frac{C}{4k^2 C_2} + C_2, \quad (40)$$

which leads to the following relation for the two values of the constant C_2 as a function of C :

$$C_2^{1,2} = \frac{ku \pm \sqrt{k^2 u_0^2 + C}}{2k}. \quad (41)$$

Considering (41) and (40), for both values $C_2^{1,2}$, the same dependence is obtained between the constants C and C_2 :

$$C = 4C_2 k^2 (C_2 - u_0). \quad (42)$$

Introducing the constant C , given by (42), in solution (39) leads to the following relation of the amplitude $u(x)$:

$$u(x) = u_0 e^{kx} - C_2 (e^{kx} - e^{-kx}). \quad (43)$$

The constant C_2 can be found considering conditions (39) and (40):

$$\rho_h S_{hL} [u_0 e^{kL} - C_2 (e^{kL} + e^{-kL})] + kL_t \rho_t S_t q u_0 = 0, \quad (44)$$

where ρ_h is the density of the horn's material, S_{hL} is the cross section at the end of the horn, S_t is the cross section of the tool ($S_L = S_t$), L_t is the tool's length, ρ_t is the density of the tool's material, and q is the gain of the initial signal u_0 .

From (44) results

$$C_2 = \frac{u_0 (\rho_h S_{hL} e^{kL} + qkL_t S_t \rho_t)}{2\rho_h S_{hL} \cosh(kL)}. \quad (45)$$

Considering dependence (42), one can define the function of the space dependent amplitude component of the signal along the horn:

$$u(x) = u_0 \left[e^{kx} - \frac{\rho_h S_{hL} e^{kL} + qkL_t S_t \rho_t}{\rho_h S_{hL} (e^{kL} + e^{-kL})} (e^{kx} - e^{-kx}) \right]. \quad (46)$$

In the following, we use the notation

$$B = \frac{\rho_h S_{hL} e^{kL} + qkL_t S_t \rho_t}{\rho_h S_{hL} (e^{kL} + e^{-kL})}, \quad (47)$$

and the signal function (46) becomes

$$u(x) = u_0 [(1 - B) e^{kx} + B e^{-kx}]. \quad (48)$$

As it is known, an important aspect in horns design is to find the position of the neutral point x_n . The neutral point defines the position of the cross section where the signal is zero. Thus, the neutral point can be easily found considering relation (48). Equalizing function (48) with zero, it is obtained that

$$x_{\text{nod}} = \frac{1}{2k} \ln \left(\frac{B}{B-1} \right). \quad (49)$$

4. Cross Section Shape Definition

From relation (48), we obtained the first two derivatives of the space component of the signal:

$$u'(x) = ku_0 [(1-B)e^{kx} - Be^{-kx}], \quad (50)$$

$$u''(x) = k^2 u_0 [(1-B)e^{kx} + Be^{-kx}].$$

Equation (9) can be rewritten as

$$\frac{S'}{S} = -\frac{u'' + k^2 \cdot u}{u'}. \quad (51)$$

Considering relations (50) in (51) results in the new formula

$$\frac{S'_x}{S_x} = -\frac{2u_0 k^2 [(1-B)e^{kx} + Be^{-kx}]}{u_0 k [(1-B)e^{kx} - Be^{-kx}]}. \quad (52)$$

Defining the function $g(x) = g_x$ as

$$g(x) = (1-B)e^{kx} - Be^{-kx}, \quad (53)$$

(52) becomes

$$\frac{S'_x}{S_x} = -2 \cdot \frac{g'_x}{g_x}. \quad (54)$$

Integrating (54) results in a constant, denoted by D :

$$\ln(S_x) = -2 \ln(g_x) + D, \quad (55)$$

which depends on the initial geometrical conditions.

At the end of the horn, for $x = L$, the cross section becomes $S_x = S_L$ and it is obtained that

$$D = \ln(S_x) + 2 \ln(g_L), \quad (56)$$

where g_L is the value of the function $g(x) = g_x$, defined by (53), for $x = L$.

Introducing the value of D obtained by (56) in (55) results in

$$\ln(S_x) = -2 \ln(g_x) + \ln(S_L) + 2 \ln(g_L), \quad (57)$$

or

$$\ln \left(\frac{S_x}{S_L} \right) = -2 \ln \left(\frac{g_x}{g_L} \right), \quad (58)$$

and the cross section at the distance x can be calculated as

$$S_x = S_L \left(\frac{g_L}{g_x} \right)^2. \quad (59)$$

FIGURE 2: The half-horn shapes along the longitudinal axis.

An interesting design aspect is represented by the value and position of the maximum cross section. Thus, we considered the condition that the first derivative of function (59) is equal to zero. This condition leads to the relation

$$g'_x = 0 \quad (60)$$

with similar solution to (49).

It can be concluded that the position of the maximum cross section is in the same place as the nodal point.

The end cross section can be found as

$$S_L = S_m \left[\frac{g_n}{g_L} \right]^2, \quad (61)$$

and introducing (61) in (59) results in the relation of the cross section value in a point situated at the distance x from the system origin, as a function of the maximum cross section:

$$S_x = S_m \left(\frac{g_n}{g_x} \right)^2. \quad (62)$$

Taking into consideration relation (62), a script in Matlab was made to generate the shape of the horn made of steel. The input parameters were the resonance frequency $f_r = 19900$ Hz, the wave velocity in steel $c = 50550$ m/s, the horn's material density $\rho_h = 7850$ kg/m³, the end radius of the horn (connection surface with the tool) $r_{hL} = 0.005$ m, the magnification coefficient $q = 5$, and tool radius equal to the end horn radius ($r_t = r_{hL}$). The tool was considered to be made of steel.

Based on the script written in Matlab, the shape presented in Figure 2 was obtained, and in Figure 3 the physical model of the horn is presented.

Based on the same script, the following design data were found: the radius in the origin of the system ($x = 0$) $r_0 = 29.256$ mm, the largest radius $r_m = 31.721$ mm, and the position of the nodal point $x_{\text{nod}} = 27.734$ mm.

Considering relation (48), using a script written in Matlab, we obtained the magnitude variation of the spatial components $u(x)$ of the considered signal (13) (Figure 4).

As it can be seen from Figure 4, the nodal point is at the coordinate found by relation (49).

FIGURE 3: The physical model of the horn.

FIGURE 4: The signal magnitude along the horn.

5. Theoretical Modal Analysis

5.1. State-Space Approach of the Optimized Model. The modal behavior can be described in many ways. Two of them are the state-space approach and transfer function representation. The differences between these two methods were described in [35, 36]. In the present paper, we considered as a theoretical modal analysis the state-space approach, and for the experimental modal analysis the FRF representation was made.

The state-space method is often used in dynamic system analysis. This method can be applied in the case of lumped systems and it is based on the concept of state vector that consists of two components: the vector of displacement and the vector of the velocities. Practically, the “ n ” second-order coupled equations that describe the dynamic behavior of a lumped system are transformed in a set of $2n$ first-order coupled equations [37–39].

The ultrasonic horn can be considered as a linear time-invariant (LTI) system that can be approximate with a lumped mass system. Thus, the obtained horn (Figure 3) is divided into “ n ” elements with the same length l_i , where $l_i = L/n$,

FIGURE 5: The lumped model of the horn.

with L being the total length of the horn equal to a quarter of the wavelength. Considering the case of the free-free system, as is in the experimental modal analysis, the horn can be modeled as a lumped system made of “ n ” masses connected with springs (Figure 5).

We considered 200 masses with a length of $l_i = 6.344 \cdot 10^{-4}$ m in each part. The obtained components are, taking into consideration the shape (Figure 3), small frusta of cone elements.

The components of the modal model are established considering the following assumptions:

- (a) The stiffness of the springs was considered to be equal to the equivalent stiffness of each element based on the relation

$$k_i = \frac{ES_{ai}}{l_i}, \quad (63)$$

where S_{ai} represents the average area of the region of length l_i :

$$S_{ai} = \frac{S_i + S_{i+1}}{2}, \quad (64)$$

with both S_i and S_{i+1} calculated using relation (62).

- (b) The mass m_i of each small frustum of cone element was found considering the classical relation of the density using for volume the areas S_i and S_{i+1} and the length l_i .

Considering the d'Alembert principle, the motion equations for the system from Figure 5 are given by

$$\begin{aligned} m_1 \ddot{x}_1 + k_1 x_1 - k_1 x_2 &= 0, \\ m_2 \ddot{x}_2 + (k_1 + k_2) x_2 - k_1 x_1 - k_2 x_3 &= 0, \\ m_3 \ddot{x}_3 + (k_2 + k_3) x_3 - k_2 x_2 - k_3 x_4 &= 0, \\ &\vdots \\ m_i \ddot{x}_i + (k_{i-1} + k_i) x_i - k_{i-1} x_{i-1} - k_i x_{i+1} &= 0, \\ &\vdots \\ m_n \ddot{x}_n + k_{n-1} x_n - k_{n-1} x_{n-1} &= 0. \end{aligned} \quad (65)$$

System (65) can be rewritten in matrix form as follows:

$$\begin{bmatrix} m_1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & m_3 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & m_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & m_n \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \vdots \\ \ddot{x}_i \\ \vdots \\ \ddot{x}_n \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 & \cdots & \cdots & \cdots & 0 \\ -k_1 & k_1 + k_2 & -k_2 & \cdots & \cdots & \cdots & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & -k_{n-1} & k_{n-1} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \quad (66)$$

or

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{0}, \quad (67)$$

from which it is obtained that

$$\ddot{\mathbf{X}} = -\mathbf{M}^{-1}\mathbf{K}\mathbf{X}. \quad (68)$$

As the state vector, we considered the matrix vector

$$\begin{aligned} \Delta &= \{\delta_1, \delta_2, \dots, \delta_{2n}\}^T = \{\mathbf{X}, \dot{\mathbf{X}}\}^T \\ &= \{x_1, x_2, \dots, x_n, \dot{x}_1, \dot{x}_2, \dots, \dot{x}_n\}^T, \end{aligned} \quad (69)$$

which, by derivation and considering (68), leads to the equation

$$\dot{\Delta} = \begin{Bmatrix} \dot{\mathbf{X}} \\ \ddot{\mathbf{X}} \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{X} \\ \dot{\mathbf{X}} \end{Bmatrix} = \mathbf{A}\Delta, \quad (70)$$

where $\mathbf{I}$ is the unit matrix and $\mathbf{0}$ is the zeros matrix, both of $n \times n$ type.

The matrix $\mathbf{A}$ is defined as the state matrix or the transmission matrix and is a property of the system being

FIGURE 6: The natural frequencies obtained for the lumped model.

FIGURE 7: The finite element model of the horn.

defined by the structure of the system. The eigenvalues of the state matrix can be found based on the relation

$$|\lambda \mathbf{I} - \mathbf{A}| = 0 \quad (71)$$

that is developed in an equation with unknowns λ .

If, in relation (71), the value $\lambda = j\omega_n$ is considered, where $j = \sqrt{-1}$ and ω_n is the natural frequency measured in rad/s, then the so-called characteristic equation is obtained, which gives the natural frequencies.

Thus, considering the equivalent system from Figure 5 with the associated motion equations (66), (67), and (70), one can find the natural frequencies of the system. The number and the precision of these frequencies depend on the number of the considered masses m_i ($i = \overline{1, n}$). Figure 6 presents the frequency values obtained for a model with 200 masses considering the method of eigenvalues of the transition matrix.

Figure 6 also presents some of the obtained natural frequencies. The first natural frequency found was $f_n = 19,837$ Hz, which is close to the designed frequency of $f_r = 19.9$ kHz.

5.2. Finite Element Model. In the next step, the finite element model was considered. The used specialized software was ANSYS. The 3D model (Figure 7) was made based on (59).

For good convergence, we considered a mapped mesh consisting of first-order solid hexaelements with 1 mm length. We obtained 87,076 nodes and 75,254 elements. Mode extraction is carried out in the frequency range 18–24 kHz using Block Lanczos option with double precision. Two pure longitudinal modes were obtained with the natural frequencies of $f = 19,817$ Hz (Figure 8) and $f = 23,151$ Hz (Figure 9).

FIGURE 8: The mode shape for the natural frequency of $f = 19,817$ Hz. FIGURE 9: The mode shape for the natural frequency of $f = 23,151$ Hz.

FIGURE 10: The modal testing setup.

FIGURE 11: The detailed connection shaker horn.

6. Experimental Modal Analysis

The aim of the experimental modal analysis is to verify whether the considered imposed working frequency of 19.9 kHz is one of the natural frequencies of the considered horn and whether the natural frequencies found by state-space approach are close to the real natural frequencies.

The method with random signal excitation given by an electrodynamic shaker (Figure 10) was considered [40]. The used equipment was a complete testing setup (Brüel & Kjær) made of an integrated platform, PULSE 12, with a force transducer (type 8230) and an accelerometer (type 4397). The excitation signal was generated by the PULSE 12 platform and was amplified using the power amplifier type 2718 and then transmitted to the tested horn by the shaker type 4810 (Figure 11).

The random generated signal (the input signal) has a frequency span up to 25.6 kHz with an excitation time of 500 ms (Figure 12). The Fourier spectrum of the input signal is shown in Figure 13.

FIGURE 12: The input random signal in time domain.

FIGURE 13: The Fourier spectrum of the input signal.

FIGURE 14: The frequency response function of the horn.

As it is known, the frequency response function estimation from the measured data represents the main step of the experimental modal analysis. The analysis is based on non-parametric estimators. In the present study, we considered the often used estimator H_1 . The obtained frequency response function, for the estimator H_1 , is presented in Figure 14.

The range of the measured frequencies was up to 20,000 Hz and the values are presented in Figure 15.

At the same time, based on the experimental modal analysis, we obtained the damping factors ζ for each natural frequency found. The distribution of these values is presented in Figure 16.

Considering the facilities of the Matlab software, a power function was defined, which can approximate the damping factor for different values of the frequency f :

$$\zeta(f) = 321.9 f^{-0.7427} - 0.07775. \quad (72)$$

Function (72) can be used to study the influence of the damping in case of horn vibration at different frequencies.

FIGURE 15: The frequencies obtained by experimental data.

FIGURE 16: The damping factor values distribution versus natural frequencies.

7. Testing the Designed Horn

The test was designed to check whether the magnification of the designed horn is the initial considered $q = 5$ for the resonance frequency of 19.9 kHz. The used setup is made of the same components as the setup used in the experimental modal analysis with the exception of the accelerometers. For testing, two Brüel & Kjær accelerometers type 4517-002 were used. The signal was generated by a magnetostrictive transducer for a harmonic signal $y(t) = A \sin \omega t$ with a frequency $f = 19.9$ kHz. The input signal, represented in time domain, is shown in Figure 17, and the output signal (response) also in time domain is designed in Figure 18.

At the same time, for frequency control, we performed a check in frequency domain for both signals: the input (Figure 19) and the output (Figure 20).

Considering the values of the input and output signals (Figures 17 and 18), the magnification obtained by experiment is given by the following relation:

$$q_{\text{real}} = \frac{u_{\text{output}}}{u_{\text{input}}} = \frac{42.78}{9.32} = 4.59, \quad (73)$$

which represents an error of about 8.2% between the considered value $q = 5$ and the obtained value q_{real} .

FIGURE 17: The input signal in time domain.

FIGURE 19: The input signal in frequency domain.

FIGURE 18: The output signal in time domain.

FIGURE 20: The output signal in frequency domain.

8. Conclusions

Based on the principle of the smallest action, an optimization of the shape of an ultrasonic horn was performed considering Webster's equation. The design was made considering a working resonance frequency of $f_r = 19.9 \text{ kHz}$ and a magnification coefficient $q = 5$, and the main assumption was that, through the horn and all attached coupling devices, only longitudinal plane waves propagate.

The considered boundary conditions were the classical ones referring to the amplitude of the signal, relations (29) and (30), to which a new one was added, relation (31), which highlights the influence of the tool mass, added at the end of the horn. This condition (31) is a particularity of the study.

The optimization procedure led to a new shape function (62), with the space dependent signal component function (48) and the position of the neutral point x_n (49) being determined at the same time. Comparing the obtained shape with the classical shape, it can be seen that it is a different one, with the largest diameter in the neutral point. In the case of the other horns' shape, the largest diameter is at one end.

The natural frequencies were determined both in simulation (theoretically) and experimentally. In the case of theoretical modal analysis, the state-space representation was used. The horn was divided into 200 masses connected by springs (Figure 5) and the eigenvalues of the transmission matrix of system (71) were calculated. The first obtained value of the natural frequency was the imposed resonant frequency $f_{1,ss} = 19,982 \text{ Hz}$.

The second considered theoretical approach in theoretical modal analysis was the finite element method. A model with 75,254 elements was designed, and based on ANSYS software facilities, two natural frequencies were found around the designed resonance frequency (Figures 8 and 9), both for the longitudinal mode shape. In order to provide clarity to the results, a run was made in double precision mode.

The experimental modal analysis offered the possibility of finding a set of natural frequencies (Figure 16) and the corresponding values of the damping factor. Considering the facilities of the Matlab tool, a function of damping factor variation was obtained according to the frequencies values (72) useful in the dynamic analysis. As it can be seen, a

natural frequency around the resonance designed frequency was found.

At the end, the horn was tested to check whether the magnification coefficient is the same as the designed one. A harmonic signal with the frequency value equal to the resonance frequency was considered as the input, and the output signal was measured. The found magnification coefficient is close to the designed one (72).

In the frame of the test, the attached tool was not considered. It is important for the assembly transducer-horn-tool to work in resonance regime and each part to work within the parameters for which it was designed.

The aim of the test was to check whether the considered horn, designed in the proposed set of boundary conditions (29), (30), and (31), generates the initial designed mechanical magnification of the input signal.

The influence of the attached tool mass is the particular design element that was taken into consideration compared with the other methods of optimization and design.

Considering the abovementioned aspects, it can be concluded that the numerical simulations and tests that were done support the validity of the assumptions used in the horn's optimization design.

The proposed model offers as an advantage a more convenient placement of the nodal point that is situated in the same place as the largest cross section.

The described design and optimization method offer the possibility of obtaining the shape based on the main working parameters.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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