Research Article

Asymptotic Stability of Delay-Controlled Nonlinear Stochastic Systems with Actuator Failures

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The problem of asymptotic stability of delay-controlled nonlinear stochastic systems with actuator failures is investigated in this paper. Such a system is formulated as a continuous-discrete hybrid system based on the random switch model of failure-prone actuator. Time delay control force is converted into delay-free one by randomly periodic characteristic of the system. Using limit theorem and stochastic averaging, an approximate formula for the largest Lyapunov exponent of the original system is then derived, from which necessary and sufficient conditions for asymptotic stability are obtained. The validity and utility of the proposed procedure are demonstrated by using a stochastically driven nonlinear two-degree system with time delay feedback and actuator failure.

1. Introduction

Modern technological systems rely on sophisticated control systems to meet increased performance requirements. In such systems, control components (e.g., actuator or sensor) are often subjected to random failures [1, 2]. It is known that control component failures can cause performance deterioration of control systems or even system instability leading to catastrophic accidents. Therefore, stability analysis for systems with control component failures has been an important and challenging research problem.

In practical applications, actuators are very important in transferring the controller output to the structure, and actuator failures are the typical faults appearing in control systems. In the last few decades, issues concerning stability and control problem of systems with actuator failures have been examined in the literatures. Mahmoud et al. [3] analyzed the stochastic stability of fault tolerant control systems in the presence of noise using the Lyapunov function approach. Gundes [4] studied the stability of the MIMO feedback system under possible failures in either the sensor-connections or actuator-connections. Aberkane et al. [5] dealt with the

problem of robust H_∞ control of active fault tolerant control systems with Markovian parameters via dynamic output feedback. A state feedback output tracking adaptive control scheme which is developed for plants with actuator failures is proposed by Tao et al. [6]. Seo and Kim [7] designed a robust and reliable H_∞ control for linear uncertain systems with actuator failures. The previous studies focused mainly on the linear systems or the one-dimensional nonlinear system. Furthermore, time delay is assumed to be ignorable. However, the phenomenon of time delay is common in control systems, which is a source of instability [8–10]. Thus, development of methodology for the stability analysis of delay-controlled nonlinear systems with actuator failures is thus much deserving.

In the present research, asymptotic stability of delay-controlled nonlinear systems with actuator failures will be investigated. The organization of this paper is as follows. In the next section, the random switch model of failure-prone actuator is established. Section 3 converts time delay control force into the delay-free one using randomly periodic characteristic of the system. Reduced averaged equations are obtained in Section 4. Section 5 gives the necessary and

sufficient conditions for asymptotic stability of the system. An illustrated example is presented in Section 6, and conclusion follows thereafter.

2. Formulation of Problem

2.1. Model of Possible Failure Actuator. The behavior of a possible failure actuator in a control system can be described as

$$u^a(t) = \kappa u(t), \quad (1)$$

where $u(t)$ is commanding signal to the actuator and u^a is the actual output of the possible failure actuator. $\kappa \in \{0, 1\}$ is the failure factor which expressed changed actuator characteristics. The case $\kappa = 0$ corresponds to complete failure of the actuator and therefore no output; $\kappa = 1$ refers to a functional actuator which is working normally. Therefore, in a real control system, the actuator may have two states: a “functional” state and a “failure” state. The “functional” actuator may change to failure actuate due to occasional breakdowns and the failure actuator can also be recovered during a finite time interval due to either efficient mobile maintenance or full redundancy. The random switches between these two states occur as a continuous-time Markov process [3, 5, 11]. Then, (1) can be rewritten in the following Markov jump form:

$$u^a(t) = \kappa^{[s(t)]} u(t), \quad (2)$$

where $s(t)$ is a Markov process, which points to state of the actuator, and it takes discrete values from a finite set $S = \{1, 2\}$. The transition rate from the functional state to failure state is assumed to be λ_1 and the backward transition rate is λ_2 . Then, the transition rate matrix Λ of $s(t)$ can be specified by [12]

$$\Lambda = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix}. \quad (3)$$

Denote by $\mu^{(u)}$ the stationary or equilibrium probability that $s(t) = u$ ($u = 1, 2$). Then $\mu^{(u)}$ satisfies the following equations [13]:

$$\begin{aligned} \sum_{u=1}^2 \mu^{(u)} \lambda_{ui} &= 0, \\ \sum_{u=1}^2 \mu^{(u)} &= 1, \\ i &= 1, 2. \end{aligned} \quad (4)$$

$\mu^{(u)}$ will be used in the next section.

2.2. Nonlinear Stochastic Systems with Actuator Failures. Consider an n degree-of-freedom (DOF) controlled stochastically driven nonlinear system with n actuators marked as $\Omega = \{1, 2, \dots, n\}$. The actuators in the system are

distinguished in two groups with regard to potential failures. One group is the set of actuators that are susceptible to failure, denoted by Ω_f . The other is a set of actuators that are robust to faults eventually, by $\Omega_n = \Omega - \Omega_f$. The motion of the system is usually governed by n second-order stochastic differential equations in the generalized displacements. These second-order equations can always be recast as $2n$ first-order equations in the Hamiltonian formulation

$$\dot{q}_i = \frac{\partial \tilde{H}'}{\partial p_i}, \quad (5a)$$

$$\begin{aligned} \dot{p}_i &= -\frac{\partial \tilde{H}'}{\partial q_i} - \varepsilon \tilde{c}_{ij}'(\mathbf{q}, \mathbf{p}) \frac{\partial \tilde{H}'}{\partial p_j} + \varepsilon \kappa_i^{[s(t)]} u_i(\mathbf{q}_\tau, \mathbf{p}_\tau) \\ &\quad + \varepsilon^{1/2} f_{ik}(\mathbf{q}, \mathbf{p}) W_k(t), \end{aligned} \quad (5b)$$

where $i, j = 1, 2, \dots, N$, $k = 1, 2, \dots, m$, and q_i and p_i are the generalized displacements and momenta, respectively; $\mathbf{q} = (q_1, \dots, q_n)^T$ and $\mathbf{p} = (p_1, \dots, p_n)^T$. ε is a small parameter. $\tilde{H}' = \tilde{H}'(\mathbf{q}, \mathbf{p})$ is the Hamiltonian. $-\varepsilon \tilde{c}_{ij}'(\mathbf{q}, \mathbf{p}) \partial \tilde{H}' / \partial p_j$ and $\varepsilon^{1/2} f_{ik}(\mathbf{q}, \mathbf{p}) W_k(t)$ represent the weak damping and pure parametric random excitations, respectively. $W_k(t)$ are independent Gaussian white noises with zero means and correction functions $E[W_k(t)W_l(t+\tau)] = 2D_{kl}\delta(\tau)$. $\mathbf{q}_\tau = \mathbf{q}(t-\tau)$ and $\mathbf{p}_\tau = \mathbf{p}(t-\tau)$ are system's states with time delay. $\kappa_i^{[s(t)]}$ are the two-state failure factors

$$\begin{aligned} \kappa_i^{(1)} &= \begin{cases} 1, & i \in \Omega_n, \\ 0, & i \in \Omega_f, \end{cases} \\ \kappa_i^{(2)} &= 1, \quad i \in \Omega. \end{aligned} \quad (6)$$

Note that, by modeling the transition between two states of the failure-prone actuator as a Markov jump process, the stochastic stability of system (5a) and (5b) can be studied in the framework of Markov jump hybrid systems.

3. Randomly Periodic Solution Approximation

Equations (5a) and (5b) can be converted to the following Itô equations by adding Wong-Zakai correction terms:

$$dq_i = \frac{\partial \tilde{H}}{\partial p_i} dt, \quad (7a)$$

$$\begin{aligned} dp_i &= \left[-\frac{\partial \tilde{H}}{\partial q_i} - \varepsilon \tilde{c}_{ij}(\mathbf{q}, \mathbf{p}) \frac{\partial \tilde{H}}{\partial p_j} \right. \\ &\quad \left. + \varepsilon \kappa_i^{[s(t)]} u_i[\mathbf{q}(t-\tau), \mathbf{p}(t-\tau)] \right] dt \\ &\quad + \varepsilon^{1/2} \sigma_{ik}(\mathbf{q}, \mathbf{p}) dB_k(t), \end{aligned} \quad (7b)$$

where $\tilde{H}$ and $\tilde{c}_{ij}(\mathbf{q}, \mathbf{p})$ are the modified Hamiltonian and coefficients of damping, respectively; $B_k(t)$ are standard Wiener processes such that $\sigma_{ik}(\mathbf{q}, \mathbf{p}) dB_k(t) = f_{ik}(\mathbf{q}, \mathbf{p}) W_k(t) dt$.

Here, we only consider the case that the associated Hamiltonian system of system (7a) and (7b) is integrable [14]. Then, the Hamiltonian $\tilde{H}$ has the form

$$\tilde{H}(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^n \tilde{H}_i(q_i, p_i), \quad (8)$$

$$\tilde{H}_i(q_i, p_i) = \frac{1}{2} p_i^2 + U_i(q_i),$$

where $\tilde{H}_1, \tilde{H}_2, \dots, \tilde{H}_n$ are n independent first integrals of the associated Hamiltonian system.

For the case of small time delay, it is proved that the following approximate expressions hold [10]:

$$\begin{aligned} q_i(t - \tau) &\approx \cos(\omega_i \tau) q_i(t) - \frac{\sin(\omega_i \tau)}{\omega_i} p_i(t), \\ p_i(t - \tau) &\approx \cos(\omega_i \tau) p_i(t) + \omega_i \sin(\omega_i \tau) q_i(t), \end{aligned} \quad (9)$$

where ω_i is the average frequency.

According to expression (9), the time delayed control forces $\kappa_i^{[s(t)]} u_i[\mathbf{q}(t - \tau), \mathbf{p}(t - \tau)]$ can be expressed in terms of current state variables with time delay as a parameter; that is, $\kappa_i^{[s(t)]} u_i[\mathbf{q}(t - \tau), \mathbf{p}(t - \tau)] = \kappa_i^{[s(t)]} u_i(\mathbf{q}, \mathbf{p}; \tau)$. Then, splitting $\kappa_i^{[s(t)]} u_i(\mathbf{q}, \mathbf{p}; \tau)$ into conservative part and dissipative part and combining them with $-\partial \tilde{H} / \partial q_i$ and $-\varepsilon \tilde{c}_{ij}(\mathbf{q}, \mathbf{p}) \partial \tilde{H} / \partial p_j$, respectively, yield

$$dq_i = \frac{\partial H'^{[s(t)]}(\tau)}{\partial p_i} dt, \quad (10a)$$

$$\begin{aligned} dp_i &= \left[-\frac{\partial H'^{[s(t)]}(\tau)}{\partial q_i} - \varepsilon \tilde{c}_{ij}^{[s(t)]}(\mathbf{q}, \mathbf{p}; \tau) \frac{\partial H'^{[s(t)]}(\tau)}{\partial p_j} \right] dt \\ &\quad + \varepsilon^{1/2} \sigma_{ik}(\mathbf{q}, \mathbf{p}) dB_k(t), \end{aligned} \quad (10b)$$

where $H'^{[s(t)]}$ and $\tilde{c}_{ij}^{[s(t)]}$ are the Markov jump Hamiltonian and damping coefficients. Since the control forces are weak, it is reasonable to assume that the conservative part of the delayed control forces may change the Hamiltonian slightly but not change the integrability of the associated Hamiltonian system. Then, the resulting system (10a) and (10b) is still a quasi integrable Hamiltonian system.

4. Averaged Equations

According to the limit theorem [15], under the assumption of ergodicity of $s(t)$, the solution of (10a) and (10b) converges in probability of the solution of the following weighted averaging equations as $\varepsilon \rightarrow 0$:

$$dq_i = \frac{\partial H(\tau)}{\partial p_i} dt, \quad (11a)$$

$$\begin{aligned} dp_i &= \left[-\frac{\partial H(\tau)}{\partial q_i} \right. \\ &\quad \left. - \varepsilon \sum_{u=1,2} \left[\tilde{c}_{ij}'^{(u)}(\mathbf{q}, \mathbf{p}; \tau) \cdot \mu^{(u)} \right] \frac{\partial H(\tau)}{\partial p_j} \right] dt \\ &\quad + \varepsilon^{1/2} \sigma_{ik}(\mathbf{q}, \mathbf{p}) dB_k(t), \end{aligned} \quad (11b)$$

where $\mu^{(u)}$ is given in (4), and

$$H(\tau) = \sum_{i=1}^n H_i(q_i, p_i; \tau), \quad (12)$$

$$H_i(q_i, p_i; \tau) = \sum_{u=1,2} \left[H_i'^{(u)}(q_i, p_i; \tau) \cdot \mu^{(u)} \right].$$

It is noted from (12) that limiting theorem will not change the integrality of the quasi Hamiltonian system. Further assume that the associated Hamiltonian system is nonresonant. Then, by using the stochastic averaging method for the quasi integrable Hamiltonian systems [14], the following averaged Itô equations associated with (11a) and (11b) can be obtained:

$$dH_i = m_i(\mathbf{H}; \tau) dt + \bar{\sigma}_{ik}(\mathbf{H}; \tau) dB_k(t), \quad (13)$$

where $\mathbf{H} = [H_1, H_2, \dots, H_n]^T$, and the drift coefficients $m_i(\mathbf{H}; \tau)$ and diffusion coefficients $\bar{\sigma}_{ik}(\mathbf{H}; \tau)$ are given by

$$\begin{aligned} m_i(\mathbf{H}; \tau) &= \frac{1}{T(\mathbf{H}; \tau)} \\ &\quad \cdot \oint \left(-\sum_{u=1,2} \left[\tilde{c}_{rj}'^{(u)}(\mathbf{q}, \mathbf{p}; \tau) \cdot \mu^{(u)} \right] \frac{\partial H}{\partial p_j} \frac{\partial H_i}{\partial p_r} \right. \\ &\quad \left. + \frac{1}{2} \sigma_{rk} \sigma_{jk} \frac{\partial^2 H_i}{\partial p_r \partial p_j} \right) \prod_{s=1}^n \left(\frac{\partial H_s}{\partial p_s} \right)^{-1} dq_s, \\ b_{is}(\mathbf{H}; \tau) &= \bar{\sigma}_{ik} \bar{\sigma}_{sk}(\mathbf{H}; \tau) = \frac{1}{T(\mathbf{H}; \tau)} \\ &\quad \cdot \oint \left(\sigma_{jk} \sigma_{ik} \frac{\partial H_i}{\partial p_j} \frac{\partial H_s}{\partial p_l} \right) \prod_{s=1}^n \left(\frac{\partial H_s}{\partial p_s} \right)^{-1} dq_s, \end{aligned} \quad (14)$$

where $\oint \cdot dq_s$ represents an n -fold loop integral and

$$T(\mathbf{H}; \tau) = \oint \prod_{s=1}^n \left(\frac{\partial H_s}{\partial p_s} \right)^{-1} dq_s. \quad (15)$$

After this approximation, the dimension of system is reduced significantly from $2n$ to n .

5. The Largest Lyapunov Exponent

Linearizing (13) at $\mathbf{H} = 0$ yields

$$dH_i = m_i^L(\mathbf{H}; \tau) dt + \bar{\sigma}_{ik}^L(\mathbf{H}; \tau) dB_k(t). \quad (16)$$

Since (5a) and (5b) contain only parametric excitations, $\mathbf{Q} = \mathbf{P} = \mathbf{0}$ is the trivial solution of (1) and $\mathbf{H} = \mathbf{0}$ is the trivial solution of (16). Then, the coefficients of (16) satisfy

$$\lim_{|\mathbf{H}| \rightarrow 0} m_i^L(\mathbf{H}; \tau) = 0, \quad (17a)$$

$$\begin{aligned} \lim_{|\mathbf{H}| \rightarrow 0} \bar{\sigma}_{ik}^L(\mathbf{H}; \tau) &= 0, \\ km_i^L(\mathbf{H}; \tau) &= m_i^L(k\mathbf{H}; \tau), \\ k\bar{\sigma}_{ik}^L(\mathbf{H}; \tau) &= \bar{\sigma}_{ik}^L(k\mathbf{H}; \tau), \end{aligned} \quad (17b)$$

$k > 0.$

Introduce the following new variables:

$$\rho = \frac{1}{2} \ln H, \quad (18a)$$

$$\alpha_i = \frac{H_i}{H}, \quad i = 1, 2, \dots, n. \quad (18b)$$

The Itô equations for ρ and α_i are then obtained from (16) by using Itô differential rule as follows:

$$d\rho = G(\boldsymbol{\alpha}; \tau) dt + \Xi_u(\boldsymbol{\alpha}; \tau) dB_u(t), \quad (19a)$$

$$d\alpha_i = \bar{G}_i(\boldsymbol{\alpha}; \tau) dt + \bar{\Xi}_{iu}(\boldsymbol{\alpha}; \tau) dB_u(t), \quad (19b)$$

where $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_n]^T$ and

$$\begin{aligned} G(\boldsymbol{\alpha}; \tau) &= \frac{1}{2} \sum_{i=1}^n m_i^L(\boldsymbol{\alpha}; \tau) \\ &\quad - \frac{1}{4} \sum_{i,j=1}^n \sum_{k=1}^m \bar{\sigma}_{ik}^L(\boldsymbol{\alpha}; \tau) \bar{\sigma}_{jk}^L(\boldsymbol{\alpha}; \tau), \\ \bar{G}_i(\boldsymbol{\alpha}; \tau) &= m_i^L(\boldsymbol{\alpha}; \tau) - \alpha_i \sum_{s=1}^n m_s^L(\boldsymbol{\alpha}; \tau) \\ &\quad + \frac{1}{2} \alpha_i \sum_{r,j=1}^n \sum_{k=1}^m \bar{\sigma}_{rk}^L(\boldsymbol{\alpha}; \tau) \bar{\sigma}_{jk}^L(\boldsymbol{\alpha}; \tau) \\ &\quad - \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^m \bar{\sigma}_{ik}^L(\boldsymbol{\alpha}; \tau) \bar{\sigma}_{jk}^L(\boldsymbol{\alpha}; \tau), \\ \bar{\Xi}_{iu}(\boldsymbol{\alpha}; \tau) &= \bar{\sigma}_{iu}^L(\boldsymbol{\alpha}; \tau) - \alpha_i \sum_{j=1}^n \bar{\sigma}_{ju}^L(\boldsymbol{\alpha}; \tau). \end{aligned} \quad (20)$$

Define the Lyapunov exponent of system (16) as the asymptotic rate of exponential growth of the square-root of the Hamiltonian; that is,

$$\Lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \log H^{1/2}. \quad (21)$$

Integrating (19a) from 0 to t and then dividing it by t yield

$$\begin{aligned} \frac{1}{2t} \log H(t) &= \frac{1}{2t} \log H(0) + \frac{1}{t} \int_0^t G(\boldsymbol{\alpha}; \tau) d\tau \\ &\quad + \frac{1}{t} \int_0^t \Xi_u(\boldsymbol{\alpha}; \tau) dB_u(s). \end{aligned} \quad (22)$$

As $t \rightarrow \infty$, the first and the last terms in the right hand side of (22) vanish and, as a consequence,

$$\Lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t G(\boldsymbol{\alpha}; \tau) d\tau. \quad (23)$$

From (18b), it is seen that $\sum_{i=1}^n \alpha_i = 1$, and only $n-1$ equations for α_i in (19b) are independent. Without loss of generality, in the following, the first $(n-1)$ equations for $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ are taken as independent equations. Let $\boldsymbol{\alpha}' = [\alpha_1, \alpha_2, \dots, \alpha_{n-1}]$ be $n-1$ dimensional vector diffusion process. Suppose that $\boldsymbol{\alpha}'$ is ergodic over the interval $0 \leq \alpha_k \leq 1$ ($k = 1, 2, \dots, n-1$). Under this condition, the time averaging in (23) can be replaced by assembly averaging; that is,

$$\Lambda = E[G(\boldsymbol{\alpha}'; \tau)] = \int G(\boldsymbol{\alpha}'; \tau) p(\boldsymbol{\alpha}') d\boldsymbol{\alpha}', \quad (24)$$

where $p(\boldsymbol{\alpha}')$ is the stationary probability density of vector process $\boldsymbol{\alpha}'$, which can be obtained by solving the Fokker-Planck-Kolmogorov (FPK) equation associated with first $n-1$ independent equations of (19b).

For an n -dimensional (16), there exist n Lyapunov exponents. However, (24) has been proved to converge to the largest Lyapunov exponent $\Lambda_{\max}$ [16]. Based on Oseledec ergodic theorem, a necessary and sufficient condition for asymptotic stability with probability one of the trivial solution of the system governed by (5a) and (5b) is approximately

$$\Lambda_{\max} = \int G(\boldsymbol{\alpha}'; \tau) p(\boldsymbol{\alpha}') d\boldsymbol{\alpha}' < 0. \quad (25)$$

6. Numerical Example

To demonstrate the application and validity of the proposed method, consider a stochastically excited nonlinear two-degree system with time delay feedback and actuator failure, governed by

$$\begin{aligned} \dot{q}_1 &= p_1, \\ \dot{p}_1 &= -\omega_1^2 q_1 - \beta_{11} p_1 - \beta_{22} p_2 + \kappa^{[s(t)]} u(t - \tau) \\ &\quad + f_{11} q_1 \xi_1(t) + f_{12} q_2^2 \xi_2(t), \\ \dot{q}_2 &= p_2, \\ \dot{p}_2 &= -\alpha q_2^3 - \beta_{21} p_1 - \beta_{22} p_2 + f_{21} q_1 \xi_1(t) \\ &\quad + f_{22} q_2^2 \xi_2(t), \end{aligned} \quad (26)$$

where $\omega_1, \alpha > 0$, and $\beta_{ij} > 0$ are constants; $\xi_i(t)$ is Gaussian white noise with zero mean and intensity $2D$; $\kappa^{[s(t)]} u(t - \tau)$ denotes the possible failure actuator with time delay. $\kappa^{[s(t)]}$ is the failure factor. The two-state Markov process $s(t)$ denotes the state of the actuator: $s(t) = 1$ is the “failure” state and $s(t) = 2$ is the “functional” state. λ_1 denotes the transition rate from the breakdown to the functional state and λ_2 backward transition state. The transition matrix $\Lambda = [\lambda_{ij}]$ has the form

of (3). The stationary distribution for $s(t) = u$ is calculated from (4):

$$\mu^{(u)} = \frac{\sum_{i=1 \neq u} \lambda_i}{\sum_{i=1,2} \lambda_i}, \quad u \in S = \{1, 2\}. \quad (27)$$

$u(t - \tau)$ can be specified as

$$u(t - \tau) = -f_1 q_1(t - \tau) - f_2 p_1(t - \tau). \quad (28)$$

According to the approximate expression as (9), (26) becomes

$$\begin{aligned} \dot{q}_1 &= p_1, \\ \dot{p}_1 &= - \sum_{u=1,2} \left[\left(\tilde{\omega}_1^{(u)} \right)^2 \mu^{(u)} \right] q_1 - \sum_{u=1,2} \left[\tilde{\beta}_{11}^{(u)} \mu^{(u)} \right] p_1 \\ &\quad - \beta_{22} p_2 + f_{11} q_1 \xi_1(t) + f_{12} q_2^2 \xi_2(t), \end{aligned} \quad (29)$$

$$\begin{aligned} \dot{q}_2 &= p_2, \\ \dot{p}_2 &= -\alpha q_2^3 - \beta_{21} p_1 - \beta_{22} p_2 + f_{21} q_1 \xi_1(t) \\ &\quad + f_{22} q_2^2 \xi_2(t), \end{aligned}$$

where

$$\begin{aligned} \left(\tilde{\omega}_1^{(u)} \right)^2 &= \omega_1^2 + [f_1 \cos(\omega_1 \tau) + f_2 \omega_1 \sin(\omega_1 \tau)] \kappa^{(u)}, \\ \tilde{\beta}_{11}^{(u)} &= \beta_{11} + \left[-\frac{f_1 \sin(\omega_1 \tau)}{\omega_1} + f_2 \cos(\omega_1 \tau) \right] \kappa^{(u)}. \end{aligned} \quad (30)$$

Note that the Hamiltonian system associated with (29) is completely integrable. The associated Hamiltonian is

$$\begin{aligned} H(\tau) &= \sum_{i=1,2} H_i(q_i, p_i; \tau), \\ H_1(q_1, p_1; \tau) &= \frac{1}{2} p_1^2 + \frac{1}{2} \sum_{u=1,2} \left[\left(\tilde{\omega}_1^{(u)} \right)^2 \cdot \mu^{(u)} \right] q_1^2, \\ H_2(q_2, p_2; \tau) &= \frac{1}{2} p_2^2 + \frac{1}{4} \alpha q_2^2. \end{aligned} \quad (31)$$

After the limit averaging and stochastic averaging, the averaged Itô equation for system (29) can be derived as the form of (13). Linearizing the averaged Itô equation yields

$$dH_i = m_i^L(H_1, H_2; \tau) dt + \bar{\sigma}_{ik}^L(H_1, H_2; \tau) dB_k(t), \quad (32)$$

where

$$\begin{aligned} m_1^L(H_1, H_2) &= - \sum_{u=1,2} \left[\tilde{\beta}_{11}^{(u)} \mu^{(u)} \right] H_1 \\ &\quad + \frac{f_{11}^2 D_1}{\sum_{u=1,2} \left[\left(\tilde{\omega}_1^{(u)} \right)^2 \mu^{(u)} \right]} H_1 \\ &\quad + \eta_1 f_{12}^2 D_2 H_2, \end{aligned}$$

$$\begin{aligned} m_2^L(H_1, H_2) &= -\eta_2 \beta_{22} H_2 \\ &\quad + \frac{f_{21}^2 D_1}{\sum_{u=1,2} \left[\left(\tilde{\omega}_1^{(u)} \right)^2 \mu^{(u)} \right]} H_1 \\ &\quad + \eta_1 f_{22}^2 D_2 H_2, \end{aligned}$$

$$\bar{\sigma}_{1k}^L \bar{\sigma}_{1k}^L(H_1, H_2) = \frac{f_{11}^2 D_1}{\sum_{u=1,2} \left[\left(\tilde{\omega}_1^{(u)} \right)^2 \mu^{(u)} \right]} H_1^2,$$

$$\bar{\sigma}_{2k}^L \bar{\sigma}_{2k}^L(H_1, H_2) = 2\eta_3 f_{22}^2 D_2 H_2^2,$$

$$\bar{\sigma}_{1k}^L \bar{\sigma}_{2k}^L(H_1, H_2) = 0, \quad (33)$$

where

$$\begin{aligned} \eta_1 &= \frac{(8/9\alpha)^{1/2}}{\int_0^1 (1-t^4)^{-1/2} dt}, \\ \eta_2 &= 2 \frac{\int_0^1 (1-t^4)^{1/2} dt}{\int_0^1 (1-t^4)^{-1/2} dt}, \\ \eta_3 &= \frac{(8/\alpha) \int_0^1 t^4 (1-t^4)^{1/2} dt}{\int_0^1 (1-t^4)^{-1/2} dt}. \end{aligned} \quad (34)$$

Note that the drift and diffusion coefficients in (33) are homogeneous in H_i of order one. Introducing the transformation as the form in (18a) and (18b), the Itô equations for ρ and α_1 are obtained

$$\begin{aligned} d\rho &= G(\alpha_1; \tau) dt + \Xi(\alpha_1; \tau) dB(t), \\ d\alpha_1 &= \bar{G}(\alpha_1; \tau) dt + \bar{\Xi}(\alpha_1; \tau) dB(t), \end{aligned} \quad (35)$$

where

$$\begin{aligned} G(\alpha_1) &= \left(\mu_1 + \frac{f_{21}^2 D_1}{\sum_{u=1,2} \left[\left(\tilde{\omega}_1^{(u)} \right)^2 \mu^{(u)} \right]} \right) \alpha_1 \\ &\quad + (\mu_2 + f_{12}^2 D_2 \eta_1) (1 - \alpha_1) \\ &\quad + \frac{A \alpha_1 (1 - \alpha_1)}{4}, \\ \bar{G}(\alpha_1) &= \left[2(\mu_1 - \mu_2) + \frac{A(1 - 2\alpha_1)}{2} \right] \alpha_1 (1 - \alpha_1) \\ &\quad + f_{12}^2 D_2 \eta_1 (1 - \alpha_1)^2 \\ &\quad - \frac{f_{21}^2 D_1}{\sum_{u=1,2} \left[\left(\tilde{\omega}_1^{(u)} \right)^2 \mu^{(u)} \right]} \alpha_1^2, \end{aligned}$$

$$\begin{aligned}
\bar{\Xi}^2(\alpha_1) &= A\alpha_1^2(1-\alpha_1)^2, \\
\mu_1 &= -\frac{1}{2} \sum_{u=1,2} [\tilde{\beta}_{11}^{(u)} \mu^{(u)}] + \frac{f_{11}^2 D_1}{\sum_{u=1,2} [(\tilde{\omega}_1^{(u)})^2 \mu^{(u)}]}, \\
\mu_2 &= -\frac{1}{2} \beta_{22} \eta_2 + \frac{1}{2} (\eta_1 - \eta_3) f_{22}^2 D_2, \\
A &= \frac{f_{11}^2 D_1}{\sum_{u=1,2} [(\tilde{\omega}_1^{(u)})^2 \mu^{(u)}]} + 2\eta_3 f_{22}^2 D_2.
\end{aligned} \tag{36}$$

Following (24), the largest Lyapunov exponent is then obtained

$$\Lambda_{\max} = \int_0^1 G(\alpha_1; \tau) p(\alpha_1) d\alpha_1. \tag{37}$$

The stationary probability density $p(\alpha_1)$ can be obtained by solving the Fokker-Planck-Kolmogorov (FPK) equation associated with Itô equation for α_1 as

$$\begin{aligned}
p(\alpha_1) &= \frac{C}{\alpha_1(1-\alpha_1)} \left(\frac{\alpha_1}{1-\alpha_1} \right)^{4(\mu_1-\mu_2)/A} \\
&\times \exp \left\{ -\frac{2}{A} \left[\frac{\eta_1 f_{12}^2 D_2}{\alpha_1} \right. \right. \\
&\quad \left. \left. + \frac{f_{21}^2 D_1}{(1-\alpha_1) \sum_{u=1,2} [(\tilde{\omega}_1^{(u)})^2 \mu^{(u)}]} \right] \right\}.
\end{aligned} \tag{38}$$

Some numerical results are obtained as shown in Figures 1–4 for system parameters

$$\begin{aligned}
\beta_{11} &= 0.01; \\
\beta_{12} &= 0.0; \\
\beta_{21} &= 0.0; \\
\beta_{22} &= 0.01; \\
f_{11} &= 1.0; \\
f_{12} &= 1.0; \\
f_{21} &= 1.0; \\
f_{22} &= 1.0; \\
\omega_1 &= 3.0; \\
\alpha &= 0.2; \\
f_1 &= 0.04; \\
f_2 &= 0.04; \\
D_1 &= 0.01; \\
D_2 &= 0.01,
\end{aligned}$$

FIGURE 1: The largest Lyapunov exponent $\Lambda_{\max}$ as function of transition rate λ_1 . The lines are obtained from numerical solution of (37) while the symbols are obtained from direct simulation of original system (26).

FIGURE 2: The largest Lyapunov exponent $\Lambda_{\max}$ as function of transition rate λ_2 . The lines are obtained from numerical solution of (37) while the symbols are obtained from direct simulation of original system (26).

$$\kappa^{(1)} = 0.0;$$

$$\kappa^{(2)} = 1.0.$$

(39)

Figure 1 shows the largest Lyapunov exponent $\Lambda_{\max}$ as function of transition rate λ_1 . It is seen that, for small delays, the largest Lyapunov exponent $\Lambda_{\max}$ is a monotonous decreasing function of λ_1 . This is understandable: for small delay, the feedback control plays positive role in stability of system (26). As λ_1 increases, the failure probability of the actuator decreases, and the stability of the system is enhanced as a result. However, for large delay, $\Lambda_{\max}$ is a monotonous increasing function of λ_1 , as also shown in

FIGURE 3: The stability boundaries in the (β_{11}, β_{22}) plane for different transition rate λ_1 when $\tau = 0.3$.

FIGURE 4: The stability boundaries in the (β_{11}, β_{22}) plane for different transition rate λ_1 when $\tau = 0.6$.

Figure 1. This implies that higher probability of actuator working in functional state deteriorates the performance of the dynamics. This is because that large delay feedback control brings negative damping to the system, as shown in (30), which has negative contribution to the stability of system. In Figure 2, the variation of $\Lambda_{\max}$ with λ_2 is displayed for different time delay. It is seen that as λ_2 increases, $\Lambda_{\max}$ increases for small delay while decreasing for large delay. The observation can be explained in a similar fashion as in Figure 1.

In Figure 3, the stability boundaries in the (β_{11}, β_{22}) plane for different transition rate λ_1 in the small delay case are constructed by imposing $\Lambda_{\max} = 0$. In small delay case, as λ_1 is increased, the stability region of the system increases. Figure 4 shows the stability boundaries in (β_{11}, β_{22}) plane for different transition rate λ_1 in the large delay case. Obviously, in large time delay case, large λ_1 will reduce the stability region of the system.

To verify the largest Lyapunov exponent calculated by using (37), the direct simulation of (26) is also obtained and shown in Figures 1 and 2 by using symbols ■. Observe that the symbols match closely with the solid lines, demonstrating the validity and accuracy of the proposed procedure.

7. Conclusions

In this paper, the asymptotic stability with probability of one of delay-controlled nonlinear systems with random actuator failures was investigated. Based on the random switch model of failure-prone actuator, the stability problem of such a system has been converted into the framework of Markov jump hybrid systems. Upon the limit principle and stochastic averaging, an approximate expression for the largest Lyapunov exponent of the original system was derived. Important results reported in the paper are summarized in the following statements:

- (1) Actuator failure has significant effects on the asymptotic stability of the delay-controlled system. For small delay, high probability of actuator failure will degrade the stability while enhancing the stability for large delay.
- (2) The validity and utility of the propose procedure have been demonstrated by using a delay-controlled nonlinear two-degree system with actuator failure. By comparison with direct system simulations, it has been observed that the procedure is fairly reliable in assessing asymptotic stability.

Disclosure

Opinions, findings, and conclusions expressed in this paper are those of the authors and do not necessarily reflect the views of the sponsors.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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